Weighted Quasi-Arithmetic Means: Utility Functions and Weighting Functions

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Abstract. This paper discusses weighted quasi-arithmetic means from viewpoint of a combined index of utility functions and weighting functions, which represent stochastic risk in economics. The combined index characterizes decision maker's attitude and background risks in stochastic environments by conditional expectation representations of weighted quasi-arithmetic means. The first-order stochastic dominance and risk premium are demonstrated using weighted quasi-arithmetic means and aggregated mean ratios, and they are characterized by the combined index. Finally, examples of weighted quasi-arithmetic mean and aggregated mean ratio for various typical utility functions are given.

1 Introduction

This paper deals with weighted quasi-arithmetic means of an interval. Weighted quasi-arithmetic means are important tools in subjective estimation of data in decision making such as management, artificial intelligence and so on ([3–5]), and it is also strongly related to utility and stochastic risk in economics ([6]). Kolmogorov [9] and Nagumo [10] studied the aggregation operators and Aczél [1] developed the theory regarding weighted aggregation. Yoshida [12–15] has studied weighted quasi-arithmetic means of an interval by utility functions and weighting functions from viewpoint of subjective decision making. In relation to decision making, a weighted quasi-arithmetic mean is defined as follows. For a continuous strictly increasing function $f : [a, b] \mapsto (-\infty, \infty)$ as decision maker's utility function and for a continuous function $w : [a, b] \mapsto (0, \infty)$ as weighting function, a *weighted quasi-arithmetic mean* on a closed interval $[a, b]$ is given by

$$f^{-1}\left(\int_a^b f(x)w(x)\,\mathrm{d}x \Bigg/ \int_a^b w(x)\,\mathrm{d}x \right).$$

Hence, it represents a *mean value* given by real number $c(\in [a, b])$ satisfying

$$f(c)\int_a^b w(x)\,\mathrm{d}x = \int_a^b f(x)w(x)\,\mathrm{d}x$$

in the *first mean value theorem for integration*. We investigate the weighted quasi-arithmetic means by a combined index regarding utility functions and

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weighting functions extending the results in Yoshida [13–15]. Weighting functions are corresponding to stochastic risk in economics. Using conditional expectation representations of weighted quasi-arithmetic means, the combined index characterizes decision maker's attitude and background risk in stochastic environments. The first-order stochastic dominance and risk premium are also demonstrated using weighted quasi-arithmetic means and aggregated mean ratios.

In Section 2, we give the definitions of *weighted quasi-arithmetic means* and *aggregated mean ratios* of weighted quasi-arithmetic means by interior ratios, and we show the relation among weighted quasi-arithmetic mean, aggregated mean ratio and decision maker's preference/attitude based on his utility and weighting. In economics, decision maker's attitudes, for example risk neutral, risk averse and risk loving, are characterized by Arrow-Pratt index of utility functions([2, 11, 7, 8]), and risks in stochastic environments are given as an index of weighing functions. In Section 3, this paper characterizes weighted quasi-arithmetic means and mean ratios by not only utility functions but also weighing functions as a combined index. Next we investigate properties of weighted quasi-arithmetic means and aggregated mean ratios regarding combinations of utility functions and weighting functions. Representing weighted quasi-arithmetic means by conditional expectations, we investigate relation between the index for stochastic risks and risk premium in economics. We also discuss the first-order stochastic dominance using weighted quasi-arithmetic means. Finally, in Section 4, we show a lot of examples of the weighted quasi-arithmetic means and the aggregated mean ratios with various typical utility functions, and we demonstrate their relations with the classical quasi-arithmetic means.

2 Weighted Quasi-Arithmetic Means and Their Properties

In this section, we introduce weighted quasi-arithmetic means and aggregated mean ratios with utility functions and weighting functions, and we discuss sufficient conditions on utility functions and weighting functions to characterize decision maker's attitude based on quasi-arithmetic mean and aggregated mean ratio. Let D be a fixed interval which is not a singleton and we call it a domain. Let $\mathcal{C}(D)$ be the set of all nonempty bounded closed subintervals of D and let $\mathcal{C}(D)_< := \{[a,b] \in \mathcal{C}(D) | a < b\}$. Let $f : D \mapsto (-\infty, \infty)$ be a continuous strictly increasing function for utility, and let $w : D \mapsto (0, \infty)$ be a continuous function for weighting. For a closed interval $[a,b] \in \mathcal{C}(D)_<$, a mapping $M_w^f : \mathcal{C}(D) \mapsto D$ given by

