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# Transversals, Topology and Colorful Geometric Results

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## 1. Introduction

Suppose we have two convex sets $A$ and $B$ in euclidean d-space $\mathbb{R}^d$. Assume the only information we have about $A$ and $B$ comes from the space of their transversal lines. Can we determine whether $A$ and $B$ have a point in common? For example, suppose the space of their transversal lines has an essential curve; that is, suppose there is a line that moves continuously in $\mathbb{R}^d$, always remaining transversal to $A$ and $B$, and comes back to itself with the opposite orientation. If this is so, then $A$ must intersect $B$, otherwise there would be a hyperplane $H$ separating $A$ from $B$; but it turns out that our moving line becomes parallel to $H$ at some point on its trip, which is a contradiction to the fact that the moving line remains transversal to the two sets. If we have three convex sets $A$, $B$ and $C$, for example, in $\mathbb{R}^3$, then our essential curve does not give us sufficient topological information. In this case, to detect whether $A \cap B \cap C \neq \phi$, we need a 2-dimensional cycle. So, for example, if we can continuously choose a transversal line parallel to every direction, then there must be a point in $A \cap B \cap C$, otherwise if not, the same is true for $\pi(A) \cap \pi(B) \cap \pi(C)$, for a suitable orthogonal projection $\pi : \mathbb{R}^3 \to H$ where $H$ is a plane through the origin (see [4, Lemma 3.1]). Hence clearly there is no transversal line orthogonal to $H$.

Suppose now we have three convex sets $A$, $B$ and $C$ in euclidean 3-space $\mathbb{R}^3$. Assume the only information we have about $A$, $B$ and $C$ comes from the space of their transversal planes. Can we determine whether $A$,

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$B$ and $C$ have a transversal line? For example, suppose the space of their transversal planes has an essential curve; that is, suppose there is a plane that moves continuously in $\mathbb{R}^3$, always remaining transversal to $A$, $B$ and $C$, and comes back to itself with the opposite orientation. If this is so, then there must be a transversal line to $\{A, B, C\}$. This time the proof is slightly more complicated from the topological point of view. Let me present it here.

Suppose there is no transversal line to $\{A, B, C\}$. Denote by $T_2$ the space of transversal planes to $\{A, B, C\}$. Consider the continuous map $\psi : A \times B \times C \to T_2$ given by $\psi(a, b, c)$, the unique plane containing $\{a, b, c\}$. The continuous map $\psi$ is well defined precisely because there is no transversal line to $\{A, B, C\}$. Furthermore, if $H \in T_2$, then $\psi^{-1}(H) = (A \cap H) \times (B \cap H) \times (C \cap H)$, which is contractible by the convexity of the two sets. The fact that the fibers of $\psi$ are contractible implies that $\psi$ is a homotopy equivalence. This implies that $T_2$ is contractible, contradicting the hypothesis that there is an essential curve in $T_2$.

We claim that for a sufficiently small family of convex sets, the topology of its transversals provide enough information to derive geometric information. To be more precise, let us state the following definition.

Let $F$ be a family of compact, convex sets. We say that $\mathcal{F}$ has a *topological $\rho$-transversal of index* $(m, k)$, $\rho < m$, $0 < k \leq d - m$, if there are, homologically, as many transversal $m$-planes to $\mathcal{F}$ as $m$-planes through a fixed $\rho$-plane in $\mathbb{R}^{m+k}$. Clearly, if $\mathcal{F}$ has a $\rho$-transversal plane, then $\mathcal{F}$ has a topological $\rho$-transversal of index $(m, k)$, for $\rho < m$ and $k \leq d - m$. The converse is not true. It is easy to give examples of families with a topological $\rho$-transversal but without a $\rho$-transversal plane. We conjecture that for a family $\mathcal{F}$ of $k + \rho + 1$ compact, convex sets in euclidean $d$-space $\mathbb{R}^d$, there is a $\rho$-transversal plane if and only if there is a topological $\rho$-transversal of index $(m, k)$. A good reference for the algebraic topology needed in this paper is [8], and [7] for the geometric transversal theory.

The purpose of this paper is to use the structure of the topology of the space of transversals to obtain geometric results in the spirit of the colourful theorems of Lovász and Bárány.

## 2. The Structure of the Space of Transversals

The purpose of this section is to state several results about the structure of the topology of the space of transversals to a family of convex sets. For the proofs see [1], [4], [5] and [9].

