Numerical and Analytical Modeling of the Stability of the Cylindrical Shell under the Axial Compression with the Use of the Non-classical Theories of Shells

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Abstract. The problem of the buckling of a transveral-isotropic cylindrical shell under axial compression by means of new non-classical shell theories is studied. The local approach is used to solve the systems of differential equations. According to this approach the buckling deflection is sought in the form of a doubly periodic function of curvilinear coordinates. The well-known solutions obtained by classical shell theories are compared with the results of non-classical shell theories. For the non-classical theories of anisotropic shell of moderate thickness the buckling equations are constructed by the linearization of nonlinear equilibrium equations. Analytical and numerical results obtained with the use of 3D theory by the FEM code ANSYS 13 are also compared.

Keywords: Cylindrical Shell, Buckling, Non-Classical Theories of Shells, Numerical and Analytical Modeling.

1 Introduction

In this paper the problem of the buckling of the transversal-isotropic cylindrical shell under the axial compression by means of different nonclassical shell theories is studied. The following non-classical theories are considered: Ambartsumian (AMB) [1] theory of anisotropic shells, Paliy-Spiro (PS) theory of moderate-thickness shells and Rodionova-Titaev-Chernykh (RTCH) [2] iteration theory. The developed buckling equations for the shell theories of PS and RTCH are constructed by linearization of nonlinear equilibrium equations. The comparison of new solutions obtained by non-classical shell theories with well-known results of classical theories - Kirchhoff-Love (KL) and Timoshenko-Reissner (TR) [3] is done. In conclusion the comparison of the analytical results of shell theories with numerical results of three–dimensional theory by the FEM code Ansys 13 is given. The main focus is on the case of small cross-section shear modulus. Also we study the influence of relative thickness and length of the shell on the value of critical load. Let us denote a polar angle by α, and the length coordinate by β.

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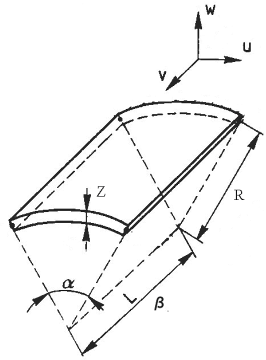

Fig. 1. An element of circular cylindrical shell

The radius of middle surface of the shell is R, the thickness is h, it's length is L , Young's modulus is E, Poisson ratio is ν and tangential shear modulus is G'. Lame coefficient and curvature coefficient which determine the geometry of cylindrical shell: $A_1 = R$, $A_2 = 1$, $k_1 = 1/R$, $k_2 = 0$.

We consider buckling equations of the shell which are constructed by the linearization of non-linear equilibrium equations. This method is very convenient in estimating of upper critical loading. It is enough to define the condition under which generalized stiffness of the construction is equal to zero. Using the method of linearization the solution of the problem is sought by summing of consequently calculated parameters of strain-stress state of the construction while loads are gradually increased. Thus at the each stage of the loading the linear shell problem is being solved.

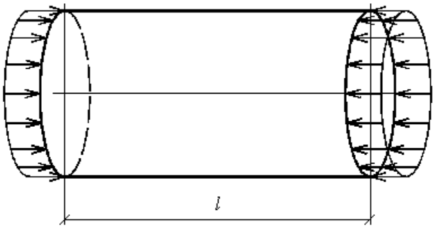

Fig. 2. The load applied to cylindrical shell

General equations are written down for increments in the components of inner forces, displacements and deformation parameters at this stage of loading. The components of the loading include parameters which describe the stress-strain state of the shell at the previous stage. If the change of the components is known and fixed, it is connected with the change of one scalar parameter. The initial state will be an implicit function of this parameter and there appears the eigenvalue problem.