$$M_w^f([a,b]) := f^{-1}\left(\int_a^b f(x)w(x)\,\mathrm{d}x \Big/ \int_a^b w(x)\,\mathrm{d}x \right) \tag{1}$$

is called *weighted quasi-arithmetic mean* with specified weighting w. For a closed interval $[a,b] \in \mathcal{C}(D)_<$ we define an interior ratio $\theta_w^f(a,b)$ from a position of weighted quasi-arithmetic mean $M_w^f([a,b])$ on the interval $[a,b]$ by

$$\theta_w^f(a,b) := \frac{M_w^f([a,b]) - a}{b-a}. \tag{2}$$

Dujmović [3–5] studied a *conjunction/disjunction degree*, which is a similar type of ratio in the power case, for computer science. This paper discusses their characterizations from viewpoint of economics. Hence we have the following results.

Lemma 1 ([13]). *Let f and g be two C^2-class utility functions on D. Let $[a,b] \in \mathcal{C}(D)_<$. Then the following (a) – (c) are equivalent.*

(a) $f''/f' \leq g''/g'$ *on* (a,b).
(b) $M_w^f([c,d]) \leq M_w^g([c,d])$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.
(c) $\theta_w^f(c,d) \leq \theta_w^g(c,d)$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.

When we may choose two utility functions f and g as decision makers' utilities, Lemma 1 says that utility f yields more risk averse results than g if $f''/f' \leq g''/g'$ on (a,b). Similarly inequality $\theta_w^f(a,b) \leq \theta_w^g(a,b)$ implies that aggregated mean ratio $\theta_w^f(a,b)$ is more risk averse than $\theta_w^g(a,b)$. Hence $-f''/f'$ is called *Arrow-Pratt index* and it implies the degree of decision maker's absolute risk aversion in micro-economics ([2, 11]). The following lemma implies the properties of weighted quasi-arithmetic mean M_w^f and ratio θ_w^f concerning weighting w.

Lemma 2 ([14, 15]). *Let $w : D \mapsto (0,\infty)$ and $v : D \mapsto (0,\infty)$ be two C^1-class weighting functions. Let $[a,b] \in \mathcal{C}(D)_<$. Then the following (a) – (c) are equivalent.*

(a) $w'/w \leq v'/v$ *on* (a,b).
(b) $M_w^f([c,d]) \leq M_v^f([c,d])$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.
(c) $\theta_w^f(c,d) \leq \theta_v^f(c,d)$ *for all* $[c,d]$ *satisfying* $[c,d] \subset [a,b]$ *and* $c < d$.

Arrow-Pratt index $-f''/f'$ indicates the degree of absolute risk aversion, and the index $-w'/w$ is related to *background risks* of stochastic environments in economics ([8, 15]). In next section, using representation of conditional expectations, we characterize a combination of Arrow-Pratt index $-f''/f'$ and *stochastic risk index* $-w'/w$.

3 Decision Making under Risk

In this paper, we focus on weighting functions w as risk factors of stochastic environments in weighted quasi-arithmetic mean (1) and we characterize it in relation to conditional expectation. Let D be a fixed domain and let $f : D \mapsto (-\infty,\infty)$ be a fixed continuous strictly increasing function for utility. Let (Ω, P) be a probability space, where P is a non-atomic probability measure on Ω.

Definition 1. For random variables X and Y on Ω, it is said that random variable X is *dominated by* random variable Y in the sense of *the first-order stochastic dominance* if

$$P(X < x) \geq P(Y < x) \text{ for any real number } x. \tag{3}$$

Hence the following result is well-known for the first-order stochastic dominance in economics (Arrow [2], Gollier [7], Eeckhoudt et al. [8]).

Lemma 3. *Let X and Y be random variables on Ω. Then, random variable X is dominated by random variable Y in the sense of the first-order stochastic dominance if and only if it holds that*

$$E(f(X)) \leq E(f(Y)) \tag{4}$$

for any increasing utility function $f : (-\infty, \infty) \mapsto (-\infty, \infty)$ satisfying tail condition $\lim_{x \to \pm\infty} f(x)(P(X < x) - P(Y < x)) = 0$.