Let $\mathcal{F}$ be a family of compact, convex sets in $\mathbb{R}^d$. By $M(d,m)$ we denote the space of $m$-planes in $\mathbb{R}^d$. It can be considered as an open subset of $G(d+1, m+1)$ and retractible to the classic Grassmanian space, $G(d,m)$, of $m$-dimensional linear subspaces of $\mathbb{R}^d$. For $0 < m < d$, we denote by $\mathcal{T}_m(\mathcal{F})$ the subspace of $M(d,m) \subset G(d+1, m+1)$ consisting of all $m$-planes transversal to $\mathcal{F}$.

We say that $\mathcal{F}$ has a *topological $\rho$-transversal of index* $(m,k)$, $\rho < m$, $0 < k \leq d-m$, if there are homologically as many transversal $m$-planes to $\mathcal{F}$ as $m$-planes through a fixed $\rho$-plane in $R^{m+k}$.

More precisely, for $\rho < m$, $0 < k \leq d-m$, the family $\mathcal{F}$ has a topological $\rho$-transversal of index $(m,k)$ if

$$i^*([0,\ldots,0,k,\ldots,k]) \in H^{(m-\rho)k}\big(\mathcal{T}_m(\mathcal{F}), \mathbb{Z}_2\big) \quad \text{is not zero,}$$

where $i^* : H^{(m-\rho)k}\big(G(d+1,m+1),\mathbb{Z}_2\big) \to H^{(m-\rho)k}\big(\mathcal{T}_m(\mathcal{F}),\mathbb{Z}_2\big)$ is the cohomology homomorphism induced by the inclusion $\mathcal{T}_m(\mathcal{F}) \subset M(d,m) \subset G(d+1,m+1)$, and

$$[0,\ldots,0,k,\ldots,k] \in H^{(m-\rho)k}\big(G(d+1,m+1),\mathbb{Z}_2\big)$$

is the Schubert-cocycle, in which the last symbol starts with $\rho+1$ zeros (see [6] for the definition of Schubert cocycle).

Clearly, if $\mathcal{F}$ has a $\rho$-transversal plane, then $\mathcal{F}$ has a topological $\rho$-transversal of index $(m,k)$, for $\rho < m$ and $k \leq d-m$. The converse is not true. It is easy to give examples of families with a topological $\rho$-transversal but without a $\rho$-transversal plane. We conjecture that for a family $\mathcal{F}$ of $k+\rho+1$ compact, convex sets in euclidean $d$-space $\mathbb{R}^d$, there is a $\rho$-transversal plane if and only if there is a topological $\rho$-transversal of index $(m,k)$.

The proof of the following theorem follows the ideas of the proof, given in the introduction, that the space of transversal planes to three convex sets, without transversal lines, in 3-space is contractible. See the proof of Theorem 3.1 in [1].

**Theorem 2.1.** *Let $0 \leq \rho < m \leq d-1$. Let $\mathcal{F} = \{A_0, \ldots, A_{\rho+1}\}$ be a family of convex sets in $\mathbb{R}^d$ and let $\alpha_i \in A_i$, $i = 0, \ldots, \rho+1$. Suppose there is no $\rho$-plane transversal to $\mathcal{F}$. Then the inclusion*

$$\mathcal{T}_m(\{\alpha_0, \ldots, \alpha_{\rho+1}\}) \subset \mathcal{T}_m(\{A_0, \ldots, A_{\rho+1}\})$$

*is a homotopy equivalence.*

*In particular, $T_m(F)$ has the homotopy type of $G(d-\rho-1, m-\rho-1)$.*

As a corollary, we have the following theorem which proves our main conjecture when $k = 1$. This theorem will allow us to transform topological information into geometric information.

**Theorem 2.2.** *Let $0 \leq \rho < m$, and let $\mathcal{F}$ be a family of $\rho+2$ compact convex sets in $\mathbb{R}^d$. Then there is a $\rho$-plane transversal to $\mathcal{F}$ if and only if there is a topological $\rho$-transversal plane of index $(m, 1)$.*

*That is, there is a $\rho$-plane transversal to $\mathcal{F}$ if*

$$[0, \ldots, 0, 1, \ldots, 1] \text{ is not zero in } \mathcal{T}_m(\mathcal{F}),$$

*where $[0, \ldots, 0, 1, \ldots, 1] \in H^{(m-\rho)}\big(G(m+1, d+1), \mathbb{Z}_2\big)$ is the $(m-\rho)$-Stiefel–Whitney characteristic class, in which the last symbol starts with $\rho+1$ zeros.*

All results in this paper can be stated in a more general setting, but to simplify the topological technicalities and to clarify the ideas, we will prove and state them only for dimensions 3 and 4. So, let us summarize in the following proposition the topology we will need in the next section.