In the given problem we use the classical hypothesis [3] holding that the basic stress-strain state of the shell before the loss of stability is membrane. Then one can take the well-known solution of the membrane shell theory as some function defining the distribution of force in a membrane shell:

$$Z = -{T_2}^0 \partial_{\beta,\beta} w \tag{1}$$

As a result, in this problem ${T_2}^0$ magnitude becomes the only one indefinite scalar parameter the eigenvalue of which should be found. In solving the systems of differential equations a local approach is used [4], according to which the bucking deflection is sought in the form of a doubly periodic function of curvilinear coordinates. The non-zero system solution is sought in the form of:

$$\begin{array}{ll} w(\alpha,\beta) = w^0 \cos(n\alpha)\sin(m\beta), & \Phi(\alpha,\beta) = \Phi^0 \cos(n\alpha)\sin(m\beta) \\ u(\alpha,\beta) = u^0 \sin(n\alpha)\sin(m\beta), & \gamma_1(\alpha,\beta) = {\gamma_1}^0 \sin(n\alpha)\sin(m\beta) \\ v(\alpha,\beta) = v^0 \cos(n\alpha)\cos(m\beta), & \gamma_2(\alpha,\beta) = {\gamma_2}^0 \cos(n\alpha)\cos(m\beta) \end{array} \tag{2}$$

where u, v, w — displacement vector components of a point of mid-surface of shell, γ_1 and γ_2 —angles of normal turn in the planes $(\alpha, z), (\beta, z)$ respectively, $\Phi(\alpha, \beta)$ — force function.

2 Kirchhoff-Love Model Solution

Let us consider the well-known solution which is obtained by the classical theory of shells which is based on the following hypothesis:

1) the straight lines normal to the mid-surface remain straight and normal to the mid-surface after deformation;

2) the thickness of the shell does not change during a deformation.

Two-dimensional equation system of the shallow shell theory [3] has a form:

$$-D\Delta\Delta w + T_2^0 \partial_{\beta,\beta} w + \frac{1}{R}\partial_{\beta,\beta}\Phi = 0, \quad \frac{1}{Eh}\Delta\Delta\Phi + \frac{1}{R}\partial_{\beta,\beta} w = 0 \tag{3}$$

where Δ - Laplace operator; $D = Eh^3/(12(1-\nu^2))$ - cylindrical stiffness; T_2^0 - desired axial force.

If we substitute the expression (2) into this system (3) for force T_2^0, we will obtain:

$$-T_2^0 = f(m,n) = \frac{D(n^2+m^2)^2}{R^2 m^2} + \frac{Ehm^2}{(n^2+m^2)^2} \tag{4}$$

The critical load value is obtained as a result of minimization by the wave parameters m and n of the function $f(m, n)$.

$$T_2^0 = \sigma_0 h, \quad \sigma_0 = -\frac{E}{\sqrt{3(1-\nu^2)}}\frac{h}{R} = \sigma_{cl} \tag{5}$$

3 Ambartsumian Model Solution

The solution (5) being constructed by KL model does not allow taking into account the effect of stiffness on cross-section shear. Let us consider Ambartsumian [1] theory which is based on the following hypothesis:

1) displacement which is normal to the shell mid-surface does not depend on the normal coordinate;

2) shear stresses or the corresponding deformations change according to a quadratic law with respect to the plane thickness;

Let us write down the equations of Ambartsumian model which takes the influence of cross-section shear into account for a transversally-isotropic shell as:

$$-\Delta\Delta u - \frac{\nu}{R}\frac{\partial^3 w}{\partial\beta^3} + \frac{1}{R}\frac{\partial^3 w}{\partial\alpha^2\partial\beta} = 0 \quad -\Delta\Delta v - \frac{2+\nu}{R}\frac{\partial^3 w}{\partial\alpha\partial\beta^2} - \frac{1}{R}\frac{\partial^3 w}{\partial\alpha^3} = 0 \quad (6)$$

$$-D\Delta^4 w + \frac{Eh}{R^2}(1-h_z\Delta)\frac{\partial^4 w}{\partial\beta^4} - T_2^0(1-h_z\Delta)\Delta^2\frac{\partial^2 w}{\partial\beta^2} = 0 \quad h_z = \frac{Eh^2}{10(1-\nu^2)G'}$$

The simplified system of differential equations of the shell buckling which is used in Ambartsumian theory was obtained basing on the equations of the shallow shell theory.

