# Chapter 5
# Collisionless Plasmas

## 5.1 Introduction: From Individual Particles to Fluids

In the previous chapters we have studied the dynamics of trapped particles in given magnetic and electric fields of sources that are *external* to the particle population of interest (such as the geomagnetic field, boundary and cross-tail currents, solar-wind-imposed electric field, etc.). On occasion we examined the electric currents and charge density generated by the particles as the result of their motion in the given fields, but neglected any retro-effects on such fields and resulting feedback on the collective behavior of the particles that caused such effects in the first place.

The population of radiation belt particles represents a very small proportion of total kinetic energy and mass in the terrestrial magnetosphere. The magnetosphere itself is shaped by currents carried by a much denser, quasi-neutral and mostly collisionless ensemble of lower energy ions and electrons confined by the magnetic field—the *magnetospheric plasma*—representing the bulk of kinetic energy flow and mass[1] in the terrestrial outer environment. In the remaining sections of this book we shall study the dynamics of these multi-species particle ensembles as a natural extension of our preceding discussion of the adiabatic theory of individual particle motion. This approach (sometimes called orbit-theory approach) will allow us to gain a better physical understanding of the sometimes intricate structure and behavior of a plasma as represented by local macroscopic temporal and/or spatial averages of particle properties—the *macroscopic fluid variables* for which our instruments provide the data and which we invoke whenever we picture mentally, describe mathematically or model numerically a collisionless plasma. Our ultimate aim is to analyze and help understand a plasma and its electromagnetic fields as one whole—a self-organizing entity with distinct but thoroughly interacting regions which in general cannot be studied and understood in isolation from each other.

[1]This sounds a bit pompous. The maximum total energy flow in the astronomically-sized magnetosphere can be estimated at barely ∼20,000 MW, its total plasma mass a mere 20 t.

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There are several ways to visualize and represent an ensemble of charged particles in a magnetic field. For pedagogical reasons, at this early stage of our discussion we will make three fundamental simplifying assumptions. First, we shall *consider separately* only one of the different species of charged particles that may constitute a plasma (electrons, protons, alpha particles, heavier ions). Second, in considering only one given species, we shall *neglect any electrostatic effects* caused by the spatial accumulation of same-sign electric charges (for instance, by assuming that in the case of a positive ion ensemble, there would be enough low-energy ambient electrons around to neutralize any such effect). Third, we shall *neglect collisions* and the field singularities in the proximity of each particle (i.e., assume a continuous, finite magnetic and electric field everywhere in the ensemble). In addition, we shall consider only *non-relativistic* particle ensembles (a limitation that excludes plasmas in the extreme environments of neutron stars and black holes). Later we shall turn to the realistic situation of an electrically quasi-neutral *mixture* of at least two different species of opposite charges. In all this, the particle distribution function (see Sect. 4.1 for definition and examples) will be the "workhorse" for the initial mathematical description of an ensemble, providing the link between microscopic properties and more intuitive and measurable macroscopic physical variables. Let us point out that as mentioned in Sect. 4.1, a distribution function already represents an average, in which an enormous number of degrees of freedom (the exact positions and velocities of each one of the particles in an ensemble) are condensed into just six variables—the coordinates of a point in 6-D *phase space* at which the distribution function represents a density (number of particles per unit volume of coordinate and momentum space).

Our point of departure will be the *kinetic theory* of an ensemble of charged particles, each species of which is described by a time-dependent particle distribution function $f(\boldsymbol{r}, \boldsymbol{p}, t)$ in phase space $\{\boldsymbol{r}, \boldsymbol{p}\}$, obeying Liouville's equation (4.25). In non-relativistic plasma physics, it is customary to define the particle distribution function in *velocity* subspace, as we did in Sect. 4.4; henceforth we shall use the general distribution function $f = f(\boldsymbol{r}, \boldsymbol{v}, t)$. Taking into account (1.1) for the local force (and neglecting other external forces such as gravitation), we write the Liouville equation in the form:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_r f + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \nabla_v f = 0 \tag{5.1}$$

This is called the *Vlasov equation*, basis of the kinetic theory of collisionless plasmas. The vector operator $\nabla_v$ has components $\partial/\partial v_i$.

As shown in Sect. 4.3, the distribution function $f$ serves to define macroscopic quantities as average values of physical variables of the ensemble particles. For a given species, we list them again:

Number density: $n(\boldsymbol{r}, t) = \int f \, d\boldsymbol{v}$
Mass density: $\rho_m(\boldsymbol{r}, t) = mn(\boldsymbol{r}, t)$
Bulk (or average) velocity: $\boldsymbol{V} = \langle \boldsymbol{v} \rangle = \int \boldsymbol{v} f \, d\boldsymbol{v} / \int f \, d\boldsymbol{v}$

Kinetic (or momentum flux) tensor (see (4.15)): $\mathbb{K}(\boldsymbol{r},t) = m\int \boldsymbol{v}\otimes\boldsymbol{v} f\, d\boldsymbol{v}$
Pressure tensor (kinetic tensor in a frame traveling with bulk velocity $\boldsymbol{V}$, (4.16)):
$\mathbb{P}(\boldsymbol{r},t) = m\int(\boldsymbol{v}-\boldsymbol{V})\otimes(\boldsymbol{v}-\boldsymbol{V}) f\, d\boldsymbol{v} = \mathbb{K} - mn\boldsymbol{V}\otimes\boldsymbol{V}$
Average kinetic energy density: $\epsilon = 1/2\, nm\langle v^2\rangle = 1/2\, m\int v^2 f\, d\boldsymbol{v}$
Internal energy density (kinetic energy density in a frame traveling with bulk velocity $\boldsymbol{V}$): $w = 1/2\, m\int(\boldsymbol{v}-\langle\boldsymbol{v}\rangle)^2 f\, d\boldsymbol{v} = \epsilon - 1/2nmV^2 = 1/2\,\mathrm{Tr}\mathbb{P}$
Charge density: $\rho_q(\boldsymbol{r},t) = q\, n(\boldsymbol{r},t)$
Electric current density: $\boldsymbol{J} = q\, n\boldsymbol{V} = q\int \boldsymbol{v} f\, d\boldsymbol{v}$

These are the fundamental macroscopic variables for a *particle fluid* (also called kinetic fluid) description of an ensemble of charged particles.[2] Notice that the electromagnetic field does not appear explicitly, except in the expression of the forces on the particles that ultimately control the distribution function via the Vlasov equation.

## 5.2 The Guiding Center Fluid Model

In many situations of collisionless magnetospheric plasmas the constituent charged particles behave adiabatically, i.e., they gyrate rapidly in cyclotron motion perpendicular to $\boldsymbol{B}$, they move parallel to $\boldsymbol{B}$ (bounce, if the field geometry is right) and they drift perpendicularly to $\boldsymbol{B}$—as long as the conditions (2.1) and (2.2) hold for the particles and the field. As we have done in the adiabatic theory of single particles in Chaps. 1–3, the mathematical description and mental visualization of the ensemble can be simplified by averaging all dynamic variables over one cyclotron turn and replacing each "madly gyrating" particle by a virtual particle at its guiding center, bouncing along and drifting across magnetic field lines. However, to accomplish

[2]A brief detour into Foundations of Physics is in order here. In the Preface we already stated that "physics is the art of modeling", and in Sect. 1.1 we introduced the model of a "guiding center particle". A fluid (any fluid!) is also a model—the model of a system in which a huge, mathematically unmanageable, number of physically real particles (molecules, atoms, electrons, nucleons, quarks, gluons, sand grains, etc., depending on the ensemble in question) has been replaced in our mental image and in the quantitative description by a virtual *continuum* (see also Appendix A.1, page 160). We speak of and quantitatively describe "parcels" of fluid and imagine how they are deformed as they move, and guided by what our physiological senses experience when exposed to liquids or flowing gases, we introduce macroscopic variables which can be used for practical purposes, like density, bulk velocity, pressure, temperature, internal energy, entropy, etc. Statistical mechanics and, as a corollary, plasma physics were developed to link approximate but intuitive macroscopic continuum descriptions of matter with their physically real microscopic structures that can only be revealed through the use of scientific instruments. In our case, distribution functions and the differential equations which they obey establish such a link. The main aim of any fluid description is to formulate physical-mathematical relationships between the macroscopic variables so as to provide a "coarse-grained" quantitative description of the dynamic state of the ensemble—regardless of the unknowable detailed state (position, velocity) of each elementary constituent.

this we must require as an additional condition that the particle distribution be *gyrotropic* (strictly speaking, gyrotropic in a local coordinate system that moves with the average guiding center velocity (2.14) at that point). This means that there should be no synchronized cyclotron phase bunching (Fig. 4.9); in presence of electromagnetic waves of the order of the particles' gyrofrequency, this assumption is no longer valid. All this allows us to dispense of one degree of freedom, the phase angle $\varphi$ of a particle's cyclotron motion, and use the distribution function

$$F = F(\boldsymbol{r}, t, v_\perp, v_\parallel) \tag{5.2}$$

where $\boldsymbol{r}$ is not the position $\boldsymbol{r}_p$ of the actual particle but that of its guiding center $\boldsymbol{r} = \boldsymbol{r}_p + (m/q)\boldsymbol{v} \times \boldsymbol{B}/B^2$ (see relation (1.25)). The quantity $\delta n = F(\boldsymbol{r}, t, v_\perp, v_\parallel)\delta\boldsymbol{r}\delta v_\perp \delta v_\parallel$ represents the number of virtual guiding centers in $\delta\boldsymbol{r}$ at point $\boldsymbol{r}$ and time $t$, whose "parent" particles have velocities $v_\perp$ and $v_\parallel$ in the element $\delta v_\perp \delta v_\parallel$. Of course, it is also possible to use derived distribution functions such as $F(\boldsymbol{r}, t, T, \mu)$ (Sect. 4.2).

Having eliminated $\varphi$ does not mean that we can neglect collective effects of the cyclotron motion. First, notice the hidden presence of the $\boldsymbol{B}$ vector: at each point in space it defines the $\perp$ and $\parallel$ directions, the natural frame of coordinates (Appendix A.1) (or, in derived distribution functions, the parameters $\mu$, $M$ or $I$). Second, the magnetic moment $\boldsymbol{M}$ (1.26) generated by the now "washed-out" cyclotron motion must be retained in the contribution of the particles to the magnetic field. Likewise, the particle's angular momentum $\boldsymbol{l} = (2m/q)\boldsymbol{M}$ (1.27) must be retained as a contribution to the macroscopic dynamic state of the fluid. Third, we must retain the contribution of a particle's cyclotron motion to the perpendicular pressure $p_\perp$ and that of its parallel motion to the parallel pressure $p_\parallel$ (4.17), as well as to the internal kinetic energy density $w$. As a consequence of all this we picture the *guiding center fluid* as a model fluid consisting of *magnetized* virtual GC particles with *intrinsic* angular momentum, and endowed with local vorticity, internal kinetic energy, temperature and perpendicular and parallel pressures. The magnetic field thus assumes in explicit form the role of a "scaffolding", an internal skeleton that greatly aids in visualizing plasmas but whose local asymmetry obliges us to always be aware of the different character of transverse and field aligned properties, respectively.

In particular, concerning the field-aligned motion of the guiding center particles, the conservation of each individual particle's magnetic moment (1.26) provides a fundamental link between points of a given field line in a guiding center fluid. For instance, great care has to be taken with the interpretation and handling of distribution functions in the guiding center fluid model. As shown in Sect. 4.4, they are causally connected along a given field line because of the bounce motion; for instance, in an equilibrium situation in which there is no particle bunching, the distribution function in a guiding center fluid can only be prescribed on a specified *surface* such as the minimum-$B$ surface which is traversed by all trapped particles on a field line (Fig. 4.12); it cannot be chosen arbitrarily all along a field line. In what follows, the position vector $\boldsymbol{r}$ in the distribution function $F$ will usually signify "distribution function at a reference (e.g., minimum-$B$) point of the field line going through point $\boldsymbol{r}$". Although in the preceding text we have been mentioning bounce motion, in plasma physics neither bounce nor drift periodicities (and the related adiabatic invariants $J$ and $\Phi$) play any direct role; mainly, because the field geometries, their time variations and the presence of multiple but mutually interacting classes of particles do not favor sustained particle trapping. Therefore, for guiding center particles in a GC fluid there will be no bounce-average drift velocities nor any drift-average quantities—only instantaneous ones. The only averaging is done over cyclotron motion.

We can list expressions for the macro-variables in the guiding center description, as we did for a kinetic fluid:

Number density: $n(\boldsymbol{r}, t) = \int F \, dv_\perp dv_\parallel$
Mass and charge densities: $\rho_m(\boldsymbol{r}, t) = m\, n(\boldsymbol{r}, t)$ and $\rho_q(\boldsymbol{r}, t) = q\, n(\boldsymbol{r}, t)$
Perpendicular and parallel pressures (refer to relations (4.17)): $p_\perp = 1/2\, mn \left(\langle v_\perp^2 \rangle - \langle v_\perp \rangle^2\right)$ and $p_\parallel = mn\left(\langle v_\parallel^2 \rangle - \langle v_\parallel \rangle^2\right)$
Magnetic moment density (refer to (4.24)): $\boldsymbol{M} = -1/2mn\langle(\boldsymbol{v}_\perp - \boldsymbol{V}_D)^2\rangle\, \boldsymbol{B}/B^2 = -p_\perp/B\, \boldsymbol{e}$
Angular momentum density: $\boldsymbol{L} = (2m/q)\, \boldsymbol{M}$

For the bulk velocity things are different. Each virtual guiding center particle has a perpendicular drift velocity $\boldsymbol{V}_D$ which, however, is not an independent variable: it is a function of $v_\perp$, $v_\parallel$ and the local magnetic field (2.14). On the other hand, the parallel velocity of a guiding center particle is a vector equal to the original particle's parallel velocity $\boldsymbol{v}_\parallel$, and it is an independent variable (2.11). Thus for a guiding center fluid we write:

Bulk perpendicular (or drift) velocity: $\boldsymbol{V}_{g\perp} = \int \boldsymbol{V}_D F \, dv_\perp dv_\parallel / \int F \, dv_\perp dv_\parallel$[3]
Bulk parallel (or field-aligned) velocity: $\boldsymbol{V}_{g\parallel} = \int \boldsymbol{v}_\parallel F \, dv_\perp dv_\parallel / \int F \, dv_\perp dv_\parallel$

As mentioned above, we will mainly deal with ensembles with symmetric pitch angle distributions in which there is no field-aligned bulk streaming (no field-aligned convection currents) and $V_{g\parallel} = \langle \boldsymbol{v}_\parallel \rangle \equiv 0$.

