Uniform Grid Approximation of Nonsmooth Solutions of a Singularly Perturbed Convection - Diffusion Equation with Characteristic Layers

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Abstract. A mixed boundary value problem for a singularly perturbed elliptic convection-diffusion equation with constant coefficients is considered in a square with the Dirichlet conditions imposed on the two sides, which are orthogonal to the flow direction, and with the Neumann conditions on the other two sides. Sufficient smoothness of the right-hand side and that of the boundary functions is assumed, which ensures the required smoothness of the solution in the considered domain, except the neighborhoods of the corner points. At the corner points themselves, zero order compatibility conditions alone are assumed to be satisfied.

For the numerical solution to the posed problem a nonuniform monotonous difference scheme is used on a rectangular piecewise uniform Shishkin grid. Non-uniformity of the scheme means that the form of the difference equations, which are used for the approximation, is not the same in different grid points but it depends on the value of the perturbing parameter.

Under assumptions made a uniform convergence with respect to ε of the numerical solution to the precise solution is proved in a discrete uniform metric at the rate $O(N^{-3/2}\ln^2 N)$, where N is the number of the grid points in each coordinate direction.

Keywords: singularly perturbed problems, condensing mesh, characteristic boundary layer, corner singularity, uniform convergence.

1 Introduction

A mixed boundary problem for a singularly perturbed convection-diffusion equation [1,2] is considered in the square $\Omega = (0,1)^2$ with the boundary $\partial\Omega$:

$$Lu \equiv -\varepsilon\Delta u + a\frac{\partial u}{\partial x} + qu = f(x,y), \quad (x,y) \in \Omega, \tag{1}$$

$$\frac{\partial u}{\partial \mathbf{n}} = \varphi(x,y), \quad (x,y) \in \partial\Omega_N, \tag{2}$$

$$u = g(x,y), \quad (x,y) \in \partial\Omega_D, \tag{3}$$

where $a = const > 0$, $q = const > 0$; $\mathbf{n}$ is the unit vector of an outer normal to $\partial\Omega_N$, and $\varepsilon \in (0,1]$ is a small parameter. The domain boundary consists

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of the parts $\partial\Omega_D = \Gamma_1 \cup \Gamma_3$ and $\partial\Omega_N = \Gamma_2 \cup \Gamma_4$, where Γ_k are the sides of the square Ω, ordered in the counterclockwise direction starting from $\Gamma_1 = \{(x,y) \in \partial\Omega \mid x = 0\}$, whereas $a_k = (x_k, y_k)$ are its vertices, ordered in the similar way being $a_1 = (0,0)$.

In the corner points a_k the boundary functions g and φ are assumed to be the subject to the following conditions

$$\frac{dg_1}{dy}(0) = -\varphi_1(0),\ \frac{dg_1}{dy}(1) = \varphi_2(0),\ \frac{dg_2}{dy}(0) = -\varphi_1(1),\ \frac{dg_2}{dy}(1) = \varphi_2(1), \qquad (4)$$

which are named [3] zero-order compatibility conditions, where $g_1(y) = g(0,y)$, $g_2(y) = g(1,y)$ and $\varphi_1(x) = \varphi(x,0)$, $\varphi_2(x) = \varphi(x,1)$.

The solution to problem (1)–(4) has a compound structure (see [2,4] e.g. and the literature cited there), which includes the regular boundary layer of width $O(\varepsilon)$ in a neighborhood of the right-hand boundary Γ_3, the two characteristic layers of width $O(\sqrt{\varepsilon})$ in neighborhoods of the upper and lower boundaries Γ_2 and Γ_4, the corner layers with corner singularities in neighborhoods of the vertices a_2, a_3 and the corner singularities in neighborhoods of the inflow vertices a_1, a_4. All this makes it difficult to solve problem (1)–(4) numerically. It is also known (see [3]) that the presence of the corner points adversely affects the smoothness of the solution.

Actually it is problem (1)–(3) though in more general setting (with variable coefficients and boundary conditions of the third kind imposed on $\partial\Omega_N$) that paper [1] is devoted to. In this paper a nonuniform monotonous scheme was made up for the equation and the scheme formally has the second order approximation with $\varepsilon \le CN^{-1}$ (compare with [6,7] in one dimensional case) and, under the assumption that the compatibility conditions at the corner points are satisfied up to the 2nd order, the convergency at the rate $O(N^{-3/2}\ln N)$ on Shishkin mesh is proved.