The first-order stochastic dominance (3) means that stochastic environment X is risky than stochastic environment Y, and it shows in (4) that all decision makers estimate stochastic environment X smaller than stochastic environment Y with respect to their expected utilities. Then decision makers prefer stochastic environment Y to stochastic environment X with their any increasing utility functions f. Let X be a real random variable on Ω with a C^1-class density function w on $(-\infty, \infty)$. Since conditional expectation of utility $f(X)$ is

$$E(f(X) \mid a < X < b) = \frac{E(f(X) 1_{\{a<X<b\}})}{P(a < X < b)} = \frac{\int_a^b f(x) w(x) \, \mathrm{d}x}{\int_a^b w(x) \, \mathrm{d}x}, \tag{5}$$

it holds that

$$M_w^f([a, b]) = f^{-1} \left(\frac{\int_a^b f(x) w(x) \, \mathrm{d}x}{\int_a^b w(x) \, \mathrm{d}x} \right) = f^{-1}(E(f(X) \mid a < X < b)) \tag{6}$$

for real numbers $a, b \, (a < b)$, where $1_{\{\cdot\}}$ implies the characteristic function of a set. From Lemma 2 and (6), we have the following result together with Lemma 3.

Lemma 4 ([13]). *Let X and Y be random variables on Ω which have C^1-class density functions w and v on $(-\infty, \infty)$ respectively. If*

$$\frac{w'}{w} \leq \frac{v'}{v} \quad \text{on } (-\infty, \infty), \tag{7}$$

then random variable X is dominated by random variable Y in the sense of the first-order stochastic dominance.

Eq. (7) is a sufficient condition for the first-order stochastic dominance (3) where stochastic environment X is risky than stochastic environment Y. Hence we find that (7) is useful to estimate risk-levels of stochastic environments and it is easy to check in actual problems. In this paper, we call $-w'/w$ *stochastic risk index*. We note that the first-order stochastic dominance (3) is a risk criterion in global area $D = (-\infty, \infty)$ for stochastic environments and it is represented by integrals in (4), however stochastic risk index $-w'/w$ can measure risks even in local areas because it is represented by differentials.

Next we discuss risk premiums regarding risk averse in financial management ([7, 8]). For simplicity, in this paper we take *initial wealth* is zero. Let $[a, b] \in \mathcal{C}(D)_<$. Let X be a random variable on Ω, which implies a *stochastic environment with some risk*. Decision making with utility f is called *risk averse on* (a, b) if

$$E(f(X) \mid a < X < b) \leq f(E(X \mid a < X < b)). \tag{8}$$

A sufficient condition for risk averse is that utility function f is concave. Let w be a density function on D for random variable X. Hence, in the following (9), real number $\pi_w^f(a, b)$ is called *risk premium on* (a, b) ([7, 8]) if it satisfies

$$E(f(X) \mid a < X < b) = f(-\pi_w^f(a, b)). \tag{9}$$

Eq.(9) means that decision maker accepts the risk arising from random variable X by paying risk premium $\pi_w^f(a, b)$.

Lemma 5 ([15]). *Let f be a continuous strictly increasing utility function on D. Let X be a random variable on Ω which has C^1-class density function w on D. The risk premium in (9) is given by*

$$\pi_w^f(a, b) = -M_w^f([a, b]). \tag{10}$$

Then we obtain the following two theorems. Theorem 1 gives an equivalence relation between combined index and weighted quasi-arithmetic means, and Theorem 2 gives an equivalence relation between combined index and risk premiums.

Theorem 1. *Let $[a, b] \in \mathcal{C}(D)_<$. Let f and g be C^2-class strictly increasing utility functions on D. Let X and Y be random variables on Ω which have C^1-class density functions w and v respectively. Then the following (a) – (c) are equivalent.*

(a) $f''/f' + 2\,w'/w \leq g''/g' + 2\,v'/v$ *on* (a, b).
(b) $M_w^f([c, d]) \leq M_v^g([c, d])$ *for all* $[c, d]$ *satisfying* $[c, d] \subset [a, b]$ *and* $c < d$.
(c) $\theta_w^f(c, d) \leq \theta_v^g(c, d)$ *for all* $[c, d]$ *satisfying* $[c, d] \subset [a, b]$ *and* $c < d$.