It is important to understand the difference between the bulk velocities in both fluid models. In the kinetic model, $\boldsymbol{V}$ is the spatial average of the instantaneous velocity vectors of *actual* particles in an element of volume, whereas in the GC fluid, $\boldsymbol{V}_g$ is a double average: the spatial average of the velocities $\boldsymbol{V}_D$, $\boldsymbol{V}_\parallel$, which are averages (over a cyclotron turn) of the velocity components of a particle: $\boldsymbol{V}_g = \langle\langle \boldsymbol{v} \rangle_\varphi\rangle$. The "missing part" of particle motion in the GC fluid model is encoded in the magnetic moment of each GC particle. The bulk velocity vector $\boldsymbol{V}_g$ of an ensemble of guiding center particles always describes *true* macroscopic mass transport, whereas the mean velocity vector $\boldsymbol{V}$ of the ensemble of the original particles may not—both velocity vectors in general will differ from each other

[3] From now on, all macro-variables in the guiding center fluid will carry the subindex $g$, whereas homologous variables in the kinetic particle fluid model will not be subindexed.

($\boldsymbol{V}_g \neq \boldsymbol{V}$); in fact, they even may be opposite to each other. Below we'll show some simple examples.

What both models *must* have in common, are the values of local current density $\boldsymbol{J}$, which links the ensemble dynamically with the local magnetic field (action of the Lorentz force $\boldsymbol{J} \times \boldsymbol{B}$ *on* the current, and contribution of the current *to* the sources $\nabla \times \boldsymbol{B}$ of the magnetic field). In the kinetic fluid model, which doesn't care whether or not individual particles in a given element of volume have mesoscopically organized motion (such as cyclotron gyration), the current density is $\boldsymbol{J} = \rho_q \boldsymbol{V}$, a pure *convection* current. In the guiding center fluid, we must use the usual $E\&M$ expression for magnetized media taking into account the *equivalent* currents $\boldsymbol{J}_{eq} = \nabla\times\boldsymbol{M}$ (Appendix A.1). The total electric current density will thus be:

$$\boldsymbol{J} = \rho_q \boldsymbol{V}_g + \nabla\times\boldsymbol{M} \tag{5.3}$$

and consequently[4]

$$\boldsymbol{V} = \boldsymbol{V}_g + \rho_q{}^{-1}\nabla\times\boldsymbol{M} = \boldsymbol{V}_g - \rho_q{}^{-1}\nabla\times(\frac{p_\perp}{B^2}\boldsymbol{B}) = \boldsymbol{V}_g + 1/2\rho_m{}^{-1}\nabla\times\boldsymbol{L} \tag{5.4}$$

The last equality stems from the definition of angular momentum density

$$\boldsymbol{L} = 2(m/q)\,\boldsymbol{M} \tag{5.5}$$

We end this section with the promised discussion of some "kindergarten" examples, to show in semi-quantitative form that the perpendicular component $\boldsymbol{V}_\perp$ of the bulk velocity of a kinetic fluid is indeed not necessarily equal to the perpendicular bulk velocity of the corresponding guiding center fluid. Quite generally, these examples are intended to shed some light on the physical nature of different, distinct classes of currents in a guiding center fluid. Consider Fig. 5.1 left side, which depicts a gyrotropic ensemble of mono-energetic 90° pitch angle particles, with a particle density gradient $\boldsymbol{\nabla} n$ in the direction of the $x$-axis, cycling in a uniform external magnetic field $\boldsymbol{B}$ directed along the $z$-axis. If we do a *cyclotron* average of the perpendicular velocity vector of any given particle to obtain its guiding center drift velocity (1.3), we obviously get $\boldsymbol{V}_D = \langle \boldsymbol{v}_\perp \rangle_{cyclotron} = 0$. The particles are all gyrating in situ and the guiding centers are all at rest—there is no flow in the guiding center fluid and there is no net transport of mass or electric charge: the guiding center convection current is $\boldsymbol{J}_g = \rho_q \boldsymbol{V}_g = 0$. But in the guiding center fluid there also will be an *equivalent* current $\boldsymbol{J}_{eq} = \nabla\times\boldsymbol{M} = \nabla\times(-1/2mnv_\perp{}^2/B)\,\boldsymbol{e} = -(1/2mv_\perp{}^2/B)\,\nabla n\times\boldsymbol{e} \neq 0$, always in the direction of $+y$ (regardless of the particles' charge $q$). This (and the next set) describes the effect

[4]The following relation (5.4) can be deduced directly for gyrotropic ensembles by linking the distribution functions $f$ (4.5) and $F$ (5.2) using (1.25) and the definitions of $\boldsymbol{V}$, $\boldsymbol{V}_g$ and $\boldsymbol{M}$ (the proof is lengthy!).

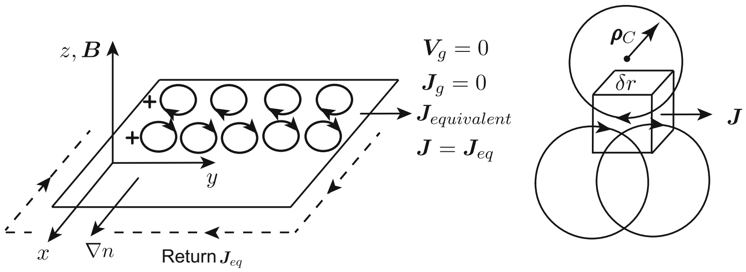


**Fig. 5.1** Sketch showing the physical origin of an equivalent current density in a non-uniform distribution of 90° pitch angle particles (zero GC current, no net charge transport)

of the magnetic field on a specific distribution of guiding center particles; later we will discuss the effect of specific ensembles (their currents) on the magnetic field.

We now turn to the kinetic model description of the same ensemble of Fig. 5.1. For this, we must look at the figure "with a magnifying glass" (or, more realistically, with a very small detector) and realize that the individual particle distribution will be anisotropic: considering a domain much smaller than ${\rho_C}^3$ (right side of the figure) we will always see (or detect) more particles traveling in the $+y$ direction than in any other. This represents a local *convection* current $\boldsymbol{J}$ along the $y$-axis.[5] Since we demand that both fluid model descriptions must be consistent with each other in terms of their macroscopic electromagnetic manifestations, for the case of Fig. 5.1 the convection current in the kinetic fluid must be the same as the equivalent current in the guiding center fluid: $\boldsymbol{J} = \nabla\times\boldsymbol{M}$.

In the example of Fig. 5.1 there is no net transport of electric charge ($\rho_q = 0$, $\partial\rho_q/dt = 0$), yet there is a current density everywhere inside the ensemble. Obviously, conservation of charge tells us that $\nabla\cdot\boldsymbol{J} = 0$, so both, the equivalent currents and (in the particle fluid picture) the convection currents must be *closed* somewhere. Observe Fig. 5.1 (left side): the distribution of particles does not extend to infinity—it must have a boundary somewhere along the $x$ and $y$ axes, which means that, eventually, somewhere there must be negative number density gradients. Such gradients represent current densities, precisely the ones that close the $J_y$ current system in the above example, as sketched in the figure. This observation shows that quite generally it is extremely dangerous to speculate qualitatively about current systems in the magnetosphere (e.g., about the neutral sheet current) without explicitly including a precise picture of all closing currents, too (e.g., the current system where the neutral sheet merges into the tail boundary).

[5]Remember that this is the usual explanation given in $E\&M$ texts to justify the appearance of an equivalent $\boldsymbol{\nabla}\times\boldsymbol{M}$ current in magnetized materials (although in ferromagnetism the magnetization is not due to "little current loops" in atoms but due to the intrinsic quantum magnetic moment (spin) of electrons). Since in an ensemble of trapped particles the magnetic moment associated to a guiding center particle is always directed antiparallel to $B$, plasmas behave like a diamagnetic gas—as we already had anticipated.

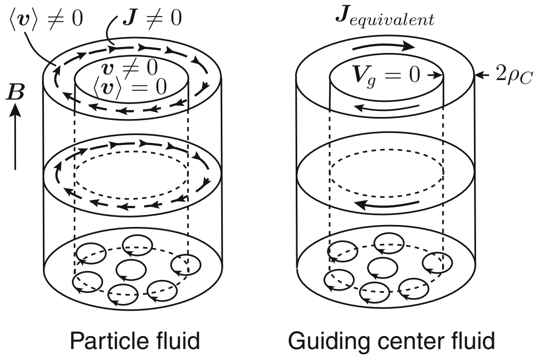


**Fig. 5.2** Flux cylinder in a uniform field with a uniform distribution of positive, mono-energetic 90° pitch angle particles. *Left*: Viewed as a guiding center fluid; there are no flows anywhere, but an equivalent current $\nabla \times \boldsymbol{M}$ in the boundary layer due to magnetic moment cut-off. *Right*: Viewed as a particle fluid at the microscopic level; laminar flow within $2\rho_C$ of the outer boundary

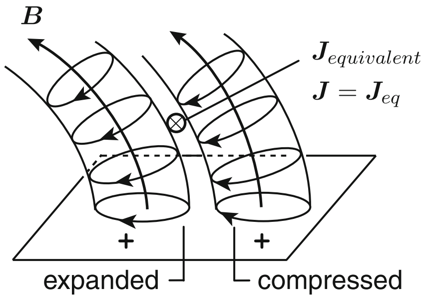


**Fig. 5.3** Origin of the "bent sausage" equivalent current

To consolidate understanding of the case of Fig. 5.1, consider a cylinder of field lines in a uniform magnetic field (a magnetic flux tube) filled with 90° mono-energetic particles with uniform guiding center density, Fig. 5.2. Viewed as a particle fluid, the distribution inside the cylinder will be isotropic everywhere, with zero average velocity except in a thin boundary layer of thickness $\delta r = 2\rho_C$, where there will be a laminar flow (macroscopically a surface current). Viewed as a guiding center particle fluid, the velocity inside the cylinder will be zero, too, but now it is the sudden jump to zero of the magnetization density in the boundary layer (due to the cut-off of guiding center density) which will lead to an *equivalent* current that must be equal to the surface convection current in the kinetic model description. Note that in this case, the current system is closed in itself.

Next consider the example of 90° pitch angle particles uniformly distributed along curved field lines as shown in Fig. 5.3, with $B$ nearly constant along and across those field lines. Such a situation is, indeed, highly artificial and can only represent an instant snapshot: the mirror force (2.8) would immediately start moving the particles along $\boldsymbol{B}$—anyway, this is just a kindergarten example! Viewed either as a guiding center fluid or a particle fluid, the respective average velocities $\boldsymbol{V}_{g\parallel}$

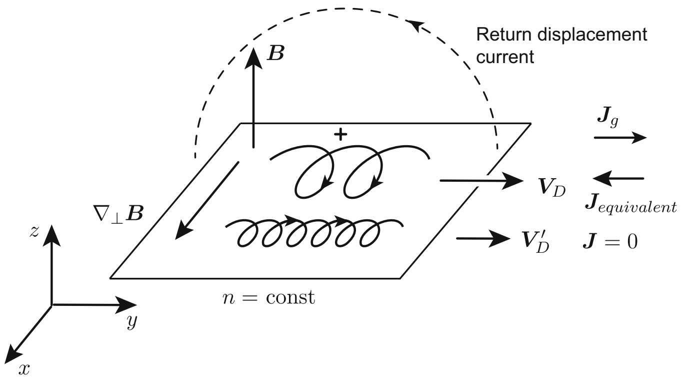


**Fig. 5.4** Origin of the equivalent counter-current $J_{eq}$ in a uniform particle distribution of gradient-$B$ drifting particles

and $\boldsymbol{V}_\parallel$ along the field lines are assumed to be zero (no field-aligned current). But there should be an equivalent current perpendicular to $\boldsymbol{B}$ (into the plane of the figure), of value $\boldsymbol{J}_{eq} = \nabla\times(-p_\perp/B)\boldsymbol{e} = p_\perp/B\,(\partial\boldsymbol{e}/\partial s \times \boldsymbol{e})$ (remember that $\partial\boldsymbol{e}/\partial s = -\boldsymbol{n}/R_C$, with $R_C$ the field line radius of curvature, relation (A.15) in Appendix A.1). Its origin is simple: just look at the figure with a magnifying glass, and you'll see that positive particles move toward you in their cyclotron motion on the convex side of the field line, and go into the paper in a slightly *compressed* fashion, on the concave side. In other words, a tiny detector would see, per unit surface, more particles going into the paper than coming out of it. This is yet another case in which $\boldsymbol{J} = \nabla\times\boldsymbol{M}$ (perpendicular components only!).