The purpose of this work is to enhance the results [1] in the following directions: getting rid of the excessive compatibility conditions, which increased the smoothness of the solution up to $u(x,y) \in C^{4,\lambda}(\bar{\Omega})$, $\lambda \in (0,1)$; working out the uniform with respect to ε estimate of the convergency rate of order $O(N^{-3/2}\ln^2 N)$ for all $\varepsilon \in (0,1]$, instead of the former estimate for $\varepsilon \le CN^{-1}$.

In the course of the paper we denote by C, $\tilde{C}$ and c some positive constants,which are different in any individual case and depend on input data, but not on N or ε.

2 Statement of the Finite Difference Problem

In paper [2] by means of the decomposition of the solution to problem (1)–(3) into smooth and boundary layer components some point-wise estimates to the solution and its derivatives are obtained, which we use here; their dependance on a small parameter ε and on the data compatibility conditions at the domain corners is shown. Before we introduce the difference scheme, let us write this solution as the following sum:

$$u(x,y) = S + E + w_1 + w_2 + w_3 + w_4 + \tilde{u},\ (x,y) \in \Omega, \qquad (5)$$

where S is a smooth component, E is a regular boundary layer, w_1, w_4 and w_2, w_3 are two characteristic layers (an upper one and a lower one respectively) and two corner layers in neighborhoods of the corners a_2, a_3; $\tilde{u}$ is the remainder term, which is exponentially small.

Thus, on the set $\bar{\Omega}$ we introduce the grid $\bar{\Omega}^h = \bar{\omega}_x \times \bar{\omega}_y$, which is a tensor product of two piecewise unform Shishkin meshes. The mesh $\bar{\omega}_x$ contains $N/2$ points on each of the half-intervals $(0, 1-\sigma_x]$ and $(1-\sigma_x, 1]$ with the steps $\mathfrak{H}_1$ and $\mathfrak{h}_1$, respectively, and the mesh $\bar{\omega}_y$ contains $N/4$ points on the half-intervals $(0, \sigma_y], (1-\sigma_y, 1]$ and $N/2$ points on the $(\sigma_y, 1-\sigma_y]$ with steps $\mathfrak{h}_2$ and $\mathfrak{H}_2$ respectively, where

$$\mathfrak{H}_1 = \frac{2(1-\sigma_x)}{N},\ \mathfrak{h}_1 = \frac{2\sigma_x}{N},\ \sigma_x = \min\left\{\frac{1}{2}; \frac{4\varepsilon \ln N}{a}\right\};\ \mathfrak{H}_2 = \frac{2(1-2\sigma_y)}{N},$$

$$\mathfrak{h}_2 = \frac{4\sigma_y}{N},\ \sigma_y = \min\left\{\frac{1}{4}; \frac{4\sqrt{\varepsilon}\ln N}{\beta}\right\};\ \beta = \min\{a/12, q/2a, \sqrt{q}\}.$$

Therefore, to find the numerical solution of the problem (1)–(4) we use the inhomogeneous monotone difference scheme, in which for small values of ε convective term is approximated by the usual directional difference at the middle point $x_{i-1/2}$ out of the regular layer, while in the regular layer we use for this the central difference.

Before we write the difference scheme, we will introduce the following notation: $\Omega^h = \bar{\Omega}^h \cap \Omega,\ \Omega_1^h = \bar{\Omega}_1^h \cap \Omega^h,\ \Omega_2^h = \bar{\Omega}_2^h \cap \Omega^h,\ \bar{\Omega}_1^h = \{0 \le x_i \le 1-\sigma_x, 0 \le y_j \le 1\},\ \bar{\Omega}_2^h = \{1-\sigma_x < x_i \le 1, 0 \le y_j \le 1\},\ u_{\bar{x},i} = (u_i - u_{i-1})/h_i,\ u_{\hat{x},i} = (u_{i+1} - u_i)/\hbar_i,\ u_{\mathring{x},i} = (u_{i+1} - u_{i-1})/2\hbar_i,\ u_{x,i} = u_{\bar{x},i+1},\ \hbar_i = (h_{i+1} + h_i)/2,$ $f_{ij}^h = f(x_i, y_j)$, and u_{ij}^h is an approximative solution to problem (1)–(4).