Theorem 2. *Let $[a, b] \in \mathcal{C}(D)_<$. Let f and g be C^2-class strictly increasing utility functions on D. Let X and Y be random variables on Ω which have C^1-class density functions w and v respectively. Then the following (a) and (b) are equivalent.*

(a) $f''/f' + 2\,w'/w \leq g''/g' + 2\,v'/v$ *on* (a, b).
(b) $\pi_w^f(c, d) \geq \pi_v^g(c, d)$ *for all* $[c, d]$ *satisfying* $[c, d] \subset [a, b]$ *and* $c < d$.

From Theorems 1 and 2, we find that combined index

$$\frac{f''}{f'} + 2\,\frac{w'}{w} \tag{11}$$

must be essential risk index of stochastic market where decision makers participates in.

4 Examples

In this section, we give examples for weighted quasi-arithmetic means which are presented in the previous sections. When we give a fixed domain D, a continuous strictly increasing function $f : D \mapsto (-\infty, \infty)$ and a fixed continuous function $w : D \mapsto (0, \infty)$, we can define weighted quasi-arithmetic mean $M_w^f([a, b])$ of an interval $[a, b] \in \mathcal{C}(D)$ by (1). We check movement of aggregated mean ratio $\theta_w^f(a, b)$, which is given by (2), with respect to parameters a and b in local regions and global regions in each example. First we discuss several examples of utility functions f.

Example 1.

(i) *(Linear case)* Let $D = (-\infty, \infty)$ and take utility function $f(x) = x$ for $x \in D$. Then $f''(x)/f'(x) = 0$. For a closed interval $[a, b] \in \mathcal{C}(D)_<$, we define *risk neutral weighted mean* $N_w([a, b])$ and its aggregated mean ratio $\nu_w(a, b)$ by

$$N_w([a, b]) := \int_a^b x\ w(x)\,\mathrm{d}x \Bigg/ \int_a^b w(x)\,\mathrm{d}x \tag{12}$$

and

$$\nu_w(a, b) := \frac{N_w([a, b]) - a}{b - a} = \int_a^b (x - a) w(x)\,\mathrm{d}x \Bigg/ \int_a^b (b - a) w(x)\,\mathrm{d}x. \tag{13}$$

Take weighting function $w(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ on $D = (0, \infty)$ with positive constants $c_0, c_1.c_2, \cdots, c_n$. Then

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} = 2\,\frac{\sum_{k=0}^{n-1}(k + 1)c_{k+1}x^k}{\sum_{k=0}^{n} c_k x^k}. \tag{14}$$

For $[a, b] \subset D$ such that $a < b$, we have

$$N_w([a, b]) = \frac{\sum_{k=0}^{n} \frac{1}{k+2} c_k (b^{k+2} - a^{k+2})}{\sum_{k=0}^{n} \frac{1}{k+1} c_k (b^{k+1} - a^{k+1})}.$$

From Yoshida [13, Theorem 5.10], it holds that $\lim_{b\downarrow a} \nu_w(a, b) = \lim_{a\uparrow b} \nu_w(a, b) = 1/2$,

$$\lim_{a\downarrow 0} \nu_w(a, b) = \frac{\sum_{k=0}^{n} \frac{1}{k+2} c_k b^{k+2}}{\sum_{k=0}^{n} \frac{1}{k+1} c_k b^{k+1}} \quad \text{and} \quad \lim_{b\to\infty} \nu_w(a, b) = \frac{n + 1}{n + 2}.$$

(ii) *(Power case)* Take utility function $f(x) = x^r$ and weighting function $w(x) = x^\alpha$ on $D = (0, \infty)$ with constants r, α satisfying $r \neq 0$. Then

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} = \frac{r - 1}{x} + \frac{2\alpha}{x}. \tag{15}$$