Using the local approach (2) for solving this system for T_2^0 we obtain:

$$-T_2^0 = f(m,n) = \frac{D(n^2+m^2)^2}{R^2m^2(1+h_z(n^2+m^2))} + \frac{Ehm^2}{(n^2+m^2)^2} \quad (7)$$

The obtained value for critical load —

$$\sigma_0 = -\frac{E}{\sqrt{3(1-\nu^2)}}\frac{h}{R} + \frac{E^2}{10G'(1-\nu^2)}\left(\frac{h}{R}\right)^2 = \sigma_{cl}\left(1 - \frac{\sqrt{3}}{10\sqrt{(1-\nu^2)}}\frac{E}{G'}\frac{h}{R}\right) \quad (8)$$

agrees completely with the one being obtained by the theory of Timoshenko-Reissner [3]. It is known [4] that for an isotropic shells and plates the TR theory being asymptotically inconsistent refines the deflection of a body. But for bodies, which are made of transversal isotropic material "in case when material stiffness in tangential directions is much larger than its stiffness in the transversal direction" the TR theory makes the KL theory more precise and gives next asymptotical approximation of the three-dimensional theory. The bodies "with moderately small transverse shear stiffness" are thin-walled bodies for which small parameter g = G'/E (where E is the Young's modulus in the tangential direction, G' is the shear modulus for plane normal to the surface of isotropy) satisfies expression $(h/R)^2 \ll g \ll 1$.

4 Paliy-Spiro Model Solution

The situation is quite different when the buckling problems are considered with the use of improved theories. In this case the old representations are not always acceptable as there appear problems related to taking into account the change of the length and the turn of the normal to mid-surface.

The Paliy-Spiro [2] theory of moderate-thickness shells accepts the following hypothesis:

1) straight fibers of the shell which are perpendicular to its mid-surface before deformation remain also straight after deformation;

2) cosine of the slope angle of these fibers to the mid-surface of the deformed shell is equal to the averaged angle of transverse shear.

The mathematical formulation of the accepted hypotheses gives following equations:

$$\begin{aligned}
&u_1 = u + \phi \cdot z, \qquad u_2 = v + \psi \cdot z,\\
&u_3 = w + F(\alpha, \beta, z),\\
&\phi = \gamma_1 + \phi_0, \qquad \psi = \gamma_2 + \psi_0,\\
&\phi_0 = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha} + k_1 u, \; \psi_0 = -\frac{1}{A_2}\frac{\partial w}{\partial \alpha} + k_2 v,
\end{aligned} \tag{9}$$

where ϕ and ψ are the angles of rotation of the normal in the planes (α, z) and (β, z); ϕ_0, ψ_0,γ_1 γ_2 — the angles of rotation of the normal to the medial surface and the angles of displacement in the same planes. The function $F(\alpha, \beta, z)$ characterizes the variation of the length of normal to the middle surface.

The shell deformations $\varepsilon_1, \varepsilon_2, \varepsilon_{13}$, η_1, η_2 are expressed by displacement components with the following formulas:

$$\varepsilon_1 = \frac{\partial_\alpha u}{R} + \frac{w}{R}, \quad \varepsilon_2 = \partial_\beta v, \quad \eta_1 = \frac{\partial_\alpha \phi}{R} \tag{10}$$

$$\eta_2 = \partial_\beta \psi, \quad \omega = \frac{\partial_\alpha v}{R} + \partial_\beta u, \quad \tau = \frac{\partial_\alpha \psi}{R} + \partial_\beta \phi$$

Substituting the mentioned dependencies (10) into the constitutive relations (11), one can obtain the equations of relation between the components of displacement and forces and moments.

$$\epsilon_1 = \frac{T_1 - \nu T_2}{Eh}, \quad \epsilon_2 = \frac{T_2 - \nu T_1}{Eh}, \quad \omega = \frac{S}{G'h}, \quad \eta_1 = \frac{12(M_1 - \nu M_2)}{Eh^3}, \tag{11}$$

$$\eta_2 = \frac{12(M_2 - \nu M_1)}{Eh^3}, \quad \tau = \frac{12H}{G'h^3}, \quad \gamma_1 = \frac{N_1}{G'h}, \quad \gamma_2 = \frac{N_2 + {T_2}^0 * \psi_0}{G'h}$$

As one can see (11), PS theory includes the characteristical parameter ${T_2}^0$ in the equation of relation between normal slope γ_2 and shear force N_2.