On the mesh $\bar{\Omega}^h$ we assign a difference problem (see [1]) to problem (1)–(4)

$$L^h u_{ij}^h \equiv -\varepsilon\left(u_{\bar{x}\hat{x};ij}^h + u_{\bar{y}\hat{y};ij}^h\right) + L_1^h u_{ij}^h + q u_{ij}^h = F_{ij}^h, \quad \Omega^h \tag{6}$$

where

$$L_1^h u_{ij}^h \equiv \begin{cases} \left(a - \dfrac{q\mathfrak{H}_1}{2}\right) u_{\bar{x};ij}^h, & \varepsilon < a\mathfrak{H}_1/2,\ \Omega_1^h, \\ a u_{\mathring{x};ij}^h, & \varepsilon < a\mathfrak{H}_1/2,\ \Omega_2^h, \\ a u_{\mathring{x};ij}^h, & \varepsilon \ge a\mathfrak{H}_1/2,\ \Omega^h, \end{cases} \quad F_{ij}^h = \begin{cases} f_{i-1/2,j}^h, & \varepsilon < a\mathfrak{H}_1/2,\ \Omega_1^h, \\ f_{ij}^h, & \varepsilon < a\mathfrak{H}_1/2,\ \Omega_2^h, \\ f_{ij}^h, & \varepsilon \ge a\mathfrak{H}_1/2,\ \Omega^h. \end{cases}$$

$$L^h u_{i0}^h \equiv \begin{cases} -u_{y;i0}^h + \dfrac{\mathfrak{h}_2}{2}\left[-u_{\bar{x}\hat{x};i0}^h + \dfrac{1}{\varepsilon}\left(a - \dfrac{q\mathfrak{H}_1}{2}\right) u_{\bar{x};i0}^h + \dfrac{q}{\varepsilon} u_{i0}^h\right] \\ -u_{y;i0}^h + \dfrac{\mathfrak{h}_2}{2}\left[-u_{\bar{x}\hat{x};i0}^h + \dfrac{a}{\varepsilon} u_{\mathring{x};i0}^h + \dfrac{q}{\varepsilon} u_{i0}^h\right] \end{cases} =$$

$$= \begin{cases} \varphi_1(x_i) + \dfrac{\mathfrak{h}_2}{2\varepsilon} f_{i-1/2,0}^h, & \varepsilon < \dfrac{a\mathfrak{H}_1}{2},\ 0 < i \le N/2, \\ \varphi_1(x_i) + \dfrac{\mathfrak{h}_2}{2\varepsilon} f_{i0}^h, & \varepsilon < \dfrac{a\mathfrak{H}_1}{2},\ N/2 < i < N \textit{ or } \varepsilon \ge \dfrac{a\mathfrak{H}_1}{2},\ 0 < i < N, \end{cases} \tag{7}$$

$$L^h u^h_{iN} \equiv \begin{cases} u^h_{\bar{y};iN} + \frac{\mathfrak{h}_2}{2}\left[-u^h_{\bar{x}\hat{x};iN} + \frac{1}{\varepsilon}\left(a - \frac{q\mathfrak{H}_1}{2}\right)u^h_{\bar{x};iN} + \frac{q}{\varepsilon}u^h_{iN}\right] \\ u^h_{\bar{y};iN} + \frac{\mathfrak{h}_2}{2}\left[-u^h_{\bar{x}\hat{x};iN} + \frac{a}{\varepsilon}u^h_{\mathring{x};iN} + \frac{q}{\varepsilon}u^h_{iN}\right] \end{cases} =$$

$$= \begin{cases} \varphi_2(x_i) + \frac{\mathfrak{h}_2}{2\varepsilon}f^h_{i-1/2,N}, \ \varepsilon < \frac{a\mathfrak{H}_1}{2},\ 0 < i \le \frac{N}{2}, \\ \varphi_2(x_i) + \frac{\mathfrak{h}_2}{2\varepsilon}f^h_{iN}, \qquad \varepsilon < \frac{a\mathfrak{H}_1}{2},\ \frac{N}{2} < i < N \ or\ \varepsilon \ge \frac{a\mathfrak{H}_1}{2},\ 0 < i < N, \end{cases} \tag{8}$$

$$u^h_{0j} = g_1(y_j), \quad u^h_{Nj} = g_2(y_j), \qquad 0 \le j \le N. \tag{9}$$