Another example, sketched in Fig. 5.4, is that of a uniform 90° pitch angle mono-energetic particle distribution on the minimum-$B$ surface of a magnetic field with a constant gradient $\nabla_\perp B \neq 0$ in the $x$ direction, but *no* gradient in number density $n$. Guiding centers will drift with velocity $\boldsymbol{V}_g$ to the right along the $x$-axis, which represents a guiding center convection current to the right $\boldsymbol{J}_g = -p_\perp/B^2\,\nabla_\perp B\times\boldsymbol{e}$. In addition, there will be an equivalent current $\nabla\times\boldsymbol{M}$, where $\boldsymbol{M} = -(p_\perp/B)\,\boldsymbol{e}$. Since for a uniform distribution only $\boldsymbol{B}$ depends on the position $\boldsymbol{r}$ this equivalent current is directed to the left and *exactly cancels* the convection drift current $\boldsymbol{J}_g$ (remember that $\nabla\times\boldsymbol{e} = -1/B(\nabla B\times\boldsymbol{e})$, so that in this case the total current density in the guiding center fluid model is zero. Therefore, the convection current in the particle fluid model should also be zero. It is a little trickier to convince oneself, by looking at the figure, that the pertinent velocity distribution of particles on the minimum-$B$ surface is indeed isotropic, and that the number of particles moving in the $+y$ direction in a very small element of volume is always the same as the number of those going in the opposite direction, for constant $p_\perp$. In summary, in the case of Fig. 5.4, the current density in the particle fluid is $\boldsymbol{J} = \boldsymbol{J}_g + \nabla\times\boldsymbol{M} \equiv 0$. Here we have an example of a particle distribution with net mass and charge transport (to the right in the figure), but in which the local average particle velocity is zero (the implication of this fact for the magnetospheric ring current will be discussed

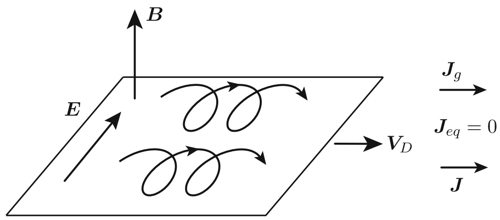


**Fig. 5.5** Case of a GC current equal to the convection current (zero equivalent current)

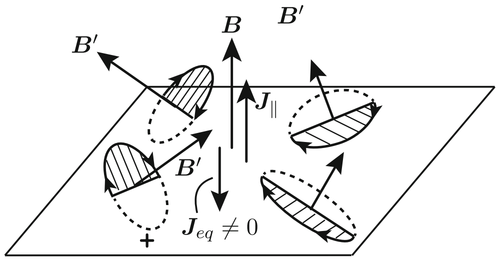


**Fig. 5.6** Sketch (not in scale!) of positive particles on field lines twisted by a parallel current along the central axis (snapshot from the guiding center system moving with velocity $V_\parallel$). The cyclotron loops are perpendicular to the local, inclined, $\boldsymbol{B}'$. Near the central axis there is a net uncompensated velocity component from the cyclotron motion of the particles into the plane ⊥ to $\boldsymbol{B}$

briefly in the next section). Note that in this example there is no equivalent current to close because there just is no current. It is an interesting exercise to figure out the electrodynamics near the right edge of the drifting particle distribution (something quite relevant to the physics of magnetospheric plasma blobs!)

A uniform gyrotropic distribution of mono-energetic particles in a uniform magnetic field, crossed by a uniform electric field $\boldsymbol{E}$ (Fig. 5.5), is a typical example where $\nabla\times\boldsymbol{M} \equiv 0$, i.e., in which the convection currents in both models are equal: $\boldsymbol{J} = \boldsymbol{J}_g$. Note however the difference in physical character of the two types of currents: the particle current $\boldsymbol{J}$ in the kinetic model is due to the fact that, because of electric field acceleration, particles moving to the right in the figure are on the "upper" arc of the cyclotron orbit where their speed is higher than that of left-moving lower arc particles (see (1.35) and Fig. 1.7). Guiding center particles drift at constant speed, equal to the average speed of the actual particles in their OFR.

Our last kindergarten example is that of a configuration in which the mono-energetic particles have a pitch angle <90° in a near-uniform magnetic field. Now there will be a component of $\boldsymbol{J}_g$ parallel to $\boldsymbol{B}$. As explained in Appendix A.1, any parallel current causes a twist of the magnetic field lines: $\nabla\times\boldsymbol{M}$ will have a component parallel to $\boldsymbol{B}$. Consider Fig. 5.6, shown in the guiding center frame of reference that moves with the velocity $\boldsymbol{v}_\parallel$ (assumed common to all particles in this example) along the central field line. As sketched in the figure, because of the torsion, neighboring cyclotron loops are slanted with respect to those of particles circling the central field line. As a result, a geometry arises such that near the axis of symmetry the neighboring particles have a component of motion *counter* the main parallel drift whenever they cross the central loops, a motion that is uncompensated by upward-moving particles from neighboring cyclotron loops. In other words, a parallel equivalent current $\nabla\times\boldsymbol{M}|_{\parallel}$ will always counteract, i.e., diminish the effect of whatever field-aligned guiding center current flows in the first place.

The discussion of these five simple examples suggests that it is always important *to clearly state which fluid model*, kinetic or guiding center, is being invoked in a plasma[6] description. In theoretical analysis and numerical calculations one mostly works with the kinetic model, but when describing intuitively and qualitatively a plasma system, or even when trying to interpret measurement results, physicists more often than not are thinking about, or visualizing, the system in guiding center model terms (often without explicitly saying so). However, there is a danger of using the guiding center fluid model, despite its greater physical intuitiveness concerning the ensemble's macroscopic properties such as electric currents and internal stresses (see next section). One too often forgets that this model is valid only provided that: (i) the guiding center approximations (2.1) and (2.2) apply everywhere (thus excluding neutral lines, sharp gradients, rapid oscillations, etc.), (ii) the collision rate is negligible (thus excluding the ionospheric regions), and (iii) particle distributions are gyrotropic (thus excluding waves in the cyclotron frequency or higher range).

## 5.3 Currents and Stresses Arising from Interactions with the Magnetic Field

In order to analyze the types of currents sustained by an ensemble of guiding center particles defined by a distribution function $F(\boldsymbol{r}, t, v_{\perp}, v_{\parallel})$, we turn to the general expression of the drift velocity (2.14) given in Sect. 2.1. We shall assume that the guiding center approximation is valid everywhere and that there is no bounce bunching ($\langle \boldsymbol{v}_{\parallel}\rangle = 0$), and we shall neglect the action of external non-electromagnetic forces ($\boldsymbol{F} \equiv 0$) as well as all higher order drift terms, i.e., we shall retain only the 1st, 3rd and 4th terms. We thus write, for the transverse drift velocity vector of a single guiding center particle:

$$\boldsymbol{V}_{g\perp} = \boldsymbol{V}_D = \left[q\boldsymbol{E} - \frac{m v_{\perp}^{*2}}{2B}\nabla B - m v_{\parallel}{}^2 \frac{\partial \boldsymbol{e}}{\partial s} - m\frac{d\boldsymbol{V}_D}{dt}\right] \times \frac{\boldsymbol{B}}{qB^2}$$

Remember that $v_{\perp}^{*}$ is the modulus of the perpendicular component of the actual particle's velocity in its guiding center system at the point in question ($\boldsymbol{v}_{\perp}{}^* = \boldsymbol{v}_{\perp} - \boldsymbol{V}_D$, relation (1.6)). It is evident that in the bracket, only the perpendicular

[6]Never mind that in these kindergarten examples we have considered only one class of particles—the results about currents thus far are independent of the electric charge of the particles involved.

components of $\boldsymbol{E}$ and $d\boldsymbol{V}_D/dt$ will contribute. The parallel guiding center velocity $\boldsymbol{V}_{g\parallel}$ will be equal to the cyclotron-average of the parallel velocity of the actual particle (2.11), but as mentioned above, for the time being we will exclude streaming along field lines ($\boldsymbol{V}_{g\parallel} \equiv 0$).

We now use this equation to determine the macroscopic quantity $\langle \boldsymbol{V}_D \rangle$ (transverse bulk velocity), by multiplying each term by the guiding center distribution function $F(\boldsymbol{r}, t, v_\perp, v_\parallel)$ and integrating over velocity space $[v_\perp, v_\parallel]$. We obtain, taking into account the definitions (4.17) of $p_\perp$ and $p_\parallel$ (the latter, with $\boldsymbol{V}_{g\parallel} = 0$):

$$\boldsymbol{V}_{g\perp} = \langle \boldsymbol{V}_D \rangle = \left[\rho_q \boldsymbol{E} - \frac{p_\perp}{B}\nabla_\perp B - p_\parallel \frac{\partial \boldsymbol{e}}{\partial s} - \rho_m \frac{d\boldsymbol{V}_{g\perp}}{dt}\right] \times \frac{\boldsymbol{B}}{\rho_q B^2} \tag{5.6}$$

To obtain the total current $\boldsymbol{J}$, we have to multiply this equation with $\rho_q$ and add $\nabla \times \boldsymbol{M}$. This latter magnetization current density is

$$\nabla \times \boldsymbol{M} = \nabla \times \left(-\frac{p_\perp}{B^2}\boldsymbol{B}\right) = -\frac{p_\perp}{B^2}\nabla \times \boldsymbol{B} - \nabla p_\perp \times \frac{\boldsymbol{B}}{B^2} + 2\frac{p_\perp}{B}\nabla B \times \frac{\boldsymbol{B}}{B^2}$$

The second and third terms are perpendicular to $\boldsymbol{B}$. Using (A.26) and (A.27) of Appendix A.1 for $\nabla \times \boldsymbol{B}|_\perp$ and $\nabla \times \boldsymbol{e}|_\perp = \boldsymbol{e} \times \partial \boldsymbol{e}/\partial s$, we obtain for the $\perp$ and $\parallel$ components of the equivalent current density:

$$\nabla \times \boldsymbol{M}|_\perp = \left[-\nabla_\perp p_\perp + p_\perp \frac{\partial \boldsymbol{e}}{\partial s} + p_\perp \frac{\nabla_\perp B}{B}\right] \times \frac{\boldsymbol{B}}{B^2} \tag{5.7}$$

$$\nabla \times \boldsymbol{M}|_\parallel = -\frac{p_\perp}{B^2}\nabla \times \boldsymbol{B}|_\parallel \tag{5.8}$$

Turning first to the transverse equations, we add expressions (5.7) and (5.6) (multiplied by $\rho_q$) to obtain for the total transverse current density $\boldsymbol{J}_\perp$ (which of course must also be the total transverse current density in the corresponding particle fluid model):

$$\begin{aligned}
\boldsymbol{J}_\perp &= \boldsymbol{J}_{g\perp} + \nabla \times \boldsymbol{M}|_\perp \\
&= \left[\rho_q \boldsymbol{E} - \nabla_\perp p_\perp - (p_\parallel - p_\perp)\frac{\partial \boldsymbol{e}}{\partial s} - \rho_m \frac{d\boldsymbol{V}_{g\perp}}{dt}\right] \times \frac{\boldsymbol{B}}{B^2} \\
&= \boldsymbol{J}_E + \boldsymbol{J}_D + \boldsymbol{J}_A + \boldsymbol{J}_I
\end{aligned} \tag{5.9}$$

where

$$\boldsymbol{J}_E = \rho_q \frac{\boldsymbol{E}}{B} \times \boldsymbol{e} \qquad \text{Electric field drift current} \tag{5.10}$$

$$\boldsymbol{J}_D = -\frac{\nabla_\perp p_\perp}{B} \times \boldsymbol{e} \qquad \text{diamagnetic current} \tag{5.11}$$

$$\boldsymbol{J}_A = -\frac{(p_\parallel - p_\perp)}{B}\frac{\partial \boldsymbol{e}}{\partial s} \times \boldsymbol{e} \qquad \text{“pressure anisotropy” current} \tag{5.12}$$

$$\boldsymbol{J}_I = \frac{\rho_m}{B}\frac{d\boldsymbol{V}_{g\perp}}{dt} \times \boldsymbol{e} \qquad \text{Inertial current} \tag{5.13}$$

Notice that an important rearrangement has taken place by adding the equivalent current that arises from the magnetization of the guiding center fluid. In particular, the gradient-$B$ drift current term (second one in (5.6)) *has dropped out*, cancelled by an homologous term in (5.7). This is exactly what happened in our “kindergarten example” shown in Fig. 5.4 in the preceding section![7] In the above list, $\boldsymbol{J}_E$ and $\boldsymbol{J}_I$ are convection currents, $\boldsymbol{J}_D$ is an equivalent current and $\boldsymbol{J}_A$ is mixed: the first part (with $p_\parallel$) is a convection current carried by the field-line curvature drift (see (2.14)) whereas the second part (with $p_\perp$) is the equivalent current whose microscopic origin was shown in the “kindergarten” example of Fig. 5.3. Notice that for isotropic pressure ($p_\parallel = p_\perp$) $\boldsymbol{J}_A \equiv 0$, which means that also the curvature drift drops out, cancelled by the (unnamed) equivalent current part of $\boldsymbol{J}_A$. All currents depend on $\boldsymbol{B}$ and the particle distribution (pressure tensor or density): the local magnetic field dictates, and the particle ensemble properties drive, the currents! Note that in (5.6) only the electric field drift is independent of the particles’ properties; thus it will not contribute to the total current density in a collisionless *charge-neutral* ensemble of two or more species.

An important point is that relations (5.9)–(5.12) are valid in both fluid models, the kinetic and the guiding center one. Concerning relation (5.13), it can be shown (rather laboriously), that although in general $\boldsymbol{V}_g \neq \boldsymbol{V}$, for the total time derivatives $d\boldsymbol{V}_g/dt \cong d\boldsymbol{V}/dt$ within the guiding center approximation, so that this relation (5.13) is valid, too, in both models. The current (5.9) is thus indeed the total current density that acts as the *source* of a magnetic field, i.e., the one that enters in Maxwell’s equations (A.49). This somewhat trivial remark will be important later.