Hence we can deal with not only $r > 0$ for increasing function $f(x) = x^r$ but also $r < 0$ for decreasing function $f(x) = x^r$. For $[a, b] \subset D$ such that $a < b$, weighted quasi-arithmetic mean is given by the following $M_{(\alpha)}^{(r)}([a, b]) := M_w^f([a, b])$:

$$M_{(\alpha)}^{(r)}([a,b]) = \left(\frac{(1+\alpha)(b^{1+\alpha+r} - a^{1+\alpha+r})}{(1+\alpha+r)(b^{1+\alpha} - a^{1+\alpha})} \right)^{1/r}$$

if $r \neq 0, \alpha \neq -1, \alpha + r \neq -1$. The limiting values regrading r and α are

$$\lim_{\alpha \to -r-1} M_{(\alpha)}^{(r)}([a,b]) = ab \left(\frac{r(\log b - \log a)}{b^r - a^r} \right)^{1/r} \qquad \text{if } r \neq 0,$$

$$\lim_{\alpha \to -1} M_{(\alpha)}^{(r)}([a,b]) = \left(\frac{r(\log b - \log a)}{b^r - a^r} \right)^{-1/r} \qquad \text{if } r \neq 0,$$

$$\lim_{r \to 0} M_{(\alpha)}^{(r)}([a,b]) = \exp \left(\frac{b^{\alpha+1} \log b - a^{\alpha+1} \log a}{b^{\alpha+1} - a^{\alpha+1}} - \frac{1}{\alpha+1} \right) \quad \text{if } \alpha \neq -1,$$

$$\lim_{\alpha \to -1} \lim_{r \to 0} M_{(\alpha)}^{(r)}([a,b]) = \sqrt{ab},$$

$$\lim_{r \to -\infty} M_{(\alpha)}^{(r)}([a,b]) = a,$$

$$\lim_{r \to \infty} M_{(\alpha)}^{(r)}([a,b]) = b.$$

From Yoshida [13, Corollary 5.4] we also have

$$\theta_w^f(a,b) \lesseqgtr \nu_w(a,b) \quad \text{if } r \lesseqgtr 1.$$

From Yoshida [13, Theorems 5.9 and 5.10], it holds that $\lim_{b \downarrow a} \theta_w^f(a,b) = \lim_{a \uparrow b} \theta_w^f(a,b) = 1/2$ and

$$\lim_{a \downarrow 0} \theta_w^f(a,b) = \lim_{b \to \infty} \theta_w^f(a,b) = \left(\frac{1+\alpha}{1+\alpha+r} \right)^{1/r}.$$

(iii) *(Logarithmic case)* Take concave utility function $f(x) = r \log x$ and weighting function $w(x) = x^\alpha$ on $D = (0, \infty)$ with constants r, α satisfying $r > 0$. Then

$$\frac{f''(x)}{f'(x)} + 2\frac{w'(x)}{w(x)} = -\frac{1}{x} + \frac{2\alpha}{x}. \tag{16}$$

For $[a, b] \subset D$ such that $a < b$, we can check

$$M_w^f([a,b]) = \exp \left(\frac{b^{\alpha+1} \log b - a^{\alpha+1} \log a}{b^{\alpha+1} - a^{\alpha+1}} - \frac{1}{\alpha+1} \right),$$

and Yoshida [13, Corollary 5.4] implies $\theta_w^f(a, b) < \nu_w(a, b)$. Yoshida [13, Theorems 5.9 and 5.10] also imply $\lim_{b \downarrow a} \theta_w^f(a,b) = \lim_{a \uparrow b} \theta_w^f(a,b) = 1/2$ and

$$\lim_{a \downarrow 0} \theta_w^f(a,b) = \lim_{b \to \infty} \theta_w^f(a,b) = \exp \left(-\frac{1}{\alpha+1} \right).$$

(iv) *(Exponential case)* Take convex utility function $f(x) = e^{sx}$ and weighting function $w(x) = x^{\alpha}$ on $D = (0, \infty)$ with constants r, α satisfying $s > 0$. Then

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} = s + \frac{2\alpha}{x}. \tag{17}$$

For $[a, b] \subset D$ such that $a < b$, we can check

$$M_w^f([a, b]) = \frac{1}{s} \log \left(\frac{(1+\alpha)(\Gamma(1+\alpha, -sb) - \Gamma(1+\alpha, -sa))}{s^{1+\alpha}(b^{1+\alpha} - a^{1+\alpha})} \right)$$

and Yoshida [13, Corollary 5.4] implies $\nu_w(a, b) < \theta_w^f(a, b)$, where we put $\Gamma(\alpha+1, z) = \int_z^{\infty} x^{\alpha} e^{-x} dx$ for $z \geq 0$. From Yoshida [13, Theorem 5.9], we obtain $\lim_{b \downarrow a} \theta_w^f(a, b) = \lim_{a \uparrow b} \theta_w^f(a, b) = 1/2$.