In order to get the convergence rate estimate let us represent the solution to difference problem (6)–(9), by analogy with the differential problem, in the form

$$u^h = S^h + E^h + \sum_{k=1}^{4} w^h_k + \tilde{u}^h, \quad (x_i, y_j) \in \bar{\Omega}^h \tag{10}$$

where each component satisfies the following discrete problems

$$\begin{cases} L^h S^h_{ij} = F^h_{ij}, \quad \Omega^h, \\ L^h S^h_{i0} = -\frac{\partial S}{\partial y}(x_i, 0) + \frac{\mathfrak{h}_2}{2\varepsilon}F^h_{i0}, \quad 0 < i < N, \\ L^h S^h_{iN} = \frac{\partial S}{\partial y}(x_i, 1) + \frac{\mathfrak{h}_2}{2\varepsilon}F^h_{iN}, \quad 0 < i < N, \end{cases} \tag{11}$$

$$S^h_{0j} = S(0, y_j), \quad S^h_{Nj} = S(1, y_j), \qquad 0 \le j \le N. \tag{12}$$

$$\begin{cases} L^h E^h_{ij} = 0, \quad \Omega^h, \\ L^h E^h_{i0} = -\frac{\partial E}{\partial y}(x_i, 0), \quad 0 < i < N, \\ L^h E^h_{iN} = \frac{\partial E}{\partial y}(x_i, 1), \quad 0 < i < N, \end{cases} \tag{13}$$

$$E^h_{0j} = E(0, y_j),\ E^h_{Nj} = E(1, y_j), \quad 0 \le j \le N. \tag{14}$$

$$\begin{cases} L^h w^h_{k;ij} = 0, \quad \Omega^h, \\ L^h w^h_{k;i0} = -\frac{\partial w_k}{\partial y}(x_i, 0), \quad 0 < i < N, \\ L^h w^h_{k;iN} = \frac{\partial w_k}{\partial y}(x_i, 1), \quad 0 < i < N, \end{cases} \tag{15}$$

$$w^h_{k;0j} = w_k(0, y_j), \quad w^h_{k;Nj} = w_k(1, y_j), \qquad k = 1, 2, 3, 4, \quad 0 \le j \le N. \tag{16}$$

The estimate for the convergence of the numerical solution to the exact one will be obtained as a sum of the estimates for each term from (10). To this effect we represent the approximation error also as a sum
$\psi_{ij}^h = \psi_{S;ij}^h + \psi_{E;ij}^h + \sum_{k=1}^4 \psi_{w_k;ij}^h + \psi_{\tilde{u};ij}^h$, $(x_i, y_j) \in \bar{\Omega}^h$, where $\psi_{S;ij}^h = L^h S(x_i, y_j) -$
$-L^h S_{ij}^h$, $\psi_{E;ij}^h = L^h \left(E(x_i, y_j) - E_{ij}^h\right)$, $\psi_{w_k;ij}^h = L^h \left(w_k(x_i, y_j) - w_{k;ij}^h\right)$.

Further in the whole article we shall use the comparison principle [1] while working out the convergence rate estimates for each component from (10).

Theorem 1 (Comparison principle). *Let V_{ij}^h and W_{ij}^h be arbitrary mesh functions, defined on the mesh $\bar{\Omega}^h$, so that $\left|L^h V_{ij}^h\right| \le L^h W_{ij}^h$ в $\Omega^h \cup \partial\Omega_N^h$, where $\Omega_N^h = \bar{\Omega}^h \cap \partial\Omega_N$ и $\left|V_{ij}^h\right| \le W_{ij}^h$ on $\partial\Omega_D^h = \bar{\Omega}^h \cap \partial\Omega_D$. Then in $\bar{\Omega}^h$ the following estimate holds true $\left|V_{ij}^h\right| \le W_{ij}^h$.*

The next theorem contains the main result of this paper.