**Proposition 2.1.** *Let $F$ be a family of convex sets in $\mathbb{R}^4$. Let $D(F) \subset \mathbb{RP}^3$ be the set of directions in $\mathbb{R}^4$ orthogonal to a transversal hyperplane of $F$ and let $d(F) \subset G(4,2)$ be the set of directions in $\mathbb{R}^4$ orthogonal to a transversal plane of $F$. Then*

a) *if the homomorphism induced by the inclusion $H^1\big(\mathbb{RP}^3, \mathbb{Z}_2\big) \to H^1\big(D(F), \mathbb{Z}_2\big)$ is not zero, there is a transversal plane to every quadruple of the convex sets of $F$,*

b) *if the homomorphism induced by the inclusion $H^2\big(\mathbb{RP}^3, \mathbb{Z}_2\big) \to H^2\big(D(F), \mathbb{Z}_2\big)$ is not zero, there is a transversal line to every triple of the convex sets of $F$,*

c) *if the homomorphism induced by the inclusion* $H^1\big(G(4,2),\mathbb{Z}_2\big) = \mathbb{Z}_2 \to H^1\big(d(F),\mathbb{Z}_2\big)$ *is not zero, there is a transversal line to every triple of the convex sets of* $F$.

**Proof.** Note that the classical retraction $M(4,3) \to \mathbb{R}P^3$ is a homotopy equivalence. Furthermore its restriction $T_3(F) \to D(F)$ is a homotopy equivalence because the fibers are contractible. So, if the homomorphism induced by the inclusion $H^1\big(\mathbb{R}P^3,\mathbb{Z}_2\big) \to H^1\big(D(F),\, \mathbb{Z}_2\big)$ is not zero, then the generator of $G(5,4)$ is not zero in $T_3(F)$ and hence by Theorem 2.2 there is a plane transversal to every quadruple of convex sets of $F$. The proofs of b) and c) are essentially the same. ■

## 3. The Colorful Geometric Results

The purpose of this section is to use the topological results developed in the previous section to obtain geometric results in the spirit of the colorful theorems of Lovász and Bárány [3].

We state the colorful Helly Theorem.

**Theorem 3.1.** *Let* $F$ *be a family of convex sets in* $\mathbb{R}^d$ *painted with* $d+1$ *colors. Suppose that every heterochromatic* $d+1$*-tuple of* $F$ *is intersecting. Then there is a color with the property that the family of all convex sets of this color is intersecting.*

In particular, if we have a collection of red and blue intervals in the line and every red interval intersects every blue internal, then either there is a point in the intersection of all red intervals or there is a point in the intersection of all blue intervals.

The colorful Helly Theorem has the following geometric interpretation: any linear embedding of a combinatorial $d$-cube in $\mathbb{R}^d$ has, in every direction, a transversal line to two opposite faces.

Let us consider the configuration of lines in the plane that consists of nine points and six lines, in which the first three red lines, $\ell_1$, $\ell_2$, $\ell_3$ are parallel and the next three blue lines $L_1$, $L_2$, $L_3$ are parallel and orthogonal to the red ones. So every line has exactly three points, and the intersection of a red and a blue line consists exactly of one point. Let us denote by $G^3$ the 2-dimensional simplicial complex describing this configuration, in which we have three red triangles corresponding to the red lines and three blue triangles corresponding to the blue lines.

**Theorem 3.2.** *In any linear embedding of $G^3$ in euclidean 3-space $\mathbb{R}^3$, there is either a transversal line to the red triangles or a transversal line to the blue triangles.*

**Proof.** The ingredients of the proof are: i) the fact that if $\mathbb{R}P^2$, the projective plane, is the union of two closed ANR sets $R$ and $B$, then either $R$ contains an essential cycle or $B$ contains an essential cycle, and ii) the colorful Helly Theorem in the line.