(v) Take utility function $f(x) = x^r$ and weighting function $w(x) = x^{\alpha} e^{-\beta x}$ on $D = (-\infty, \infty)$, where $r \neq 0$. Then

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} = \frac{r-1}{x} + 2\left(\frac{\alpha}{x} - \beta\right) \tag{18}$$

for $x \in D$. Then for $[a, b] \in \mathcal{C}(D)_<$ we have

$$M_w^f([a, b]) = \left(\frac{\Gamma(1+\alpha+r, \beta b) - \Gamma(1+\alpha+r, \beta a)}{\beta^r(\Gamma(1+\alpha, \beta b) - \Gamma(1+\alpha, \beta a))} \right)^{1/r},$$

where $\Gamma(\cdot, \cdot)$ is defined by $\Gamma(c, z) := \int_z^{\infty} x^{c-1} e^{-x}\, \mathrm{d}x$ for $c > 0$.

(vi) Take utility function $f(x) = e^{sx}$ and weighting function $w(x) = x^{\alpha} e^{-\beta x}$ on $D = (-\infty, \infty)$, where $s \neq 0$. Then

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} = s + 2\left(\frac{\alpha}{x} - \beta\right) \tag{19}$$

for $x \in D$. Let $[a, b] \subset D = (-\infty, \infty)$ such that $a < b$. Then, for $[a, b] \in \mathcal{C}(D)_<$ we have

$$M_w^f([a, b]) = \frac{1}{s} \log \left(\frac{\beta^{1+\alpha}(\Gamma(1+\alpha, (\beta-s)b) - \Gamma(1+\alpha, (\beta-s)a))}{(\beta-s)^{1+\alpha}(\Gamma(1+\alpha, \beta b) - \Gamma(1+\alpha, \beta a))} \right).$$

(vii) Take utility function $f(x) = x^r e^{sx}$ and weighting function $w(x) = x^{\alpha} e^{-\beta x}$ on $D = (-\infty, \infty)$, where $r \neq 0$. Then

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} = \frac{-r + (r+sx)^2}{(r+sx)x} + 2\left(\frac{\alpha}{x} - \beta\right) \tag{20}$$

for $x \in D$. Let $[a, b] \subset D = (-\infty, \infty)$ such that $a < b$. Then for $[a, b] \in \mathcal{C}(D)_<$ we have $M_w^f([a, b]) =$

$$\frac{r}{s} L \left(\frac{s}{r} \left(\frac{\beta^{1+\alpha}(\Gamma(1+\alpha+r, (\beta-s)b) - \Gamma(1+\alpha+r, (\beta-s)a))}{(\beta-s)^{1+\alpha+r}(\Gamma(1+\alpha, \beta b) - \Gamma(1+\alpha, \beta a))} \right)^{1/r} \right),$$

where L is the inverse function of function $x \mapsto xe^x$.

(viii) Take utility function $f(x) = e^{sx}$ and weighting function $w(x) = e^{-\beta x - \gamma x^2}$ on $D = (-\infty, \infty)$, where $s \neq 0$. We have

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} = s - 2(\beta + 2\gamma x)$$

for $x \in D$. Let $[a, b] \subset D = (-\infty, \infty)$ such that $a < b$. Then for $[a, b] \in \mathcal{C}(D)_<$ it holds that

$$M_w^f([a,b]) = \frac{1}{s}\log\left(\frac{\exp\left(\frac{(\beta-r)^2-\beta^2}{4\gamma}\right)\left(\operatorname{erf}\left(\frac{\beta-r+2\gamma b}{2\sqrt{\gamma}}\right) - \operatorname{erf}\left(\frac{\beta-r+2\gamma a}{2\sqrt{\gamma}}\right)\right)}{\operatorname{erf}\left(\frac{\beta+2\gamma b}{2\sqrt{\gamma}}\right) - \operatorname{erf}\left(\frac{\beta+2\gamma a}{2\sqrt{\gamma}}\right)}\right),$$

where $\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}\,\mathrm{d}x$.