Theorem 2. *Let $u(x_i, y_j)$ be a solution to the original problem (1)-(4), and u_{ij}^h be a solution to the discrete problem (6)-(9) on a piecewise unform Shishkin mesh. Then for $\varepsilon \in (0, 1]$ the following rate convergence estimate holds true*

$$\left|u(x_i, y_j) - u_{ij}^h\right| \le C N^{-3/2} \ln^2 N, \qquad (x_i, y_j) \in \bar{\Omega}^h. \tag{17}$$

Proof. The proof follows from Theorem 3 and Remark 1, which are given in the next section. □

3 The Uniform Convergence of Numerical Solutions

The reasoning we use in this section when proving convergence for the smooth component S_{ij}^h, of the regular layer E_{ij}^h, and also for boundary layer components $w_{k;ij}^h$ $(k = \overline{1,4})$ outside the domains of the characteristic layer $\Omega_{w_1}^h = \{0 \le x_i \le 1, 0 \le y_j < \sigma_y\}$ and of the corner layer $\Omega_{w_2}^h = \{1 - \sigma_x < x_i \le 1, 0 \le y_j < \sigma_y\}$ ($\Omega_{w_4}^h$ и $\Omega_{w_3}^h$ are dealt in the similar way), do not differ from those, which are given in [1].

Hence, proceeding as in [1] and taking the estimates for derivatives [2], we arrive at the following estimates

$$\left|S(x_i, y_j) - S_{ij}^h\right| \le C N^{-2}, \qquad (x_i, y_j) \in \bar{\Omega}^h, \tag{18}$$

$$\left|E(x_i, y_j) - E_{ij}^h\right| \le C \begin{cases} N^{-2}, & (x_i, y_j) \in \bar{\Omega}_1^h, \\ N^{-2} \ln^2 N, & (x_i, y_j) \in \bar{\Omega}_2^h, \end{cases} \tag{19}$$

$$\left|w_k(x_i, y_j) - w_{k;ij}^h\right| \le C N^{-2}, \qquad (x_i, y_j) \in \bar{\Omega}^h \backslash \Omega_{w_k}^h. \tag{20}$$

Some additional investigation is needed for the convergence rate estimates for $w_{1;ij}^h$ and $w_{2;ij}^h$ (estimates for $w_{4;ij}^h$ and $w_{3;ij}^h$ are obtained by analogy) in the domains of the characteristic layer and the corner layer respectively, including the corner singularities as well.

3.1 The Characteristic Layer

Let us consider the discrete function $w^h_{1;ij}$ of the lower characteristic layer. Using the derivative estimates [2], taking into consideration the approximation error of problem (6)–(9) and estimate (20) and also proceeding as in [5], we obtain

$$\left|\psi^h_{w_1;ij}\right| \le C \begin{cases} N^{-3/2} r^{-1}_{1;ij} \ln N, \; x_i \in (0, 1-\sigma_x], \; y_j \in [0, \sigma_y), \\ N^{-2} \ln^2 N, \qquad x_i \in (1-\sigma_x, 1), y_j \in [0, \sigma_y), \end{cases} \tag{21}$$

$$\begin{aligned} \left|w_1(x_i, y_{N/4}) - w^h_{1;iN/4}\right| &= O(N^{-2}), & 0 < i < N, \\ \left|w_1(0, y_j) - w^h_{1;0j}\right| = \left|w_1(1, y_j) - w^h_{1;Nj}\right| &= 0, & 0 \le j \le N. \end{aligned} \tag{22}$$

We choose the barrier function (see [5]) in view of the corner singularity in a neighborhood of the point $a_1 = (0, 0)$. to estimate the convergence rate of $w^h_{1;ij}$. Thus, let us consider the function

$$\tilde{B}_{w_1}(x, y) = N^{-3/2} \ln N \left(CB_1(x, y) + \tilde{C} b_1(y) \ln N\right) + CN^{-2}, \quad (x, y) \in \Omega^h_{w_1},$$

where $B_1(x, y) = \ln(r_1^{'}/\mathfrak{H}_1) + \left(-\varphi^{'2} - \varphi^{'} + \pi/4 + \pi/2 + 1\right), b_1(y) = e^{-\frac{\beta y}{2\sqrt{\varepsilon}}}, y^{'} = y$, $x^{'} = x + b\mathfrak{H}_1$, $r_1^{'} = \sqrt{x^{'2} + y^{'2}}$, $\varphi^{'} = \arctan \dfrac{y}{x^{'}}$, $\tilde{C} = 64a/27(2\pi + 7)^3 \beta^2$, $b =$ *const* > 1. The barrier function $\tilde{B}_{w_1}(x, y)$, unlike [5], contains an additional term $b_1(y)$ to enhance. The condition of choice for the constant b is given below.