The obtained equations of relation between components of displacement and forces and moments are substituted into equilibrium equations:

$$\frac{\partial_\alpha T_1}{R} + \partial_\beta S + \frac{N_1}{R} = 0, \quad \frac{\partial_\alpha S}{R} + \partial_\beta T_2 = 0, \quad \frac{\partial_\alpha N_1}{R} + \partial_\beta N_2 - \frac{T_1}{R} = 0 \tag{12}$$

$$\frac{\partial_\alpha M_1}{R} + \partial_\beta H - N_1 = 0, \quad \frac{\partial_\alpha H}{R} + \partial_\beta M_2 - N_2 - {T_2}^0 * \psi_0 = 0.$$

Resolution matrix in this case will be of [5*5] dimension. Nevertheless the value of critical load obtained is similar to the value of Ambartsumian theory (8). This is the factor of the second coefficient of asymptotical expansion by small parameter h/R that makes the results different. One can see that this numerical factor reduces the influence of transversal shear.

$$\sigma_0 = -\frac{E}{\sqrt{3(1-\nu^2)}}\frac{h}{R} + \frac{E^2}{12G'(1-\nu^2)}\left(\frac{h}{R}\right)^2 = \sigma_{cl}\left(1 - \frac{\sqrt{3}}{12\sqrt{(1-\nu^2)}}\frac{E}{G'}\frac{h}{R}\right) \tag{13}$$

5 Rodionova-Titaev-Chernykh Model Solution

The use of RTCH shell theory yields more interesting results [2]. This is a linear theory of non-homogeneous anisotropic shells which takes into account low transversal shear compliance and deformation towards the normal to the middle surface. It also takes into account transversal normal strains and supposes non-linear distribution of displacement vector component over shell thickness.

1) transverse tangential and normal stresses are distributed along the shell thickness according to quadratic and cubic laws respectively;

2) tangential and normal components of the displacement vector are distributed along the shell thickness according to quadratic and cubic laws;

The functions which describe shell displacement $u_1(\alpha,\beta,z)$,$u_2(\alpha,\beta,z)$, $u_3(\alpha,\beta,z)$ according to RTCH theory are supposed to be sought in the form of Legendre polynomial series P_0, P_1, P_2, P_3 from normal coordinate $z \in \left[-\frac{h}{2}, \frac{h}{2}\right]$.

$$\begin{aligned}
u_1(\alpha,z) &= u(\alpha,\beta)^* P_0(z) + \gamma_1(\alpha,\beta)^* P_1(z) + \theta_1(\alpha,\beta)^* P_2(z) + \varphi_1(\alpha,\beta)^* P_3(z),\\
u_2(\alpha,z) &= v(\alpha,\beta)^* P_0(z) + \gamma_2(\alpha,\beta)^* P_1(z) + \theta_2(\alpha,\beta)^* P_2(z) + \varphi_2(\alpha,\beta)^* P_3(z),\\
u_3(\alpha,z) &= w(\alpha,\beta)^* P_0(z) + \gamma_3(\alpha,\beta)^* P_1(z) + \theta_3(\alpha,\beta)^* P_2(z)
\end{aligned} \tag{14}$$

$$P_0(z) = 1, \quad P_1(z) = \frac{2z}{h}, \quad P_2(z) = \frac{6z^2}{h^2} - \frac{1}{2}, \quad P_3(z) = \frac{20z^3}{h^3} - \frac{3z}{h} \tag{15}$$

where γ_3 and θ_3 characterize normal length variation to this surface, magnitudes θ_1 and φ_1, describe normal curvature in the plane (α,z) of a fiber, θ_2 φ_2, describe normal curvature in the plane (β,z)which before the deformation were perpendicular to the medial surface of the shell.

The shell deformations $\varepsilon_1, \varepsilon_2, \varepsilon_{13}$, η_1, η_2 are expressed by displacement components:

$$\epsilon_1 = \frac{\partial_\alpha u}{R} + \frac{w}{R}, \quad \epsilon_2 = \partial_\beta v, \quad \eta_1 = \frac{\partial_\alpha \gamma_1}{R} + \frac{\gamma_3}{R}, \quad \eta_2 = \partial_\beta \gamma_2, \quad \vartheta_0 = \partial_{\alpha,\alpha} w \tag{16}$$

$$\omega = \frac{\partial_\alpha v}{R} + \partial_\beta u, \quad \tau = \frac{\partial_\alpha \gamma_2}{R} + \partial_\beta \gamma_1, \quad \epsilon_{13} = \frac{\partial_\alpha w}{R} - \frac{u}{R} + \frac{2\gamma_1}{h}, \quad \epsilon_{23} = \partial_\beta w + \frac{2\gamma_2}{h}$$