Next we give three examples as applications of Theorems 1 and 2. The following examples show the cases that two decision making with utilities f and g are compared.

Example 2.

(i) *(Square root case and logarithmic case)* Let domain $D = (0, \infty)$. Take concave utility functions $f(x) = \sqrt{x}$ and $g(x) = \log x$ on D and take weighting functions $w(x) = x^\alpha e^{-\beta x - x^2/4}$ and $v(x) = \lambda e^{-\lambda x}$. Then we have

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} = -\frac{1}{2x} + \frac{2\alpha}{x} - 2\beta - x, \tag{21}$$

$$\frac{g''(x)}{g'(x)} + 2\,\frac{v'(x)}{v(x)} = -\frac{1}{x} - 2\lambda. \tag{22}$$

Therefore it follows

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} \le \frac{g''(x)}{g'(x)} + 2\,\frac{v'(x)}{v(x)} \iff x^2 - 2(\lambda - \beta)x - 2\alpha - \frac{1}{2} \ge 0$$

for $x \in D$. If $(\lambda - \beta)^2 + 2\lambda + 1/2 \le 0$,

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} \le \frac{g''(x)}{g'(x)} + 2\,\frac{v'(x)}{v(x)} \qquad \text{for all } x \in D.$$

If $(\lambda - \beta)^2 + 2\lambda + 1/2 > 0$,

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} \le \frac{g''(x)}{g'(x)} + 2\,\frac{v'(x)}{v(x)} \iff x \in (-\infty, x_-] \cup [x_+, \infty),$$

where $x_\pm := \lambda - \beta \pm \sqrt{(\lambda - \beta)^2 + 2\lambda + 1/2}$. From Theorem 1, we obtain $\theta_w^f(a, b) < \theta_v^g(a, b)$ for $[a, b] \in \mathcal{C}(D)$ such that $a < b$, where $\theta_w^f(a, b)$ is aggregated mean ratio given by $f(x)$ and $\theta_v^g(a, b)$ is aggregated mean ratio given by $g(x)$. This shows that $f(x)$ is more risk averse than $g(x)$ as decision making.

(ii) *(Exponential case and logarithmic case)* Let domain $D = (0, \infty)$. Take concave utility functions $f(x) = 1 - e^{-2\lambda x}$ and $g(x) = \log x$ on D and take weighting functions $w(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\,\sigma^2}\right)$ and $v(x) = \sqrt{x}\, e^{-\beta x - x^2/2}$. Then we have

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} = -\frac{2(x - \mu + \lambda\sigma^2)}{\sigma^2}, \tag{23}$$

$$\frac{g''(x)}{g'(x)} + 2\,\frac{v'(x)}{v(x)} = -2(x + \beta). \tag{24}$$

Therefore, if $\sigma^2 \neq 1$,

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} \leq \frac{g''(x)}{g'(x)} + 2\,\frac{v'(x)}{v(x)} \iff x \begin{cases} \leq x_3 & \text{if } \sigma^2 > 1 \\ \geq x_3 & \text{if } \sigma^2 < 1, \end{cases}$$

where $x_3 = \frac{(\lambda-\beta)\sigma^2-\mu}{\sigma^2-1}$. If $\sigma^2 = 1$,

$$\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} \leq \frac{g''(x)}{g'(x)} + 2\,\frac{v'(x)}{v(x)} \text{ for all } x \in D \iff \mu + \beta - \lambda \leq 0.$$

From Theorem 1, we obtain $\theta^f_w(a,b) < \theta^g_v(a,b)$ for $[a,b] \subset (0,1]$ such that $a < b$ and we also obtain $\theta^f_w(a,b) > \theta^g_v(a,b)$ for $[a,b] \subset [1,\infty)$ such that $a < b$, where $\theta^f_w(a,b)$ is aggregated mean ratio given by $f(x)$ and $\theta^g_v(a,b)$ is aggregated mean ratio given by $g(x)$. This shows that $f(x)$ is more risk averse than $g(x)$ in the region $(0,1)$ and that $f(x)$ is more risk loving than $g(x)$ in the region $(1,\infty)$. This example shows that decision makers' attitudes are comparable in each local area using the index $f''/f' + 2\,w'/w$.