Lemma 1. *If $w^h_{1;ij}$ is a solution to difference problem (15)–(16) at $k = 1$ and $\varepsilon < a\mathfrak{H}_1/2$, while $w_1(x_i, y_j)$ is a solution to the corresponding differential problem, then the following estimate is valid*

$$\left|w_1(x_i, y_j) - w^h_{1;ij}\right| \le C\left(N^{-3/2} \ln^2 N + N^{-2}\right), \quad (x_i, y_j) \in \Omega^h_{w_1}. \tag{23}$$

Proof. Validity of estimate (23) follows from the validity of inequality (see [5])

$$L^h \tilde{B}_{w_1;ij} \ge C(N^{-3/2} \ln N / r_{1;ij}) + CN^{-2}, \quad \Omega^h_{w_1} \backslash \partial\Omega^h_D, \tag{24}$$

and this implies the result by applying comparison principle for the approximation error in $\Omega^h_{w_1}$, in view of the estimates (21)–(22). Estimate (24) at $0 < j < N/4$ is obtained by the same method as in ([5], see (3.18), (3.19)), if we assume that $Lb_{1;j} \ge C(q, \beta)$ holds true and instead of (3.19) from [5] we require the fulfilment of condition
$r^{'}_{1;ij} \ge \max\limits_{ij} \left\{\left(6\mathfrak{H}_1\sqrt{2\pi + 5}\right)/\sqrt{3}; (4q\mathfrak{H}_1^2)/a; 6\mathfrak{H}_1(2\pi + 7)\right\}$. Since $r^{'}_{1;ij} \ge (1 + b)\mathfrak{H}_1$ in domain $\Omega^h_{w_1} \backslash \partial\Omega^h_D$, then the last inequality certainly holds true provided $6\mathfrak{H}_1(2\pi + 7) \le (1 + b)\mathfrak{H}_1$, by which b is determined. Similar reasoning also holds true at $y = 0$ if we take into consideration the derivative estimates from [2], choice of the constant $\tilde{C}$ and fulfilment of the following relations

$$LB_{1;i0} = \frac{1}{r^{'}_{1;i0}}, \; Lb_{1;i0} \ge \frac{\beta}{2\sqrt{\varepsilon}}, \; 0 < i < N; \; \left|\frac{\partial^3 b_1(y)}{\partial y^3}\right| \le \frac{C}{\varepsilon\sqrt{\varepsilon}}, \; \left|\frac{\partial^4 b_1(y)}{\partial y^4}\right| \le \frac{C}{\varepsilon^2}.$$

And the following expression will be an analogue for inequality (3.19) from [5], which represents the Neumann boundary condition. And the expression $r'_{1;i0} \geq \frac{4}{\sqrt{3}} \max_{0<i<N} \left\{ \mathfrak{h}_2 \sqrt{2(\pi+3)}; 2\sqrt{2\mathfrak{H}_1 \mathfrak{h}_2} \right\}$, will be an analogue of the mentioned earlier inequality (3.19) from [5], which stands for the boundary the Neumann condition and holds true with our choice of b.

In case $N/2 < i < N$ the approximation error of the difference scheme contains the fourth order derivatives, but at the expense of multipliers $\mathfrak{h}_1^2, \mathfrak{h}_2^2$ all the reasoning holds true. We finish proving the lemma with the estimates $b_1(y) = e^{-\frac{\beta y}{2\sqrt{\varepsilon}}} \leq 1,\ C \leq B_1(x,y) \leq C(\ln N + 1)$ at $0 < x < 1,\ 0 \leq y < \sigma_y$, the second of which is the implication of inequalities $N^{-1} < \mathfrak{H}_1 < 2N^{-1}$, $(1+b)\mathfrak{H}_1 \leq r'_{1;ij} \leq \sqrt{2}(1 + b\mathfrak{H}_1)$. □