The characteristical parameter ${T_2}^0$ is also included into the constitutive relations.

$$T_1 = \frac{Eh}{1-\nu^2}(\epsilon_1 + \nu\epsilon_2), \quad T_2 = \frac{Eh}{1-\nu^2}(\nu\epsilon_1 + \epsilon_2), \quad M_1 = \frac{Eh^2}{6\,(1-\nu^2)}(\eta_1 + \nu\eta_2),$$

$$M_2 = \frac{Eh^2}{6\,(1-\nu^2)}(\nu\eta_1 + \eta_2), \quad S = G'h\omega, \quad H = G'\frac{h^2}{6}\tau, \quad N_1 = \frac{5hG'}{6}\epsilon_{13}, \quad (17)$$

$$N_2 = \frac{5hG'}{6}\epsilon_{23} + \frac{{T_2}^0\vartheta_0}{6}, \quad \gamma_3 = -\frac{\nu}{1-\nu}\frac{h}{2}(\epsilon_1 + \epsilon_2),$$

The equations of relation forces and moments and displacement components were substituted into equilibrium equations (12).

In minimizing the determinant the following solution was obtained:

$$\sigma_0 = -\frac{E}{\sqrt{3(1-\nu^2)}}\sqrt{1 - \frac{E^2}{60G'^2(1-\nu^2)}\left(\frac{h}{R}\right)^2}\left(\frac{h}{R}\right) + \frac{E^2}{15G'(1-\nu^2)}\left(\frac{h}{R}\right)^2 \quad (18)$$

This expansion into a series by a small parameter yields succeedent terms of expansion:

$$\sigma_0 = -\frac{E}{\sqrt{3(1-\nu^2)}}\frac{h}{R} + \frac{E^2}{15G'(1-\nu^2)}\left(\frac{h}{R}\right)^2 + O\left[\frac{h}{R}\right]^3 \quad (19)$$

6 The Comparison with Numerical Results

Let us compare the results which are obtained with the use of developed analytical formulae of shell theory and numerical results for three-dimensional theory. Unfortunately, the formulae for the critical load of the shell theory do not take into account the tube length. Being applied to the buckling problems the obtained solutions well agree with medium-length shells. For example, a three-dimensional model of steel tube under the influence of axial compression was studied under the following parameters $h/R = 2/15, \nu = 0.3$. The cross-section shear modulus is equal to $G' = E/(2(1+\nu))$. For modeling the three-dimensional problem in package Ansys 13 the finite element Solid186 was used. This is a higher order 3-D 20-node solid element that exhibits quadratic displacement behavior. The element is defined by 20 nodes having three degrees of freedom per node: translations in the nodal x, y, and z directions. The element supports plasticity, hyperelasticity, creep, stress stiffening, large deflection, and large strain capabilities. It also has mixed formulation capability for simulating deformations of nearly incompressible elastoplastic materials, and fully incompressible hyperelastic materials. [5] During mesh construction the tube thickness was split for five

Table 1. The comparison of critical load values

h/R	0.025	0.05	0.1	0.133	0.162
KL	0.01532	0.03103	0.0637	0.08069	0.09814
Amb	0.01513	0.03028	0.06054	0.07561	0.09063
PS	0.01516	0.03041	0.06107	0.07646	0.09188
RTCH	0.01519	0.03053	0.06155	0.07722	0.09297
Ansys	0.01445	0.02875	0.055	0.0595	0.0635

elements. Thus the value of the critical load according to the three-dimensional theory is smaller than shell theories. The value of a critical load for the considered tubes with length ranged from 1.5 to 3 diameters of mid-surface does not change considerably. Table 1 shows dimensionless values of critical load σ_0/E for different ratios of tube thickness to the radius of its middle surface.

7 Conclusions

As one can see in the table, as shell thickness increases, the values of critical load obtained by shell theories are not consistent with the results of three-dimensional theory. It can be noticed that error increases as the thickness grows. It is possible to claim that in spite of improvements of non-classical hypotheses reliable results can be obtained only for thin shells. However, as it was shown in [2] similar hypotheses suit well for defining stress-strain state of a shell.

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