Let $\mathcal{R} \subset \mathbb{R}P^2$ be the collection of directions orthogonal to transversal planes to the red triangles and let $\mathcal{B} \subset \mathbb{R}P^2$ be the collection of directions orthogonal to transversal planes to the blue triangles. First note that $\mathcal{R} \cup \mathcal{B} = \mathbb{R}P^2$, because if $L$ is any line through the origin, we may project the three red triangles and the three blue triangles orthogonally onto $L$. Thus we have three red intervals and three blue intervals in $L$ with the property that every red interval intersects a blue interval, but this means that either there is a point in the intersection of all red intervals or there is a point in the intersection of all blue intervals. Therefore there is, orthogonally to $L$, either a plane transversal to the three red triangles or a plane transversal to the three blue triangles. Since $\mathcal{R} \cup \mathcal{B} = \mathbb{R}P^2$, either $\mathcal{R}$ contains an essential cycle or $\mathcal{B}$ contains an essential cycle. This immediately implies that there is an essential cycle of planes (see the introduction) transversal to the red triangles or an essential cycle of planes transversal to the blue triangles. Thus there is either a transversal line to the red triangles or a transversal line to the blue triangles. ∎

We have essentially proved that if we have three red convex sets and three blue convex sets in 3-space and every red set intersects every blue set, then there is either a transversal line to the red sets or a transversal line to the blue sets. Now we want to prove a similar theorem but this time using more than two colors. For this purpose we need the following proposition, which essentially claims that $G(4,2)$ cannot be covered by three null homotopic sets.

**Proposition 3.1.** *Let $G(4,2) = A_1 \cup A_2 \cup A_3$ be a closed cover of the 4-dimensional Grassmanian space $G(4,2)$ of planes through the origin in $\mathbb{R}^4$. For some $i \in \{1,2,3\}$, the homomorphism induced by the inclusion $H^*\big(G(4,2),\mathbb{Z}_2\big) \to H^*\big(A_i,\mathbb{Z}_2\big)$ is not zero.*

**Proof.** The strategy is to prove first that there are $\gamma_i \in H^*\big(G(4,2),\mathbb{Z}_2\big)$, $i = 0,1,2$ such that $\gamma_0 * \gamma_1 * \gamma_2 \neq 0$. Recall (see [6] that the product structure in $H^*\big(G(4,2),\mathbb{Z}_2\big)$ can be totally described by the following formula:

$$[\lambda_1,\lambda_2][0,1] = \sum [\xi_1,\xi_2],$$

where the summation extends over all combinations $\xi_1$, $\xi_2$ such that

i) $0 \le \xi_1 \le \xi_2 \le 2$,

ii) $\lambda_1 \le \xi_1 \le \lambda_2$, $\lambda_2 \le \xi_2 \le 2$, and

iii) $\xi_1 + \xi_2 = \lambda_1 + \lambda_2 + 1$.

Let $\gamma_0 = [1,1]$ and $\gamma_1 = [0,1]$. Then $\gamma_0 * \gamma_1 = [1,2]$ and then $\gamma_0 * \gamma_1 * \gamma_1 = [2,2] \neq 0$.

Suppose that the homomorphism induced by the inclusion

$$H^*\big(G(4,2),\mathbb{Z}_2\big) \to H^*\big(A_i,\mathbb{Z}_2\big)$$

is zero, for $i \in \{1,2,3\}$. Hence by exactness

$$H^*\big(G(4,2),A_i;\mathbb{Z}_2\big) \to H^*\big(G(4,2),\mathbb{Z}_2\big)$$

is an epimorphism. We can pull $\gamma_0$, $\gamma_1$, $\gamma_2$ back to $H^*\big(G(4,2),A_i;\mathbb{Z}_2\big)$ and hence pull the product $\gamma_0 * \gamma_1 * \gamma_1$ back to $H^*\big(G(4,2),A_1 \cup A_2 \cup A_3;\mathbb{Z}_2\big) = 0$, which is a contradiction. ■

We are ready for the following theorem:

**Theorem 3.3.** *Suppose we have three red convex sets, three blue convex sets and three green convex sets in $\mathbb{R}^4$ and every heterochromatic triple is intersecting. Then there is one color that has a line transversal to all convex sets of this color.*