3.2 The Corner Layer

Let us do convergence rate estimates for the function $w^h_{2;ij}$ of the corner layer and analyze the approximation error. Proceeding by analogy with characteristic layers, we obtain

$$\left|\psi^h_{w_2;ij}\right| \leq C\frac{N^{-2}\ln^2 N}{\varepsilon}, \qquad (x_i, y_j) \in \Omega^h_{w_2} \backslash \partial\Omega^h, \tag{25}$$

$$\left|\psi^h_{w_2;i0}\right| \leq C \begin{cases} \dfrac{N^{-2}\ln^2 N}{r_{2;i0}} + \dfrac{N^{-3}\ln^3 N}{\varepsilon}, & r_{2;i0} < \varepsilon, \\ \dfrac{N^{-3}\ln^3 N}{\varepsilon}, & r_{2;i0} \geq \varepsilon, \end{cases} \quad N/2 < i < N, \tag{26}$$

$$\begin{aligned} &\left|w_2(x_{N/2}, y_j) - w^h_{2;N/2j}\right| = O(N^{-2}),\ \left|w_2(1, y_j) - w^h_{2;Nj}\right| = 0,\ 0 \leq j \leq \frac{N}{4}, \\ &\left|w_2(x_i, y_{N/4}) - w^h_{2;iN/4}\right| = O(N^{-2}), \qquad \frac{N}{2} \leq i \leq N, \end{aligned} \tag{27}$$

where $r_{2;ij} = \sqrt{(1-x_i)^2 + y_j^2}$.

So the following lemma gives the convergence rate estimate for the $w^h_{2;ij}$.

Lemma 2. *If $w^h_{2;ij}$ is a solution to difference problem (15)–(16) at $k = 2$ and $\varepsilon < a\mathfrak{H}_1/2$, and $w_2(x_i, y_j)$ is a solution to the corresponding differential problem, then the following estimate is valid*

$$\left|w_2(x_i, y_j) - w^h_{2;ij}\right| \leq CN^{-2}\ln^3 N, \quad (x_i, y_j) \in \Omega^h_{w_2}. \tag{28}$$

Proof. Let us introduce the notation $z^h_{ij} = w_2(x_i, y_j) - w^h_{2;ij}$ and assume $z^h_{ij} = z^h_{1;ij} + z^h_{2;ij}$, where $z^h_{1;ij}$, $z^h_{2;ij}$ satisfies the conditions (27) and the following inequalities (see (25), (26))

$$\left|L^h z^h_{1;ij}\right| \leq C \begin{cases} \dfrac{N^{-2}\ln^2 N}{\varepsilon}, & N/2 < i < N,\ 0 < j < N/4, \\ \dfrac{N^{-3}\ln^3 N}{\varepsilon}, & j = 0, \end{cases} \tag{29}$$

$$\left|L^h z^h_{2;ij}\right| \le C\frac{N^{-2}\ln^2 N}{r_{2;ij}}, \qquad N/2 < i < N,\ 0 \le j < N/4. \tag{30}$$

Therefore the error estimate for z^h_{ij} will be obtained as the sum of the estimates for $z^h_{1;ij}$, $z^h_{2;ij}$.

We shall begin our investigation with $z^h_{1;ij}$. Let $B^h_{z_1;ij} = C(N^{-2}\ln^2 N)B^h_{E;ij}$, where

$$B^h_{E;ij} = \begin{cases} \prod\limits_{s=i+1}^{N}(1+ah_s/2\varepsilon)^{-1}, 0 \le i < N,\ 0 \le j \le N, \\ 1, \qquad\qquad\qquad\qquad i = N, \qquad 0 \le j \le N. \end{cases}$$

Applying the difference operator from (6)–(8) to the barrier $B^h_{E;ij}$ and, using the approximation error bounds (27), (29), we shall get the inequalities

$$\begin{aligned}
&L^h B^h_{z_1;ij} \ge \frac{C(a)}{\varepsilon}B^h_{z_1;ij} \ge \left|L^h z^h_{1;ij}\right|, \qquad (x_i,y_j) \in \Omega^h_{w_2}\backslash\partial\Omega^h,\\
&L^h B^h_{z_1;i0} \ge \frac{C(a)}{\sqrt{\varepsilon}}B^h_{z_1;i0} \ge \left|L^h z^h_{1;i0}\right|, \qquad N/2 < i < N,\\
&B^h_{z_1;N/2j} \ge \left|z^h_{1;N/2j}\right|,\ B^h_{z_1;Nj} \ge \left|z^h_{1;Nj}\right|, \quad 0 \le j \le N/4,\\
&B^h_{z_1;iN/4} \ge \left|z^h_{1;iN/4}\right|, \qquad\qquad\qquad N/2 \le i \le N.
\end{aligned}$$