Concerning the parallel bulk velocity, let us lift for a moment the initial assumption that it is zero. We shall have

$$V_{g\parallel} = \langle v_\parallel \rangle, \tag{5.14}$$

basically an independent variable in the sense that at one given point it *only* depends on the particle distribution function there—which, however, as we will show in the next section, varies in a specific manner along any given magnetic field line. For an equation for $\boldsymbol{J}_\parallel$, complement to (5.9), we write

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[7]This dropout, predicted by theoreticians in the early days of magnetospheric physics, caused confusion among experimentalists studying ring current data, who from the beginning assumed this West-East current to be due to the *convective* E-W and W-E drift of trapped protons and electrons, respectively. However, the ring current is the superposition of a E-W convection drift current with an equivalent *diamagnetic* current (5.11), the latter with an W-E inner ring (radially outward directed density or pressure gradient in (4.24)) and a E-W outer ring where the density gradient is reversed.

$$\boldsymbol{J}_{\|} = \rho_q \boldsymbol{V}_{g\|} + \nabla \times \boldsymbol{M}|_{\|} = \rho_q \boldsymbol{V}_{g\|} - p_{\perp}/B^2 \nabla \times \boldsymbol{B}|_{\|}$$

or, taking into account that under stationary or slowly varying conditions $\nabla \times \boldsymbol{B}|_{\|} = \mu_0 \boldsymbol{J}_{\|}$, where $\boldsymbol{J}_{\|}$ is the total field-aligned current,

$$\boldsymbol{J}_{\|} = \boldsymbol{J}_{g\|} \frac{1}{1 + p_{\perp}/(B^2/\mu_0)} \tag{5.15}$$

Note that always $J_{\|} \leq J_{g\|}$. This confirms what we have anticipated in the fifth kindergarten example of the previous section (see Fig. 5.6). The equation shows that if $V_{g\|} = 0$ (no average parallel velocity of guiding centers, symmetric pitch angle distribution), the total field-aligned current density is always zero—in other words, a field-aligned current cannot "be made of" an equivalent current alone. If on the other hand, there is GC field-aligned streaming ($\boldsymbol{J}_{g\|} \neq 0$) and the transverse particle pressure $p_{\perp}$ is much smaller than the magnetic energy density $B^2/2\mu_0$ (Appendix A.1, relation (A.40)—the ratio $p/u$ is called the *beta* of the plasma), which in general implies low particle number density, the total field-aligned current density $J_{\|}$ is maximum and equal to it. If in the other extreme $p_{\perp} \sim B^2/2\mu_0$ (high number density), $J_{\|} \to 0$ again, regardless of the parallel streaming of guiding centers (cancelled by the equivalent current in the guiding center model). This is an example of the above-mentioned special nature of parallel motions in the fluid descriptions. As we have seen in Appendix A.1, the field-aligned current is responsible for a twist of magnetic field lines; in the present example it also controls the proportion between convection and equivalent currents in the guiding center fluid. But remember that field-aligned currents, while causing torsion in the magnetic field ((A.28) and (A.29)), cannot sustain any magnetic stresses: $\boldsymbol{J}_{\|} \times \boldsymbol{B} \equiv 0$.

We now turn to the general stresses, i.e., the average macroscopic Lorentz force densities acting inside the guiding center fluid, $\boldsymbol{J} \times \boldsymbol{B} = \boldsymbol{J}_{\perp} \times \boldsymbol{B} = (\boldsymbol{J}_E + \boldsymbol{J}_D + \boldsymbol{J}_A + \boldsymbol{J}_I) \times \boldsymbol{B}$ (5.9). Let us begin again with the kindergarten example of 90° pitch angle particles filling a cylindrical flux tube in a uniform $\boldsymbol{B}$-field (Fig. 5.2). Regardless of the fluid model considered (left or right in the figure), there will be a thin layer of current on the surface of the cylinder, as shown in that figure. If there are "many, many" particles, two things will happen: (i) the magnetic field inside the cylinder will decrease noticeably due to the solenoidal surface currents (diamagnetic property of the ensemble), and (ii) an average non-negligible Lorentz force will appear acting on the outer equivalent current-carrying part of the ensemble. This latter outward-directed force density $\boldsymbol{J} \times \boldsymbol{B}$ represents an internal *stress* in the ensemble, quite similar to the magnetostriction acting on equivalent $\nabla \times \boldsymbol{M}$ currents inside condensed matter with magnetization density $\boldsymbol{M}$. Our kindergarten example can be carried further qualitatively: as the magnetic field in the cylinder decreases with time, an induced electric field will appear (see example with the case of an increasing field on page 21!) and the associated outward drift will expand the particle ensemble. But equivalently, in the GC model we could well attribute the expansion to the action of an outward Lorentz force stress! A similar quantitative analysis can be made with the example of Fig. 5.6: here we have a convection current density; the average Lorentz force on it counteracts exactly the electric field force—a good example to convince a skeptic that a plasma exposed to an electric field is *not* accelerated in the direction of the field but will drift perpendicularly to it!

If we cross Eq. (5.9) with $\boldsymbol{B}$ and rearrange terms, we are led to the following *dynamic equation for the perpendicular bulk flow* in a guiding center fluid:

$$\rho_m \frac{d\boldsymbol{V}_g}{dt}\bigg|_{\perp} = \rho_q \boldsymbol{E}_{\perp} - \nabla_{\perp} p_{\perp} - (p_{\parallel} - p_{\perp})\frac{\partial \boldsymbol{e}}{\partial s} + \boldsymbol{J}_{\perp} \times \boldsymbol{B} \tag{5.16}$$

Observe that this is not a "true" dynamic equation which, by integration, would lead to the calculation of $\boldsymbol{V}_g$; it merely serves to display the stresses or force densities responsible for the transverse acceleration of parcels in the guiding center fluid model.

To derive a dynamic equation of flow parallel to the magnetic field, complement to Eq. (5.16), we can convert the single-particle parallel equation (2.20) (without non-electric forces) into a macroscopic equation for an ensemble of particles by multiplying it with the guiding center distribution function $F$ and integrating, to obtain:

$$\rho_m \frac{d\boldsymbol{V}_g}{dt}\bigg|_{\parallel} = \rho_m \frac{d\boldsymbol{V}_g}{dt} \cdot \boldsymbol{e} = \rho_q \boldsymbol{E}_{\parallel} - \frac{\partial p_{\parallel}}{\partial s} + \frac{(p_{\parallel} - p_{\perp})}{B}\frac{\partial B}{\partial s} \tag{5.17}$$

The two above equations can be combined into one by taking into account (4.21):

$$\rho_m \frac{d\boldsymbol{V}_g}{dt} = \rho_q \boldsymbol{E} - \nabla \mathbb{P} + \boldsymbol{J} \times \boldsymbol{B} \tag{5.18}$$

This general equation for the bulk velocity of an ensemble of guiding centers explicitly reveals the dynamic action of three types of physical causes: (i) non-magnetic forces (the first term, to which any non-electromagnetic force density could be added), (ii) the "mechanical stresses" represented by the pressure tensor, and (iii) the action of the magnetic field on the ensemble through Lorentz-type forces (the third term). It is important to remember that at this stage of our discussion, we are still dealing with just *one species* of particles and that the magnetic and electric fields are given, i.e., that the contribution to the fields of charges and currents in the ensemble are being neglected.

Before we get real and drop this limitation, we end this section with an analysis of the physical meaning of the stresses in a guiding center fluid, as illustrated by the hypothetical example of a "magnetohydrostatic" equilibrium state of the ensemble: no time dependence, no total current density, no external forces, no electric field. The condition of equilibrium implies that transverse equation (5.16) now should be:

$$\nabla_\perp p_\perp + (p_\parallel - p_\perp)\frac{\partial \boldsymbol{e}}{\partial s} = 0$$

This equation is identical to (4.33), only that here it has been derived in a more general way. According to (5.9), this equation also implies that $\boldsymbol{J}_D + \boldsymbol{J}_A = 0$ in the case of a stationary state.

Along the respectively binormal and normal $x$ and $y$ axes (in the natural coordinate system, Appendix A.1),

$$\frac{\partial p_\perp}{\partial x} = 0$$

$$\frac{\partial p_\perp}{\partial y} - (p_\parallel - p_\perp)\frac{1}{R_c} = 0$$

where $R_c = |\partial \boldsymbol{e}/\partial s|^{-1}$ is the field line's radius of curvature. These two equations show how the perpendicular pressure and the pressure anisotropy $p_\parallel - p_\perp$ must obey stringent conditions of spatial variability in a magnetostatic field to remain in equilibrium (indeed, it is useful to re-examine the example given in Sect. 4.4, where the equilibrium conditions were derived for a specific case.) Under the same static equilibrium conditions, the following relation is obtained for the parallel stresses from (5.17), in partnership with the transverse equation (4.33):

$$\frac{\partial p_\parallel}{\partial s} - \frac{(p_\parallel - p_\perp)}{B}\frac{\partial B}{\partial s} = 0$$

This equation is identical to (4.32).

To interpret the detailed physical meaning of the various terms in the above equilibrium relations (4.33) and (4.32), consider a guiding center fluid element in a magnetic flux tube, as sketched in Fig. 5.7. In this figure, the axes represent the natural reference frame, with $z \parallel \boldsymbol{e}$, $y$ along the normal $\boldsymbol{n}$ and $x$ along the binormal $\boldsymbol{b}$ (Appendix A.1). As always, $R_C$ is the radius of curvature of the field lines, and we have the following relations between the side areas: $\delta A_x^* = \delta A_x$ (binormal axis); $\delta A_y^* = \delta A_y(1 + \delta y/R_C)$ (field-geometric factor) and $\delta A_z^* = \delta A_z(1 - 1/B\,(\partial B/\partial s)\,\delta s)$ (conservation of magnetic flux).

Along the $y$-axis, the following forces act on the GC fluid element in hydrostatic balance:

$$-(p_\perp + \frac{\partial p_\perp}{\partial y}\delta y)\delta A_y(1 + \frac{\delta y}{R_c}) + p_\perp \delta A_y + p_\parallel \left|\frac{\partial \boldsymbol{e}}{\partial s}\right| \delta A_y \delta y = 0$$

in which the third term represents the total centrifugal force on the particles. This means that

$$\frac{\partial p_\perp}{\partial y} - (p_\parallel - p_\perp)\left|\frac{\partial \boldsymbol{e}}{\partial s}\right| = 0$$

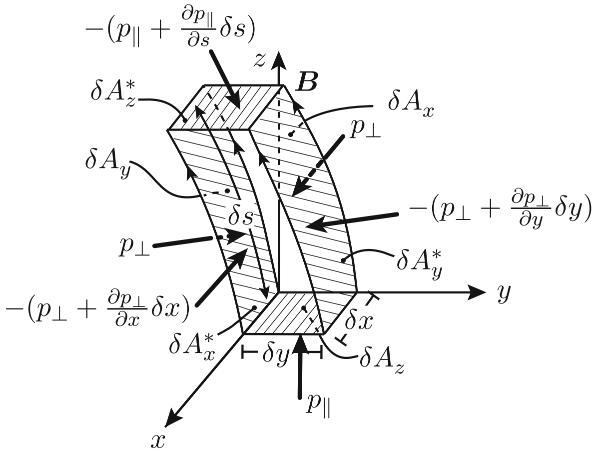


**Fig. 5.7** Flux tube element filled with guiding center fluid in "hydrostatic" equilibrium

To find the other term $\partial p_\perp/\partial x$, we have along the $x$ axis:

$$-\left(p_\perp + \frac{\partial p_\perp}{\partial x}dx\right)\delta A_x + p_\perp \delta A_x = 0$$

which means that

$$\frac{\partial p_\perp}{\partial x} = 0$$

The above relations represent the vector equilibrium condition (4.33) along the $y$ and $x$ axes perpendicular to $\boldsymbol{B}$.

For the parallel equation of equilibrium (along the $z$-axis), consider again Fig. 5.7. With the total mirror force density on the guiding center particles in the flux tube element $(-p_\perp/B\ (\partial B/\partial s))$, we have the following equilibrium condition:

$$-\left(p_\| + \frac{\partial p_\|}{\partial s}\delta s\right)\left(1 - \frac{1}{B}\frac{\partial B}{\partial s}\delta s\right) + p_\| - \frac{p_\perp}{B}\frac{\partial B}{\partial s}\delta s = \left(-\frac{\partial p_\|}{\partial s} + \frac{(p_\| - p_\perp)}{B}\frac{\partial B}{\partial s}\right)\delta s = 0$$

leading to (4.32).

Finally, we may relax a bit the a priori conditions in our example and admit transverse currents that, however, are independent of time—i.e., limited to a stationary ensemble. This does not change the parallel equation (4.32) and its physical meaning (Fig. 5.7). The perpendicular equation (5.16) now becomes, taking into account (A.36) of Appendix A.1:

$$\nabla_\perp p_\perp + (p_\| - p_\perp)\frac{\partial \boldsymbol{e}}{\partial s} = \boldsymbol{J}_\perp \times \boldsymbol{B} = \frac{B^2}{\mu_0}\frac{\partial \boldsymbol{e}}{\partial s} - \nabla_\perp\left(\frac{B^2}{2\mu_0}\right) \qquad (5.19)$$

or:

$$\nabla_{\perp}\left(p_{\perp}+\frac{B^2}{2\mu_0}\right)+\left(p_{\parallel}-p_{\perp}-\frac{B^2}{\mu_0}\right)\frac{\partial \boldsymbol{e}}{\partial s}=0 \tag{5.20}$$

In this equation, the magnetic energy density $B^2/2\mu_0$ (A.40) plays the role of transverse magnetic field pressure; $(B^2/\mu_0)\ \partial \boldsymbol{e}/\partial s$ is the perpendicular magnetic tension of Maxwell's theory.