**Proof.** The proof is essentially that of the previous theorem but using the colorful Helly Theorem in the plane and Proposition 3.3. Let $\mathcal{R} \subset G(4,2)$ be the set of directions in $G(4,2)$ orthogonal to transversal planes to the red convex sets, let $\mathcal{B} \subset G(4,2)$ be the set of directions orthogonal to transversal planes to the blue convex sets, and finally let $\mathcal{G} \subset G(4,2)$ be the set of directions orthogonal to transversal planes to the green convex sets. First note that $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G} = G(4,2)$, because if $H$ is any plane through the origin, we may project our nine convex sets orthogonally onto $L$. So, by the colorful Helly Theorem 3.2 in the plane, there is a color, say red, such that there is a point in common to the projection of all convex sets of that color. Therefore there is, orthogonally to $H$, a plane transversal to the three red sets. Since $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G} = G(4,2)$, by Proposition 3.3 one of these closed sets, say $\mathcal{B}$ without loss of generality, has the property that the homomorphism induced by the inclusion $H^*\big(G(4,2), \mathbb{Z}_2\big) \to H^1(\mathcal{B}, \mathbb{Z}_2)$ is not zero, but if this is so $H^1\big(G(4,2), \mathbb{Z}_2\big) \to H^1(\mathcal{B}, \mathbb{Z}_2)$ is not zero. By Proposition 2.3c), this implies that there is a transversal line to the three blue convex sets, as required. ■

Now we will use a variant of the colorful Helly Theorem.

**Proposition 3.2.** *Let $F$ be a family of red, blue and green intervals in $\mathbb{R}^1$. Suppose that for every heterochromatic triple, one of the intervals intersects the other two. Then there is a color such that there is a point in common to all intervals of this color.*

**Proof.** If every pair of red and every blue intervals intersects, then by the colorful Helly Theorem in the line, either there is point common to all red intervals or there is a point common to all blue intervals. If not, there is a red interval $I_R \in F$ and a blue interval $I_B \in F$, such that $I_R \cap I_B = \phi$. Therefore every green interval of $F$ intersects both $I_R$ and $I_B$, which implies that there is point in common to all green intervals, as required. ■

This variant of the colorful Helly Theorem and the fact that $\mathbb{RP}^3$ can not be covered by three null homotopic closed sets together give rise to the following theorem:

**Theorem 3.4.** *Suppose we have four red convex sets, four blue convex sets and four green convex sets in $\mathbb{R}^4$ and for every heterochromatic triple one of the sets intersects the other two. Then there is a color such that there is a plane transversal to all convex sets of this color.*

**Proof.** Let $\mathcal{R} \subset \mathbb{R}P^3$ be the collection of directions orthogonal to transversal hyperplanes to the red convex sets, let $\mathcal{B} \subset \mathbb{R}P^3$ be the collection of directions orthogonal to transversal hyperplanes to the blue convex sets, and let $\mathcal{G} \subset \mathbb{R}P^3$ be the collection of directions orthogonal to transversal hyperplanes to the green convex sets. Note that $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G} = \mathbb{R}P^3$, because if $L$ is any line through the origin, we may project our twelve convex sets orthogonally onto $L$, obtaining four red intervals, four blue intervals and four green intervals in $L$ with the property that for every heterochromatic triple, one of the intervals intersects the other two. Then by Proposition 3.5, there is a color such that there is a point common to all intervals of this color. Therefore there is, orthogonally to $L$, a hyperplane transversal to the three convex sets of this color. Since $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G} = \mathbb{RP}^3$, the Lusternik Schnirelmann category $\mathbb{RP}^3$ implies that one of these closed sets, say $\mathcal{B}$ without loss of generality, has the property that the homomorphism induced by the inclusion $H^1(\mathbb{RP}^3, \mathbb{Z}_2) \to H^1(\mathcal{B}, \mathbb{Z}_2)$ is not zero. By Proposition 2.3a), this implies that there is a transversal plane to the blue convex sets as required. ■

It is well known that the projective plane is not the union of two null homotopic closed sets, but can be the union of three null homotopic closed sets. As a consequence, the following topological proposition, whose proof is an interesting application of the Mayer–Vietoris exact sequence in homology, will allow us to obtain two interesting results.