Hence, by virtue of comparison principle we have

$$\left|z^h_{1;ij}\right| \le B^h_{z_1;ij} \le CN^{-2}\ln^2 N, \quad (x_i,y_j) \in \Omega^h_{w_2}. \tag{31}$$

In order to estimate the second term $z^h_{2;ij}$, which contains a corner singularity, by analogy with a characteristic layer we shall choose the following barrier function

$$B_{z_2}(x,y) = C(N^{-2}\ln^2 N)B_{w_2}(x,y), \qquad (x,y) \in \Omega^h_{w_2},$$

where $B_{w_2}(x,y) = 4\ln(c\sigma_y/r_2^{'}) + \left(-\varphi^{'2} + 4\varphi^{'} + \pi^2/4 - 2\pi + 1\right)$, $r_2^{'} = \sqrt{x^{'2}+y^{'2}}$, $\varphi^{'} = \arctan(y^{'}/x^{'})$, $x^{'} = 1-x$, $y^{'} = y + \tilde{b}\mathfrak{h}_2$, $\tilde{b} = const > 1$.

Following the same reasoning as in the case of the characteristic layer and taking into account the following inequalities $r^{'}_{2;ij} \ge \max\{c\mathfrak{h}_1; c\mathfrak{h}_2; 2c\mathfrak{h}_1/a\}$, $B_{w_2} > 1$ in domain $\Omega^h_{w_2}\backslash\partial\Omega^h_D$, which hold true in the case of our choice of $\tilde{b}$, we can obtain the estimate

$$L^h B^h_{w_2;ij} \ge C\left(1/r_{2;ij}\right), \qquad \Omega^h_{w_2}\backslash\partial\Omega^h_D. \tag{32}$$

Then, in virtue of estimates (27), (30), (32) and of the choice of the barrier, also in virtue of application of comparison principle, the following estimate is valid

$$\left|z^h_{2;ij}\right| \le C(N^{-2}\ln^2 N)B^h_{w_2;ij}, \qquad (x_i,y_j) \in \Omega^h_{w_2}. \tag{33}$$

Since in $\Omega^h_{w_2}$ the inequalities $c\mathfrak{h}_2 \le r^{'}_{2;ij} \le c\sigma_y$ hold true, then the estimate $B_{w_2} \le C\ln N$ is valid. Substituting this estimate into (33), we obtain

$$\left|z^h_{2;ij}\right| \le CN^{-2}\ln^3 N, \qquad (x_i,y_j) \in \Omega^h_{w_2}. \tag{34}$$

Combining (31) and (34), in $\Omega^h_{w_2}$ we obtain the final error estimate. $\square$

3.3 The Final Results

Thus we have all the necessary information to obtain the rate estimate for the uniform in ε convergence of the solution to scheme (6)–(9) to the exact solution.

Theorem 3. *Let $u(x_i, y_j)$ be the solution to problem (1)–(4), and let u_{ij}^h be the solution to difference problem (6)–(9) at $\varepsilon < a\mathfrak{H}_1/2$. Then, for $N > N_0$, where N_0 is a positive integer, which does not depend on ε, the following estimates are valid*

$$\left|u(x_i, y_j) - u_{ij}^h\right| \leq C \begin{cases} N^{-3/2}\ln^2 N, & \Omega_{w_1}^h \cup \Omega_{w_4}^h, \\ N^{-2}\ln^3 N, & N/2 < i \leq N,\ N/4 \leq j \leq 3N/4, \\ N^{-2}, & 0 \leq i \leq N/2,\ N/4 \leq j \leq 3N/4. \end{cases}$$

Proof. The proof follows from (10), (18)–(20), (23) and (28). □

Remark 1. The case $\varepsilon \geqslant a\mathfrak{H}_1/2$ is not investigated here in details. However we should note, that in this case the solution to problem (6)–(9) is also uniformly convergent, but this time the rate is $O(N^{-2}\ln^3 N)$. The general proof scheme for this fact remains the same as in the case of small values of ε. Only S_{ij}^h in Ω^h and $w_{1;ij}^h$, $w_{2;ij}^h$ require some additional investigation ($w_{4;ij}^h$ and $w_{3;ij}^h$ are examined by analogy) in corresponding boundary layer domains.

In conclusion we shall note that numerical calculations were performed, which corroborate the theoretical results.

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