To summarize, we end this section with a "kindergarten" view of the preceding equilibrium expressions (4.32) and (4.33). Refer again to Fig. 5.7. The element of fluid is subjected to pressure forces on its sides ($p_{\perp}$-related) and a "buoyancy force" ($p_{\parallel}$-related, which can be interpreted as the differential of parallel pressure forces on the tops of the flux element), and to two internal magnetostrictive forces: the mirror force (which can be interpreted as being responsible for the "slippery soap" effect of a narrowing magnetic flux tube squeezing the incoming particles and bouncing them back in their parallel motion) and the inertial centrifugal force (on the guiding center particles in a bent flux tube while they travel up and down in their bounce motion). If this force system is in hydrostatic equilibrium, there is no macroscopic bulk acceleration in any direction (perpendicular or parallel to $B$). As a result, the particle ensemble is stationary; a locally time-independent guiding center bulk flow $\boldsymbol{V}_{g\perp}$ is allowed, but only in such a way that no total current $\boldsymbol{J}_{\perp}$ occurs. No net field-aligned bulk flow (or current) is allowed.

This entire discussion involved the guiding center fluid model—the kinetic model does not care about what the individual particles do elsewhere (like whether they are executing a systematic cyclotron gyration and come back to the same volume element repeatedly to be counted each time as a contribution to a current, or whether they fly away and are replaced by other incoming particles); what counts in the kinetic fluid model is what happens *locally* to each particle at any given point in space and instant of time. In this more general kinetic formalism one loses track of the integral, macrophysical picture and related intuitive understanding. Remarkably, however, as we shall see in the next section the equations discussed above are valid also in the kinetic particle model, with the velocities and pressures defined in the list on page 124. This is a relief because, as hinted before, the kinetic fluid model is the only recourse available for the quantitative study of regions in which the adiabatic conditions break down for the particles in question, such as in the vicinity of neutral sheets and lines, boundaries and shocks.

## 5.4 From the Guiding Center Fluid to a Quasi-neutral Center-of-Mass Fluid

It is high time to turn to *quasi-neutral mixtures* of positive and negative plasma particles. Most of the examples to be considered will be, for simplicity, singly-charged positive ions and negative electrons. We also must turn our attention to the fact that, as hinted at the beginning of this chapter, in a real plasma we cannot neglect the contribution of the plasma currents to the field: there is a circular cause-effect relationship: particle dynamics ⇒ currents ⇒ magnetic field ⇒ particle dynamics—in other words, we must activate the link between Maxwell's equations and fluid dynamics, in which the magnetic field still plays the grand role of a common framework holding the different components and regions of a plasma together. It is important to point out, as we shall see in a later section, that neither fields nor particles come first (a "chicken-and-egg" situation)—except when one or the other has separate and dominating *externally controlled* sources or sinks (e.g., the internal geomagnetic field; solar wind particle injections, atmospheric losses). And since thus far we were dealing with collisionless ensembles in which the only interaction between particles is mediated by the macroscopic electromagnetic field, at one point we must get real and turn inter-particle collisions on.

First of all, we start with Eq. (5.18) for one species and note that it really can also be derived *directly* from Vlasov's equation (5.1) for collisionless ensembles: just multiply all terms of this equation tensorially by $m\boldsymbol{v}$ and integrate over velocity space! This means that it is valid for a kinetic fluid, too, provided one accepts the fact that, as mentioned on page 135, for the bulk accelerations $d\boldsymbol{V}_g/dt = d\boldsymbol{V}/dt$ despite both velocities being different. For that reason, we shall drop the subindex "g" from the velocity vector $\boldsymbol{V}$.[8] This means that (5.18) has more general validity than the perpendicular and parallel guiding center fluid equations from which we extracted it.

It is our task now to merge two ensembles with mutually opposite charges, each one representing a class of particles under one common electromagnetic field. In this way we obtain yet another fluid which provides a quantitative macroscopic description of the overall system, and from which one can extract some useful information about the behavior of each one of the merged ensembles. To develop this "grand" new fluid model, we shall use + and − as subindices characterizing each species. With this notation, we rewrite (5.18) in the forms

$$n_+ m_+ \frac{d\boldsymbol{V}_+}{dt} = n_+ q_+ \boldsymbol{E} - \nabla \mathbb{P}_+ + \boldsymbol{J}_+ \times \boldsymbol{B} \tag{5.21}$$

$$n_- m_- \frac{d\boldsymbol{V}_-}{dt} = n_- q_- \boldsymbol{E} - \nabla \mathbb{P}_- + \boldsymbol{J}_- \times \boldsymbol{B} \tag{5.22}$$

To these we must add a continuity equation for each species (we are assuming that there are no sources or sinks of particles in our collisionless mixed ensembles):

[8] A question still subsists: How can two different solutions, either $\boldsymbol{V}_g$ or $\boldsymbol{V}$, be obtained for the two different fluid models from one and the same equation? The answer is that $\boldsymbol{V}_g$ or $\boldsymbol{V}$ sit inside $\boldsymbol{J}$, which in the case of the *magnetized* guiding center fluid model also contains $\boldsymbol{\nabla} \times \boldsymbol{M}$, with $\boldsymbol{M}$ in turn being a function of $\boldsymbol{B}$ and $p_\perp$.

$$\frac{\partial n_\pm}{\partial t} + \nabla \cdot (n_\pm \boldsymbol{V}_\pm) = 0 \tag{5.23}$$

What specific properties must we expect from our new single-fluid mixed-species model? First of all, its mass, charge and current densities should be the sum of the individual densities:

$$\rho_m = n_+ m_+ + n_- m_- \tag{5.24}$$

$$\rho_q = n_+ q_+ - n_- |q_-| \tag{5.25}$$

$$\boldsymbol{J} = \boldsymbol{J}_+ + \boldsymbol{J}_- = n_+ q_+ \boldsymbol{V}_+ - n_- |q_-| \boldsymbol{V}_- \tag{5.26}$$

Second, the bulk velocity of the fluid $\boldsymbol{V}$ should be such that the momentum density $\boldsymbol{G} = \rho_m \boldsymbol{V}$ is equal to the sum of the momentum densities of each component:

$$\boldsymbol{G} = n_+ m_+ \boldsymbol{V}_+ + n_- m_- \boldsymbol{V}_- \tag{5.27}$$

For that purpose we now introduce the *center of mass velocity* of the two fluids:

$$\boldsymbol{V} = \frac{n_+ m_+ \boldsymbol{V}_+ + n_- m_- \boldsymbol{V}_-}{n_+ m_+ + n_- m_-} \tag{5.28}$$

We can now officially introduce the *center of mass fluid* as one with mass density $\rho_m$ (5.24), charge density $\rho_q$ (5.25), current density $\boldsymbol{J}$ (5.26) and momentum density $\boldsymbol{G} = \rho_m \boldsymbol{V}$. A continuity equation can be derived from (5.23),

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \boldsymbol{V}) = 0 \tag{5.29}$$

with a charge continuity equation

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot (\rho_q \boldsymbol{V}) = 0 \tag{5.30}$$

We have sketched the situation of a center of mass fluid in Fig. 5.8:Consider an element of volume $\delta \boldsymbol{r}^3$ at time $t$ with two classes of particles of opposite charge, $\circ$ and $\bullet$.[9] Each class came, in principle, from a different volume element at time $t - \delta t$, and each will end up in a different parcel at time $t + \delta t$. The centers of mass of the parcel pairs at these different times are shown (please note that in reality these parcels are only infinitesimal time intervals and distances apart!). With the center of mass fluid model we have replaced two distinct, intercrossing $\circ$ and $\bullet$ fluids with

[9] Of course, we can show only a subgroup of particles of each class in the central element of volume at time $t$; it may be crossed by many other particles coming from other pairs of pre-$t$ parcels.

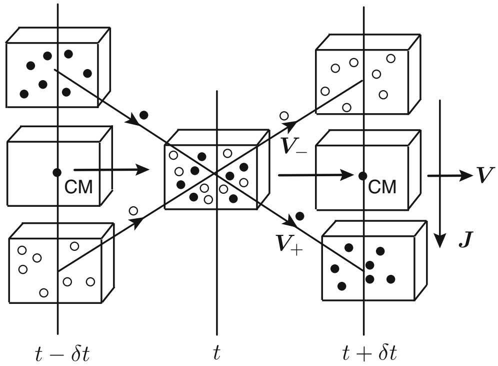


**Fig. 5.8** Sketch of two guiding center fluids in their partial motions and that of the virtual center of mass fluid. The corresponding electric current density vector is shown. In an ion-electron fluid, the center of mass motion would be nearly identical with the bulk motion of the ions

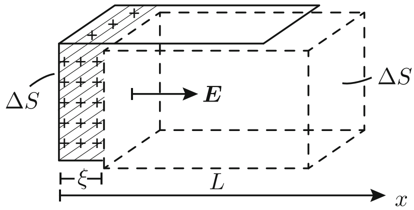


**Fig. 5.9** Mutual displacement of positive and negative particle ensembles; generation of a local electrostatic field

another virtual fluid whose parcels follow the center of mass line (horizontal in the figure), whose (virtual) density is the sum of the individual fluid densities, and which sustains a current density given by the relative convection of opposite charges (vertical in the figure).

Figure 5.8 shows two ensembles of oppositely charged particles running through each other—What prevents them from separating, what keeps them together? Evidently, it must be the electric field that would build up rapidly between oppositely charged "clouds" of particles if a local charge density fluctuation occurs in an electrically neutral distribution. As a matter of fact, just a tiny collective charge separation in a limited volume would build up a space charge and generate a large local electric field (thanks to the large value of the constant $\epsilon_0$, Appendix A.1) acting against any further separation. Refer to Fig. 5.9 showing a portion of the electron population displaced by an amount $\xi$ to the right ($\xi \ll L$), and assume for a moment that all particles are "frozen" into their instantaneous position. A positive electrostatic space charge will appear in the thin rectangular element of volume $\Delta S\, \xi$ ($\xi^2 \ll \Delta S$).

The electric field on the right of this thin, flat element would be approximately uniform and of value $E = (ne\xi)/\epsilon_0$, directed as shown ($e$: absolute value of the elementary charge; $ne\xi$: charge per unit surface $S$ of the thin element). This field will exert a total force $F = qE = -(neL\Delta S)(ne\xi)/\epsilon_0$ on the electron cloud of total mass $M = nm_e L\Delta S$. Newton's equation $F = M\ddot{\xi}$ turns out to be that of an harmonic oscillator $\ddot{\xi} + {\omega_e}^2\, \xi = 0$, where

$$ {\omega_e}^2 = \frac{ne^2}{\epsilon_0 m_e} \tag{5.31} $$

Despite the very artificial nature of the model used in the derivation, this frequency of coherently oscillating electrons (and also of the oscillating local electric field $\boldsymbol{E}$) is a basic plasma parameter which plays a fundamental role in certain types of plasma waves (e.g., Langmuir and upper hybrid waves) and affects the propagation velocity of electromagnetic waves in dense plasmas. It is called the *electron plasma frequency*. One also defines an *ion plasma frequency* ${\omega_i}^2 = ne^2/\epsilon_0 m_i$, and, combining the two, the *plasma frequency* $\omega_p^2 = \omega_e^2 + \omega_i^2$. However, since ${\omega_i}^2$ is at least three orders of magnitude smaller than ${\omega_e}^2$, one also uses the term plasma frequency for $\omega_e$.

In our artificial model, the cloud of slightly displaced electrons will oscillate about the neutral charge position (where $\rho_e = \rho_i$) with a frequency $\omega_e$ that only depends on their number density (5.31). By loosening the initial restrictions and allowing the particles to be in thermal motion (no collisions and still no magnetic field), it is possible to estimate the upper limit of the amplitude $A_{max}$ noting that the electrostatic oscillation energy must compete with the kinetic energy of the particles. A reasonable limit should therefore be $A_{max} \cong \langle|v|\rangle/\omega_e$, where $\langle|v|\rangle$ is the average thermal velocity of the electrons. Taking into account (5.31) and (4.22), we introduce the *Debye length* as another important plasma parameter, which in our specific example would be equal to $A_{max}$:

$$ {\lambda_D}^2 = \frac{\epsilon_0 k T_e}{ne^2} \tag{5.32} $$

Removing the last artificial restriction and allowing the presence of a magnetic field $\boldsymbol{B}$ will complicate the model of Fig. 5.9 considerably because of the additional action of the Lorentz force[10]; one should anticipate that the behavior will be different for a magnetic field parallel to the $x$-axis than for a perpendicular field. If we call $\rho_{Ce}$ the average cyclotron radius of the electrons, we can write

$$ \frac{\omega_c}{\omega_e} = \frac{\lambda_D}{\rho_{Ce}} = \frac{1}{c}\frac{B}{\sqrt{(\mu_0 n m_e)}} \tag{5.33} $$

The lower limit of this ratio in the magnetosphere is about 0.001; only in some very limited regions it may exceed 1. This is an indication that the electrostatic effect in our simplified model sketched in Fig. 5.9 will in general affect only a small portion of a typical Larmor orbit in the magnetosphere; the "thermal" motion is then given by the random cyclotron phases (even for mono-energetic particles)—neglecting the magnetic field in the derivation of (5.32) was not such an unrealistic choice after all!

[10]In most plasma physics books, the Debye length and the plasma frequency are introduced at the very beginning, without any mention of the magnetic field (and often assuming a Maxwellian distribution). This sometimes confuses the student, especially if the book mainly deals with *magnetized* plasmas.