(iii) *(Weighted quasi-arithmetic means and conditional expectations)* Finally we show the relation between weighted quasi-arithmetic means and conditional expectations and their application to economics. We give an example for Theorems 1 and 2 by normal distributions on stochastic environments. Let domain $D = (0, \infty)$. Take concave utility functions $f(x) = x^r$ and $g(x) = x^s$ on D. Let random variables X and Y have normal distributions on Ω with density functions w and v respectively as follows: Let μ_X and μ_X be the means and let σ_X and σ_Y be the standard deviations for w and v respectively,

$$w(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{(x-\mu_X)^2}{2\,\sigma_X^2}\right)$$

and

$$v(x) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(x-\mu_Y)^2}{2\,\sigma_Y^2}\right)$$

for real numbers x. Then we have

$$
\begin{aligned}
&\frac{f''(x)}{f'(x)} + 2\,\frac{w'(x)}{w(x)} \leq \frac{g''(x)}{g'(x)} + 2\,\frac{v'(x)}{v(x)} \\
&\iff \frac{r-1}{x} - \frac{2(x-\mu_X)}{\sigma_X^2} \leq \frac{s-1}{x} - \frac{2(x-\mu_Y)}{\sigma_Y^2} \\
&\iff (r-s)\sigma_X^2\sigma_Y^2 - 2(\sigma_X^2\mu_Y - \sigma_Y^2\mu_X)x + 2(\sigma_X^2 - \sigma_Y^2)x^2 \leq 0 \\
&\iff x \in D_{\leq} := \begin{cases} (-\infty,\infty) & \text{if } \sigma_X < \sigma_Y \text{ and } \eta \leq 0 \\ (-\infty, x_-] \cup [x_+,\infty) & \text{if } \sigma_X < \sigma_Y \text{ and } \eta > 0 \\ [x_-, x_+] & \text{if } \sigma_X > \sigma_Y \text{ and } \eta < 0 \\ \emptyset & \text{if } \sigma_X > \sigma_Y \text{ and } \eta \geq 0 \\ [x_4,\infty) & \text{if } \sigma_X = \sigma_Y \text{ and } \mu_X < \mu_Y \\ (-\infty, x_4] & \text{if } \sigma_X = \sigma_Y \text{ and } \mu_X > \mu_Y \\ (-\infty,\infty) & \text{if } \sigma_X = \sigma_Y \text{ and } \mu_X = \mu_Y, \end{cases}
\end{aligned}
$$

where $x_4 = \frac{(r-s)\sigma_X^2}{2(\mu_Y-\mu_X)}$, $\eta := (\sigma_X^2\mu_Y - \sigma_Y^2\mu_X)^2 - 2(r-s)(\sigma_X^2-\sigma_Y^2)\sigma_X^2\sigma_Y^2$, $x_\pm := \frac{\sigma_X^2\mu_Y - \sigma_Y^2\mu_X \pm \sqrt{\eta}}{(r-s)\sigma_X^2\sigma_Y^2}$. By Theorems 1 and 2 we get $M_w^f([a,b]) \leq M_v^f([a,b])$ and $\pi_w^f(a,b) = -M_w^f([a,b]) \geq -M_v^f([a,b]) = \pi_v^f(a,b)$ for subintervals $[a,b] \subset D_{\leq}$. Further if $\sigma_X < \sigma_Y$ and $\eta \leq 0$, all agents prefers stochastic environment Y to stochastic environment X for his any increasing utility f, i.e. it holds that $E(f(X)) \leq E(f(Y))$ for any increasing utility function f, which is equivalent that X is dominated by Y in the sense of the first-order stochastic dominance.

5 Conclusions

We have analyzed weighted quasi-arithmetic means with utility functions and weighting for random factors in stochastic environments. We have investigated a lot of examples of weighted quasi-arithmetic means and aggregated mean ratio for various typical utility functions. Stochastic dominance is a risk criterion in a global area for stochastic environments. Using combined index $f''/f' + 2w'/w$, we can analyze risks even in local areas. Combined index $f''/f' + 2w'/w$ will be useful and easy to calculate in actual problems.

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