The requirement that $\lambda_D \ll$ scale of the system is usually taken as the very *definition of a plasma*. For what follows, we need not concern ourselves with the detailed mechanism by means of which quasi-neutrality $\rho^+ \simeq \rho^-$ is maintained; we simply make the assumption that this condition is upheld at all times through some local mechanism. This is quite similar to what one does in analytical mechanics: setting predetermined constraints without in any way specifying how such limiting conditions are physically maintained!

Returning to our center of mass fluid, it is important to be aware of the relationships between the species' bulk velocities and the total current:

$$V_\pm = \boldsymbol{V} + \frac{m_\mp}{q_\pm}\boldsymbol{J}/\rho_m \tag{5.34}$$

Carefully note the $+$ and $-$ correspondences. If the negative particles are electrons, and the ensemble is quasi-neutral, we can write:

$$\begin{aligned} n &\simeq n^+ \simeq n^- \\ \rho_m &\simeq n_+ m_+ \\ \rho_q &\simeq 0 \\ \boldsymbol{J} &= \boldsymbol{J}_+ + \boldsymbol{J}_- \simeq ne(\boldsymbol{V}_+ - \boldsymbol{V}_-) \simeq -ne\boldsymbol{V}_- \\ \boldsymbol{V}_+ &\simeq \boldsymbol{V} \quad \text{and} \quad \boldsymbol{V}_- \simeq \boldsymbol{V} - \boldsymbol{J}/ne \end{aligned} \tag{5.35}$$

We now have to come up with a single dynamic equation for the center of mass fluid. Unfortunately, we cannot simply add algebraically the species-specific equations (5.21) and (5.22). There are several reasons. First, they contain total time derivatives, which follow each species of particles traveling with different bulk speeds in different directions. So we need to break them up into local and convective operators: $d/dt = \partial/\partial t + \nabla \cdot \boldsymbol{V}_\pm$. Second, the pressure tensor components are not additive either: according to the definition (4.16) and the discussion in Sect. 4.3, the pressure tensor is controlled by the velocity dispersion of a species of particles in a frame of reference moving at each point with the bulk velocity $\boldsymbol{V}_\pm$ of *that* particular species. It is thus necessary to introduce another entity, a pressure tensor which for each species links the particles' velocity distribution with the common center of mass frame of reference. For that purpose the *partial pressure* tensor $\mathbb{P}^*_\pm$ is introduced, defined as

$$\mathbb{P}^*_\pm = m_\pm \int f_\pm(\boldsymbol{v} - \boldsymbol{V}) \otimes (\boldsymbol{v} - \boldsymbol{V}) d\boldsymbol{v}^3 \tag{5.36}$$

in which $\boldsymbol{V}$ now is the center of mass velocity (5.28). Note the algebraic relationship with the respective species-specific kinetic tensors (4.15) which indeed *are* additive:

$$\mathbb{P}^*_\pm = \mathbb{K}_\pm - n_\pm m_\pm \boldsymbol{V} \otimes \boldsymbol{V} \tag{5.37}$$

Therefore, the partial pressure tensors $\mathbb{P}^*_\pm$ are legitimately additive for different species, too.

To accomplish all this, let us go back to Eqs. (5.21) and (5.22) and replace the original pressure tensors $\mathbb{P}_\pm$ with their relation to their kinetic tensors and species-specific bulk velocities: $\mathbb{P}_\pm = \mathbb{K}_\pm - n_\pm m_\pm \boldsymbol{V}_\pm \otimes \boldsymbol{V}_\pm$ (4.16). We then have

$$\nabla\mathbb{P}_\pm = \nabla\mathbb{K}_\pm - \nabla(n_\pm m_\pm \boldsymbol{V}_\pm \otimes \boldsymbol{V}_\pm)$$

One can verify by components that

$$\nabla(n_\pm m_\pm \boldsymbol{V}_\pm \otimes \boldsymbol{V}_\pm) = n_\pm m_\pm (\boldsymbol{V}_\pm \cdot \nabla)\boldsymbol{V}_\pm + \boldsymbol{V}_\pm[\nabla \cdot (n_\pm m_\pm \boldsymbol{V}_\pm)]$$

Using the continuity equation (5.23) for the last term, we obtain

$$\nabla\mathbb{P}_\pm = \nabla\mathbb{K}_\pm - n_\pm m_\pm (\boldsymbol{V}_\pm \cdot \nabla)\boldsymbol{V}_\pm + \boldsymbol{V}_\pm \frac{\partial(n_\pm m_\pm)}{\partial t}$$

Inserting in (5.21) and (5.22), rearranging terms and remembering that $d/dt = \partial/\partial t + \boldsymbol{V} \cdot \nabla$,

$$\begin{aligned}\frac{\partial}{\partial t}(n_\pm m_\pm \boldsymbol{V}_\pm) &= n_\pm m_\pm \frac{d\boldsymbol{V}_\pm}{dt} - n_\pm m_\pm (\boldsymbol{V}_\pm \cdot \nabla)\boldsymbol{V}_\pm + \boldsymbol{V}_\pm \frac{\partial(n_\pm m_\pm)}{\partial t} \\ &= n_\pm q_\pm \boldsymbol{E} - \nabla\mathbb{K}_\pm + \boldsymbol{J}_\pm \times \boldsymbol{B}\end{aligned}$$

With $\boldsymbol{G}_\pm = n_\pm m_\pm \boldsymbol{V}_\pm$ as the momentum density of each partial fluid, we finally have a pair of momentum equations for the two fluids which are indeed summable:

$$\begin{aligned}\frac{\partial \boldsymbol{G}_+}{\partial t} &= n_+ q_+ \boldsymbol{E} - \nabla\mathbb{K}_+ + \boldsymbol{J}_+ \times \boldsymbol{B} \\ \frac{\partial \boldsymbol{G}_-}{\partial t} &= n_- q_- \boldsymbol{E} - \nabla\mathbb{K}_- + \boldsymbol{J}_- \times \boldsymbol{B}\end{aligned} \tag{5.38}$$

Adding the two equations, we obtain

$$\frac{\partial \boldsymbol{G}}{\partial t} = \rho_q \boldsymbol{E} - \nabla\mathbb{K} + \boldsymbol{J} \times \boldsymbol{B} \tag{5.39}$$

with $\mathbb{K} = \mathbb{K}_+ + \mathbb{K}_-$ the total kinetic tensor of the center of mass fluid. This is the *momentum* magnetohydrodynamic equation.

Now we can revert to a true dynamic equation, with a total time derivative that represents the acceleration of a fluid element in the new model as it flows. This can be done by starting with (5.39) and "undoing" some of the previous steps, to obtain the familiar *magnetohydrodynamic equation*

$$\rho_m \frac{d\boldsymbol{V}}{dt} = \rho_q \boldsymbol{E} - \nabla \mathbb{P} + \boldsymbol{J} \times \boldsymbol{B} \tag{5.40}$$

Note that it looks just as the Eq. (5.18) for one species, but here $\boldsymbol{V}$ is the *center of mass velocity* and $\mathbb{P} = \mathbb{P}^*_+ + \mathbb{P}^*_-$ is the *total pressure tensor* (5.36), sum of partial pressure tensors (5.37).

At once, with this equation we can retrieve several earlier relations that we have deduced for conditions of stationary equilibrium. In particular, for a given static magnetic field and $\boldsymbol{V} =$ const. there are strong restrictions on the admissible particle distributions of a quasi-neutral ensemble. Remembering relations (A.37) and (A.38) of Appendix A.1, we can write (5.40) in the form $\nabla(\mathbb{P} - \mathbb{S}) = 0$ or, in general, the equilibrium between the plasma pressure tensor and Maxwell's magnetic stress tensor $\mathbb{P} = \mathbb{S}$.

It is easy to extend the center of mass fluid equations to a mixture of particles with more components than two: if we replace the $+$ and $-$ subindices in the preceding derivations with the subindex $s$, we just have to sum everything over $s$. We end up with the following list of macroscopic variables for a center of mass fluid of any number of constituents:

Total mass density: $\rho_m = \sum_s n_s m_s$
Total charge density: $\rho_q = \sum_s n_s q_s$ ($\simeq 0$ in quasi-neutrality)
Total current density: $\sum_s n_s q_s \boldsymbol{V}_s$
Bulk or center of mass velocity: $\boldsymbol{V} = (\sum_s n_s m_s \boldsymbol{V}_s)/(\sum_s n_s m_s)$
Total momentum density: $\boldsymbol{G} = \rho_m \boldsymbol{V}$
Total pressure tensor: $\mathbb{P} = \int \sum_s m_s f_s (\boldsymbol{v} - \boldsymbol{V}) \otimes (\boldsymbol{v} - \boldsymbol{V}) d\boldsymbol{v}^3$ (sum of partial pressure tensors).

With these macroscopic variables, the continuity equation (5.29) remains unchanged, and so does the magnetohydrodynamic equation (5.40).

## 5.5 Collisions and the Generalized Ohm Equation

It is prudent to take stock of what we have accomplished so far in the development of quantitative relationships between macroscopic variables for quasi-neutral ensembles of electrically charged particles, and the dynamic equations governing their time changes. Electrostatic forces which may appear on a mesoscopic scale (the Debye length (5.32), large compared to inter-particle distances but small with respect to the overall scale of the system) overwhelm the local magnetic field forces that normally dominate the behavior of a collisionless particle ensemble and prevent any local charge density fluctuations from growing to a macroscopic scale. This omnipresent mechanism justifies adopting quasi-neutrality as one of the defining properties of a plasma.

To arrive at the momentum and magnetohydrodynamic equations we have followed two possible routes by introducing two models. (1) The model of a guiding center fluid ruled by adiabatic theory, restricted to situations in which the guiding center approximation, gyrotropicity and trapping conditions are valid at all points of the fluid. (2) A particle or kinetic fluid model that follows directly from the Vlasov equation (5.1), with no restrictions such as adiabatic conditions. The latter, however, does not offer the intuitive visualization of "what particles are really doing at the microscopic level, and why". Both approaches lead to identical results wherever the adiabatic conditions are satisfied, and both involve a distribution function (of six and seven variables, respectively) as the fundamental physical and measurable quantity at the mesoscopic level. Initially formulated for just one species of particles, we combined two oppositely charged species into a quasi-neutral mixture by introducing yet another model, the center of mass fluid. As a fundamental result we obtained a single center of mass fluid momentum equation (5.39) and the magnetohydrodynamic (MHD) equation (5.40).

The principal aim of this formalism is to be able to predict or retrodict the behavior of a given plasma, eventually subjected to some externally controlled electromagnetic field and particle sources and sinks, which at a given initial time is found in a given macroscopic state. In more practical terms for magnetospheric physics, the aim is to develop a mathematical framework that, given some observed large-scale phenomena such as the trigger and development of a magnetospheric substorm, an auroral breakup, a sudden energetic trapped particle injection, etc., would allow us to pinpoint the ultimate external cause, understand the quantitative evolution, and formulate associated prediction algorithms. Taken in isolation, the MHD equation would be useful *only* to address some oversimplified situations, where there is an a priori set symmetries and isotropies, absence of collisions, and a priori imposed equilibrium conditions. In fact, to use it at this stage, the electromagnetic field vectors must be *pre-specified*, and all retro-effects on the plasma on the field must be ignored. The real problem is that we still have too many unknowns but not enough equations: we have not yet properly *linked* the MHD equation with the overall electromagnetic field!

The MHD equation was derived by manipulating the two momentum equations (5.38) for oppositely charges species. From the mathematical point of view, we are still allowed to extract one more independent equation from those two, which explicitly reflects the local interaction between plasma and field. But before we do so, we shall introduce elastic collision processes (Coulomb scattering) between the particles of the ensemble, thus dropping yet another of the restrictions imposed at the beginning of this chapter. Let us call $\boldsymbol{k}_{rs}$ the average *momentum transfer density* per unit time from the fluid of $s$ particles to the $r$-particle species. Obviously, $\boldsymbol{k}_{rs} = -\boldsymbol{k}_{sr}$ for elastic collisions. These quantities would then have to be added, respectively, to each momentum equation (we assume that although particles may collide with their own kind, there should be no net average momentum transfer between them). It should be clear, then, that this addition would not affect at all the procedure followed on page 146 (the extra collision terms would cancel each other), which means that the MHD equation (5.40) *is valid even in presence of elastic collisions*. For our next purpose, however, we have to come up with a quantitative expression for $\boldsymbol{k}$; to simplify the argument, we shall do it for a quasi-neutral mixture of singly-charged positive ions and electrons. If $\boldsymbol{V}_i$ and $\boldsymbol{V}_e$ are the average bulk velocities of ions and electrons, it is reasonable to assume that $\boldsymbol{k}_{ie} = C(\boldsymbol{V}_e - \boldsymbol{V}_i)$, where $C$ should be proportional, again on the average, to the mass density of electrons times their collision frequency $\nu_{coll}$ with ions: $n_e m_e \nu_{coll}$.[11] And, for Coulomb interactions, it *also* should be proportional on the average to the absolute value of their charge densities $e^2 n_i n_e$, which for charge neutrality amounts to $e^2 n^2$. In summary, we can set

$$\boldsymbol{k}_{ie} = \eta\, e^2\, n^2 (\boldsymbol{V}_e - \boldsymbol{V}_i) \;\text{ with }\; \eta = \frac{1}{\sigma} = \nu_{coll}\frac{m_e}{e^2 n} = \frac{\nu_{coll}}{\epsilon_0 \omega_e^2} \tag{5.41}$$

$\omega_e$ is the electron plasma frequency (5.31), $\eta$ is called the plasma *resistivity* and $\sigma$ its conductivity.

We return to the momentum equation as it appears in (5.38) and write it for a generic species $s$, adding the collision momentum exchange term $\boldsymbol{k}_s$, which now represents the total momentum transfer density to fluid $s$ from collisions with *all other* constituents. Multiplying all terms by $q_s/m_s$, adding over all $s$ and rearranging terms, we obtain, remembering the expression (4.15) for the kinetic tensor $\mathbb{K}$ and the multispecies expression for the current density on page 147:

$$\begin{aligned}\frac{\partial \boldsymbol{J}}{\partial t} \; &+ \nabla \int (\sum_s q_s f_s) \boldsymbol{v} \otimes \boldsymbol{v}\, d\boldsymbol{v}^3 \\ &= \sum {q_s}^2 n_s/m_s\, \boldsymbol{E} + (\sum {q_s}^2 n_s/m_s \boldsymbol{V}_s) \times \boldsymbol{B} + \sum (q_s/m_s) \boldsymbol{k}_s \end{aligned} \tag{5.42}$$

The integral can be re-written by considering the following relation involving the partial pressures $\mathbb{P}^*_s$ (5.36):

$$\begin{aligned}\sum (q_s/m_s)\mathbb{P}^*_s &= \int \sum_s q_s f_s (\boldsymbol{v} - \boldsymbol{V}) \otimes (\boldsymbol{v} - \boldsymbol{V})\, d\boldsymbol{v}^3 \\ &= \int \sum_s q_s f_s \boldsymbol{v} \otimes \boldsymbol{v}\, d\boldsymbol{v}^3 + \rho_q \boldsymbol{V} \otimes \boldsymbol{V} - \boldsymbol{V} \otimes \boldsymbol{J} - \boldsymbol{J} \otimes \boldsymbol{V}\end{aligned}$$

This leads us to the weird-looking *generalized Ohm equation*:

$$\begin{aligned}\frac{\partial \boldsymbol{J}}{\partial t} \; &+ \nabla \Big( \boldsymbol{V} \otimes \boldsymbol{J} + \boldsymbol{J} \otimes \boldsymbol{V} - \rho_q \boldsymbol{V} \otimes \boldsymbol{V} \Big) \\ = \; &\Big( \sum {q_s}^2 n_s/m_s \Big) \boldsymbol{E} + \Big( \sum {q_s}^2 n_s/m_s \boldsymbol{V}_s \Big) \times \boldsymbol{B} - \nabla \sum q_s/m_s \mathbb{P}^*_s + \sum (q_s/m_s)\boldsymbol{k}_s\end{aligned} \tag{5.43}$$

---

[11] It is assumed that the actual momentum transfer can vary with equal probability distribution between 0 and a maximum of $2 n_e m_e \nu_{coll}$.

What we have here is a companion equation to the dynamic fluid equation (5.40), connecting field sources $\boldsymbol{J}$ and $\rho_q$ *internal* to the center of mass fluid to the macroscopic particle ensemble variables $\boldsymbol{V}$, $n_s$, $\mathbb{P}_s^*$ and $\boldsymbol{k}_s$. This connection is *local*, but the overall connecting agent is the electromagnetic $\boldsymbol{E}$ and $\boldsymbol{B}$ field, which, as discussed in Appendix A.1, depends on *all* charges and currents, including those externally controlled which have nothing to do with the plasma under consideration. This means that to Eqs. (5.40) and (5.43) we must add Maxwell's equations (Appendix A.1, (A.48)–(A.51)) in which the current density $\boldsymbol{J}$ (and the charge density $\rho_q$) must also include *all external sources*, and complete the set with the conservation equations (5.29) and (5.30).[12] Note the distinct character of each one: (1) The MHD equation controls the dynamics of the particle ensemble—it must be integrated like any dynamics equation to provide information on temporal behavior. (2) The generalized Ohm equation binds together local properties of plasma and field—there is nothing there to integrate, but it leads to the electric field which then appears in the Eq. (A.53) defining $\partial \boldsymbol{B}/\partial t$. (3) Maxwell's equations tie together concurrent behavior at distant points—concurrent in the relativistic sense (however, retardation ((A.41) and (A.42)) usually plays no role in plasmas of planetary system dimension). (4) Conservation equations are the "balance sheets" for the movements of mass and electric charge.

The resulting equation framework is, unfortunately, unmanageable, and we must first trim some fat from Ohm's general equation before we can turn to some simple examples. Instead of first doing a rigorous comparative analysis of the order of magnitude of different terms under different conditions, our first step will be to again limit ourselves to electrons and singly charged positive ions, under guaranteed quasi-neutrality $n_i = n_e = n$, $\rho_q = n(q_i + q_e) = 0$ (the term "quasi" meaning eventual allowance for little departures from charge neutrality within a Debye domain). Under these conditions, we have the following relations for some of the coefficients in Eq. (5.43):

$$\sum {q_s}^2 n_s/m_s = e^2 n \frac{m_i + m_e}{m_i m_e} = \epsilon_0(\omega_e^2 + \omega_i^2)$$

$$\sum (q_s/m_s)\boldsymbol{k}_s = \eta e^3 n^2 \frac{m_i + m_e}{m_i m_e}(\boldsymbol{V}_i - \boldsymbol{V}_e) = e^2 n \frac{m_i + m_e}{m_i m_e}\eta \boldsymbol{J}$$

and

$$\sum {q_s}^2 n_s/m_s \boldsymbol{V}_s = e^2 n \frac{m_e \boldsymbol{V}_i + m_i \boldsymbol{V}_e}{m_i m_e}$$

We took into account relations (5.31) and (5.41). Multiplying the generalized Ohm equation for a two-component plasma by the first factor above, we obtain:

[12] Equation (5.43) only includes plasma-driven currents (5.9) and (5.15)—herein lies the crux of understanding correctly the "chicken-and-egg" question of what comes first, $\boldsymbol{B}$ or $\boldsymbol{J}$?! See also [1]. The set of equations (5.40), (5.43), (5.29) and (5.30) is usually called *the MHD equations* (plural!).

$$\frac{m_i m_e}{m_i + m_e} \frac{1}{n\, e^2} \left[ \frac{\partial \boldsymbol{J}}{\partial t} + \nabla (\boldsymbol{V} \otimes \boldsymbol{J} + \boldsymbol{J} \otimes \boldsymbol{V}) \right]$$
$$= \boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} - \eta \boldsymbol{J} - \frac{m_i - m_e}{m_i + m_e} \frac{\boldsymbol{J} \times \boldsymbol{B}}{n\, e}$$
$$- \frac{1}{(m_i + m_e)} \frac{1}{ne} \nabla \big( m_e \mathbb{P}_i^* - m_i \mathbb{P}_e^* \big) \qquad (5.44)$$

For isotropic pressures $p_i$ and $p_e$, the divergence vectors of the partial pressure tensors become gradient vectors of the respective scalar pressures. As an aside, note that if instead of ions and electrons we had a positron-electron or an antiproton-proton plasma ($m_+ = m_-$), the last two terms in the right side would drop out, and we would be left with a very simple generalized Ohm equation. Unfortunately, it is too dangerous to play with such plasmas, especially if $\eta \neq 0$, so our next great simplification will rather be to stick to an ion-electron plasma and take into account that $m_e \ll m_i$. Hence, $\boldsymbol{V} \simeq \boldsymbol{V}_i$, which leads to the following equation (using (5.31)):

$$\frac{1}{\epsilon_0 \omega_p^2} \left[ \frac{\partial \boldsymbol{J}}{\partial t} + \nabla (\boldsymbol{V} \otimes \boldsymbol{J} + \boldsymbol{J} \otimes \boldsymbol{V}) \right]$$
$$= \boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} - \eta \boldsymbol{J} - \frac{\boldsymbol{J} \times \boldsymbol{B}}{n\, e} + \frac{1}{n\, e} \nabla \big( \mathbb{P}_e^* \big) \qquad (5.45)$$

For an ion-electron plasma, this reduced Ohm equation has a basic physical interpretation, namely, that it is equivalent to a dynamic equation for the electron fluid (with a collision term) *as seen from a reference system fixed to, and traveling with the ion fluid*. In other words, we can imagine the ion-electron plasma as a mass fluid (the ions) and, embedded in it, a massless negative charge fluid (the electrons) guided by its own dynamic equation, and whose flow confers the main electromagnetic properties to the coupled system. The transformation of Eq. (5.45) from the original frame of reference to the ion fluid frame is simple but lengthy [2]; here we will just ask ourselves how such an equation would look. For that purpose, we start with an equation of the type (5.18) (plus the collision term). Calling $\boldsymbol{V}^* = \boldsymbol{V}_e - \boldsymbol{V}_i$ the velocity of the electron fluid, we have $\boldsymbol{J} = ne\boldsymbol{V}^*$ and $\boldsymbol{E}^* = \boldsymbol{E} + \boldsymbol{V}_i \times \boldsymbol{B}$ the electric field seen in the frame moving with the ion fluid, the equation in the moving ion frame should obviously be

$$n_e\, m_e \frac{d\boldsymbol{V}^*}{dt} = -n_e e(\boldsymbol{E}^* + \boldsymbol{V}^* \times B) - \nabla \mathbb{P}_e^* - \eta n_e^2 e^2 \boldsymbol{V}^* - n_e\, m_e (\boldsymbol{V}^* \cdot \nabla) \boldsymbol{V}_i$$

The additional last term is the *inertial force* density acting on the electron fluid due to the acceleration of the frame of reference used (motional change in velocity $\boldsymbol{V}_i$). Transforming the relevant quantities back to the original frame of reference, leads indeed to Eq. (5.45)! One might argue whether this invalidates the earlier assertion that the generalized Ohm equation is not a dynamic equation, but one which brings out local relations in the center of mass fluid. It does not, because although it is a dynamic equation in the ion frame, it describes only a *part* of the whole system.

We may now venture to discuss some simple examples. First we shall examine Eq. (5.45), neglecting the left side on the grounds that it is a quantity divided by the square of the electron plasma frequency. We then can write for the natural system components of the electric field *in the center of mass fluid*:

$$\begin{aligned} \boldsymbol{E}_{\perp}^{cm} &= \boldsymbol{E}_{\perp} + \boldsymbol{V} \times \boldsymbol{B} = \eta \boldsymbol{J}_{\perp} + \frac{\boldsymbol{J} \times \boldsymbol{B}}{ne} - \frac{\nabla_{\perp} \mathbb{P}_e^*}{ne} \\ \boldsymbol{E}_{\parallel}^{cm} &= \boldsymbol{E}_{\parallel} = \eta \boldsymbol{J}_{\parallel} - \frac{\nabla_{\parallel} \mathbb{P}_e^*}{ne} \end{aligned} \tag{5.46}$$

The quantity $\boldsymbol{E}_{\perp}^{cm}$ is the electric field seen in the center of mass fluid; the first term on the right side represents the *ohmic resistance field*; the second term is the *Hall field* (which exists in *any* current-carrying conductor placed in a magnetic field); and the third term is called the *ambipolar electric field* (similar to the field responsible for the e.m.f. in a battery). We now turn to the MHD equation, which in stationary state (and charge neutrality) leads to $\boldsymbol{J} \times \boldsymbol{B} = \nabla_{\perp} \mathbb{P}$ ($\mathbb{P} = \mathbb{P}_i^* + \mathbb{P}_e^*$). Taking this into account, the transverse component in (5.46) becomes

$$\boldsymbol{E}_{\perp} + \boldsymbol{V} \times \boldsymbol{B} = \eta \boldsymbol{J}_{\perp} - \frac{\nabla_{\perp} \mathbb{P}_i^*}{ne} \tag{5.47}$$

Multiplying vectorially by $\boldsymbol{B}/B^2$, we obtain an expression for the perpendicular component of the bulk velocity of the center of mass fluid (nearly equal to that of the ion fluid):

$$\boldsymbol{V}_{\perp} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} - \frac{\eta}{B^2} \nabla_{\perp} \mathbb{P} - \frac{\nabla \mathbb{P}_i^* \times \boldsymbol{B}}{neB^2} \tag{5.48}$$

The first term is the pure electric drift velocity—the velocity a near-zero energy probe particle would have, and therefore also the velocity of a magnetic field line (1.38), provided no potential electric fields are present. The second term (with its sign) is called *diffusion velocity* and the third term (also with its sign) is the *diamagnetic ion drift velocity* (think of the surface currents on the cylinders in Fig. 5.2!).

Next, we shall neglect the term containing the vector divergence of the total pressure tensor divided by the number density $n$ in (5.46). This leaves us with the following pair for the electric field natural components in the center of mass fluid:

$$\begin{aligned} \boldsymbol{E}_{\perp}^{cm} &= \boldsymbol{E}_{\perp} + \boldsymbol{V} \times \boldsymbol{B} = \eta \boldsymbol{J}_{\perp} + \frac{\boldsymbol{J} \times \boldsymbol{B}}{ne} \\ \boldsymbol{E}_{\perp}^{cm} &= \eta \boldsymbol{J}_{\parallel} \end{aligned} \tag{5.49}$$

With a little vector algebra we arrive at the following:

$$\boldsymbol{E}^{cm} \times \boldsymbol{B} = \eta ne\boldsymbol{E}_{\perp}^{cm} - \eta^2 ne\boldsymbol{J}_{\perp} - \boldsymbol{J}_{\perp}\frac{B^2}{ne}$$

or

$$\boldsymbol{J}_{\perp} = \frac{\eta ne\boldsymbol{E}_{\perp}^{cm} - \boldsymbol{E}^{cm} \times \boldsymbol{B}}{B^2/ne + \eta^2 ne}$$

We now introduce a series of plasma parameters, particularly important in ionospheric physics:

*Field-aligned conductivity*: $\sigma_{\parallel} = \sigma = 1/\eta$
*Hall coefficient* (take into account (1.21) and (5.41)):

$$H = \frac{\omega_C}{\nu_{ei}} = \frac{B}{\eta ne} \tag{5.50}$$

*Transverse conductivity*: $\sigma_T = \sigma/(1 + H^2)$
*Hall conductivity*: $\sigma_H = \sigma/(H + 1/H)$

With these designations, the expression for the perpendicular component of the current density becomes:

$$\boldsymbol{J}_{\perp} = \sigma_T \boldsymbol{E}_{\perp}^{cm} + \sigma_H \boldsymbol{e} \times \boldsymbol{E}^{cm} \tag{5.51}$$

All this, including the parallel equation, can be condensed into one tensor equation

$$\boldsymbol{J} = \boldsymbol{\sigma}\boldsymbol{E}^{cm} \tag{5.52}$$

where $\boldsymbol{\sigma}$ is the grand *conductivity tensor*[13]:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_T & -\sigma_H & 0 \\ \sigma_H & \sigma_T & 0 \\ 0 & 0 & \sigma_{\parallel} \end{pmatrix}$$

It is important to note that even in absence of collisions ($\eta = 0$), there is a relation between the current density and the electric field in the center of mass system: $\boldsymbol{J}_{\perp} = ne/B\boldsymbol{e} \times \boldsymbol{E}^{cm}$. This is why $\boldsymbol{E}^{cm}$ is also called the *Hall field*. In this case of zero resistivity, and considering that $\boldsymbol{V} \simeq \boldsymbol{V}_i$, we can conclude from the second equality in (5.49) that

$$0 = \boldsymbol{E}_{\perp} + \boldsymbol{V}_i \times \boldsymbol{B} - (\boldsymbol{V}_i - \boldsymbol{V}_e) \times \boldsymbol{B} = \boldsymbol{E}_{\perp} + \boldsymbol{V}_e \times \boldsymbol{B} \tag{5.53}$$

[13] Radio propagation engineers define the Hall coefficient and Hall conductivity with *opposite sign*.

But this just tells us that the electric field in the electron fluid is zero in this case! In other words, a near-zero probe particle in the OFR will drift *with* the electron fluid. Calling up the image of moving field lines, in a collisionless plasma *magnetic field lines are "frozen" into the electron fluid.*

Our next example will have a further simplification in the reduced Ohm equation (5.45): not only will we neglect the left-hand side, but by comparing the order of magnitude of the Hall term ($[JB/ne] = [H\eta J]$, see (5.50) and (5.41)) with that of the resistive term $[\eta J]$, we see that it, too, can be neglected when the Hall coefficient $H$ is sufficiently small (resistivity sufficiently high). And if we neglect the Hall term, we can also neglect $\nabla \mathbb{P}/ne$, because if we assume a stationary state $dV/dt = 0$, we have $\boldsymbol{J} \times \boldsymbol{B} = \nabla \mathbb{P}$. In the Maxwell equations, we shall consider $\partial \boldsymbol{E}/\partial t = 0$, and that there are *no external sources* of the field. These "fat-cutting" measures leave us with the following set of equations:

$$\begin{aligned} \boldsymbol{E} &= \eta \boldsymbol{J} - \boldsymbol{V} \times \boldsymbol{B} \\ \nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J} \\ \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \end{aligned} \tag{5.54}$$

Of course, we always must consider $\nabla \cdot \boldsymbol{B} = 0$, $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{E} = \rho_q/\epsilon_0 \simeq 0$. Inserting $\boldsymbol{E}$ into the last equation and taking into account that $\nabla \times (\nabla \times \boldsymbol{B}) = -\nabla^2 \boldsymbol{B}$, we obtain the following partial differential equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 \boldsymbol{B} + \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) \tag{5.55}$$

This equation tells us that, under these simplified conditions, the magnetic field in a resistive plasma can change *locally* in time because of the local resistivity *and* because of the local hydrodynamic flow pattern. It is a relationship that, by the way, is valid for all conducting fluids!

When the plasma is at rest ($\boldsymbol{V} \simeq 0$), (5.55) becomes a regular *diffusion equation*, with solutions that have a factor $e^{-t/\tau}$, with a decay time $\tau = \mu_0 L^2/\eta$ ($L$: scale size of the system). In absence of any external sources, the self-generated magnetic field of a plasma will decay exponentially. Since in Appendix A.1 we have assigned primary physical "reality" to the currents that sustain a magnetic field, we should re-state this: in a plasma under these conditions, the *currents* $\nabla \times \boldsymbol{B}$ will decay exponentially! The physical reason is easy to understand: collisions destroy the adiabatic behavior of the electrons; they diffuse and "smear out" the equivalent currents. Consider Fig. 5.2, and suppose that instead of sparsely populated by cycling particles, the cylinder is filled with denser, colliding ions and electrons. The boundary equivalent currents $\nabla \times \boldsymbol{M}$ are mainly carried by electrons. Collisions with the ions will disperse them and decrease exponentially the boundary surface current system. Since there is an overall uniform external field which originally was reduced inside the cylinder (diamagnetic effect), the total field intensity inside will increase back to the external field value. In the picture of moving field lines, originally outward-displaced field lines will straighten and move back into the cylinder.

The case of zero resistivity (collisionless plasma) in (5.55) should be reconsidered from the beginning. We arrived at this equation by neglecting the Hall term when compared to $\eta \boldsymbol{J}$. When the latter is zero, we must compare the Hall term with $\boldsymbol{V} \times \boldsymbol{B}$ in the simplified Ohm's equation. The current density is $\boldsymbol{J} = -ne(\boldsymbol{V}_e - \boldsymbol{V}_i)$; if $\boldsymbol{V} \simeq \boldsymbol{V}_i \gg (\boldsymbol{V}_e - \boldsymbol{V}_i)$ we can neglect the Hall term in a collisionless plasma, and (5.55) becomes

$$\frac{\partial B}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) \tag{5.56}$$

This equation tells us that magnetic field flux tubes will move with the guiding center fluid. Indeed, the time-change of the magnetic flux through a contour whose points move with the fluid will be

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot dA + \oint \boldsymbol{B} \cdot (\boldsymbol{V} \times \boldsymbol{dl}) = \int_S \left[ \frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) \right] \cdot \boldsymbol{dA} = 0$$

It can be easily shown that *any* contour on a flux tube will conserve the enclosed flux while moving with the plasma, thus preserving the identity of the entire flux tube. This was the feature that led Alfvén to formulate the concept of "frozen-in magnetic fields".

Speaking of Alfvén, we come to the last example and with it to the end of this chapter (and the book). It describes a wave process in collisionless plasmas, alluded to earlier in Chap. 4; however, we will only mention the most basic concepts (plasma waves deserve an entire book!) Consider a collisionless plasma in equilibrium in a uniform magnetic field $\boldsymbol{B}_0$ directed along the $z$-axis under the same conditions as in the previous paragraph ($\boldsymbol{V} = 0$; $\rho_m = 0$). We introduce a small perturbation $\boldsymbol{v}$ and $\boldsymbol{b}$ perpendicular to the uniform magnetic field. The equations to be used for the perturbations will be to first order in the perturbations:

$$\begin{aligned} \rho_m \frac{\partial \boldsymbol{v}}{\partial t} &= \frac{1}{\mu_0} \big(\nabla \times \boldsymbol{b}\big) \times \boldsymbol{B}_0 \\ \frac{\partial \boldsymbol{b}}{\partial t} &= (\boldsymbol{B}_0 \cdot \nabla) \boldsymbol{v} \end{aligned} \tag{5.57}$$

If $\boldsymbol{b}$ is directed along the $x$-axis, $\boldsymbol{v}$ will also be directed along that axis and we can find two solutions $b = b(z)$ and $v = v(z)$ that obey

$$\rho_m \frac{\partial v}{\partial t} = \frac{B_0}{\mu_0} \frac{\partial b}{\partial z}$$

and

$$\frac{\partial b}{\partial t} = B_0 \frac{\partial v}{\partial z}$$

Taking $\partial/\partial t$ of both we end up with two wave equations for $v$ and $b$:

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{V_A^2}\frac{\partial^2}{\partial t^2}\right](v; b) = 0$$

with

$$V_A = \frac{B_0}{\sqrt{\mu_0 \rho_m}} \tag{5.58}$$

the *Alfvén velocity*, with which any perturbation propagates in a collisionless plasma in the direction of the originally unperturbed magnetic field. It can be shown geometrically that in this process the elicited displacement pattern of the field line is proportional to the pattern of field and velocity perturbation and propagates with them. Alfvén waves are commonly interpreted as transverse oscillations of magnetic field lines. A bit more precisely, Alfvén waves are transverse fluid oscillations which, as they propagate in the direction of the main field, distort the field lines in their oscillatory motion. The oscillatory electric field is perpendicular to the magnetic field variations, and both are mutually out of phase by $\pi/2$, and perpendicular to the propagation vector. The Alfvén wave velocity only depends on the local magnetic field intensity and plasma mass density; thus there is no dispersion (frequency dependence) and the wave profile remains the same as it propagates. Historically, it was soon recognized that the so-called micropulsations of the ground-based geomagnetic field were standing oscillations of field-aligned Alfvén waves—making the magnetosphere a planetary-scale "musical instrument" of vibrating field lines. At the time of the discovery of these ultra low frequency (ULF) Alfvén waves, it was quite difficult to imagine the possibility of a wave propagation process in a collisionless gas—Alfvén waves became a prime example of the intricate interplay between currents and fields in a collisionless plasma. And, as we have mentioned in Chap. 4, they indeed play a fundamental role in the dynamics of the radiation belt.

In this chapter we just gave a somewhat superficial description aimed at showing how collisionless plasmas can be understood intuitively by focusing on the fundamental properties of the adiabatic behavior of charged particles in magnetic and electric fields. Formal and detailed descriptions can be found, for instance, in [2] (includes the most important relativistic equations), [3] and [4].

## 5.6 Epilogue

In his waning days Alfvén insistently lamented to one of us (JGR) about having promulgated the concept of "frozen-in magnetic field" and "moving field lines" too much during the early times of space plasma physics. He fully recognized that the field line is a purely geometric concept that can be very helpful in visualizing magnetic field geometry and, in *certain* situations, its time-changes, but that this image must be handled with great care. Field lines do not drag plasma, nor does plasma drag field lines—plasma moves in response to magnetic and electric forces acting on currents and charges embedded in the fluid, a process mathematically described by linking plasma and Maxwell's equations. It so happens that *under certain circumstances* we can visualize in our minds this motion as that of continuously changing magnetic field lines, co-moving with the plasma or, rather, its constituent electron fluid (see (5.53)).

As a tribute to Alfvén, let us end the book with the "grand finale" of a kindergarten example. Turn back to Fig. 5.2, and assume that now we have a *neutral* dense ensemble of 90° particles evenly distributed in that cylinder, in an external homogeneous magnetic field $\boldsymbol{B}$. To avoid undesirable equivalent polarization charges and other complications, we'll assume it to be a low-beta plasma ($p \ll B^2/(2\mu_0)$, page 136). The field inside the cylinder $\boldsymbol{B}^* = \boldsymbol{B} + \boldsymbol{b}$ will be reduced in intensity due to the diamagnetic effect of the boundary equivalent currents; the field topology of the self-field $\boldsymbol{b}$ is in effect that of a solenoid, opposed to $\boldsymbol{B}$ inside. This means that the total field will exhibit field lines bent somewhat outwards all along the lateral boundary surface, leaving a reduced flux inside. Now we turn on a uniform electrostatic field, say, perpendicular into the paper in Fig. 5.2, $\boldsymbol{E} \perp \boldsymbol{B}$. Obviously, the circling particles will all drift to the right with the same speed $V_E = E/B$, independent of their energy, mass and charge (Sect. 1.3). Will they carry with them the magnetic field lines? According to our probe particle definition of field line velocity (1.38), the answer is *no*! This definition indeed mandates (see page 20) that we turn off all contributions from potential electric fields, and examine the probe particle drift *exclusively* under the action of the $-\partial \boldsymbol{A}/\partial t$ induced electric field. And, carefully depicting in our mind the rigidly drifting axisymmetric $\boldsymbol{A}$-vector field configuration of the equivalent current system, regions of appreciable $-\partial \boldsymbol{A}/\partial t$ will only be found in the vicinities of this *moving* cylindrical surface current system. In other words, the plasma will drift to the right in the figure (a kindergarten version of plasma propulsion motor!) and open its way through the external $\boldsymbol{B}$-field lines as if you were walking though a corn field by bending the stocks around you. Field lines will never detach from the original magnet but just bend out and snap back as the cylindrical surface current system moves by; this applies to field lines both outside and inside the cylinder. *At no time will this plasma be carrying any frozen magnetic field lines with it*![14]

[14] All this is valid only for a low energy density, i.e., low-beta plasma. At higher densities the situation changes considerably. For instance, the equivalent current envelope may be intense enough so that the inner field $\boldsymbol{B}^*$ is so weak that the guiding center approximation breaks down and a kinetic description is necessary; in that case we can no longer talk about a common electric drift. Moreover, if the boundary current is intense enough, field line *loops* may appear enclosing parts of the equivalent current system and indeed move together with the bulk motion of the latter; this happens with the plasmoids in the magnetospheric tail or the solar magnetic loops that detach from photospheric loops to form the initial stage of a solar mass ejection. In summary, whether "magnetic field lines carry plasma" or "plasma carries magnetic field lines" depends entirely on the characteristics and the dynamic behavior of the *currents* around which those field lines are wound (remember that a magnetic field line is always part of a closed loop because of $\nabla \cdot \boldsymbol{B} = 0$—even if that closure involves an infinite number of turns or occurs at infinity!).

## References

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