**Proposition 3.3.** *Let $A \cup B \cup C = \mathbb{RP}^2$ be a closed, null homotopic cover of projective 2-space. Then $A \cap B \cap C$ is non-empty. Moreover, $A \cap B \cap C$ has at least four non-empty components.*

**Theorem 3.5.** *Suppose we have three red convex sets, three blue convex sets and three green convex sets in $\mathbb{R}^3$ and every heterochromatic triple is intersecting. Then either there is a color such that there is a line parallel to the $xy$-plane transversal to all convex sets of this color, or else there are three parallel transversal hyperplanes, one for the red sets, one for the blue sets and one for the green sets.*

**Proof.** As before, let $\mathcal{R} \subset \mathbb{R}P^2$ be the collection of directions orthogonal to transversal hyperplanes to the red convex sets, let $\mathcal{B} \subset \mathbb{R}P^3$ be the collection of directions orthogonal to transversal hyperplanes to the blue convex sets and let $\mathcal{G} \subset \mathbb{R}P^3$ be the collection of directions orthogonal to transversal hyperplanes to the green convex sets. Note that $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G} = \mathbb{R}P^2$, because if $L$ is any line through the origin, we may project our nine convex sets orthogonally onto $L$, obtaining three red intervals, three blue intervals and three green intervals in $L$ with the property that every heterochromatic triple is intersecting. Then, by the colorful Helly Theorem 3.1 in the plane, there is a color such that there is a point in common to all intervals of this color. Therefore there is a hyperplane orthogonal to $L$ transversal to the three convex sets of this color. Hence $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G} = \mathbb{RP}^2$ is a closed cover.

Suppose that for any of the three colors, there is no line parallel to the $xy$-plane transversal to all convex sets of this color. Hence $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G} = \mathbb{RP}^2$ is a closed, null homotopic cover of projective 2-space. By Proposition 3.7, there is at least one line $L$ through the origin whose direction lies in $\mathcal{R} \cap \mathcal{B} \cap \mathcal{G}$. Then there is a transversal hyperplane to the red sets orthogonal to $L$, a transversal hyperplane to the blue sets and a transversal hyperplane to the green sets. ■

**Theorem 3.6.** *Suppose we have three red convex sets, three blue convex sets and three green convex sets i* $\mathbb{R}^3$ *and for every heterochromatic triple one of the sets intersects the other two. Then either there is a color such that there is a line transversal to the all convex sets of this color or else there is a color, say green, and two parallel planes* $H_1$ *and* $H_2$ *such that* $H_1$ *is a transversal plane to all green and red convex sets and* $H_2$ *is a transversal plane to all green and blue convex sets.*

**Proof.** Define $\mathcal{R}$, $\mathcal{B}$ and $\mathcal{G}$ as in the proof of the previous theorem. Note that $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G} = \mathbb{R}P^2$, because if $L$ is any line through the origin, we may project our nine convex sets orthogonally onto $L$, obtaining three red intervals, three blue intervals and three green intervals in $L$ with the property that for every heterochromatic triple, one of the intervals intersects the other two. Then by Proposition 3.5, there is a color such that there is a point common to all intervals of this color. Therefore there is a hyperplane orthogonal to $L$ transversal to the three convex sets of this color. Hence $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G} = \mathbb{RP}^2$. Suppose that for any of the three colors, there is no line transversal to all convex sets of this color. Hence $\mathcal{R} \cup \mathcal{B} \cup \mathcal{G} = \mathbb{RP}^2$ is a closed, null homotopic cover of projective 2-space. By Proposition 3.7, there is at least one line $L$ through the origin whose direction lies in $\mathcal{R} \cap \mathcal{B} \cap \mathcal{G}$.

Let us project our nine convex sets orthogonally onto $L$, obtaining three red intervals, three blue intervals and three green intervals in $L$. Note that since the direction of $L$ lies in $\mathcal{R} \cap \mathcal{B} \cap \mathcal{G}$, every pair of intervals of the same color intersect. If every pair of differently-colored intervals intersects, then the collection of our nine intervals intersects pairwise and hence by the Helly Theorem in the line, there is a point $x_0 \in L$ common to all nine intervals. Then if $H_1 = H_2$ is the plane orthogonal to $L$ through $x_0$, we are done. If not, let $I_1$ and $I_2$ be the two intervals with different color that are farthest apart. Suppose without loss of generality that $I_1 = [a_1, b_1]$ is red, $I_2 = [a_2, b_2]$ is blue and $a_1 \leq b_1 < a_2 \leq b_2$. By the hypothesis, every green interval contains both $b_1$ and $a_2$. Furthermore, every red interval $I_R$ contains $b_1$, otherwise the distance from $I_R$ to $I_2$ would be greater than the distance from $I_1$ to $I_2$. Similarly, every blue interval $I_B$ contains $b_1$. Consequently the plane $H_1$ orthogonal to $L$ through $b_1$ and the plane $H_2$ orthogonal to $L$ through $a_2$ satisfy our requirements. ■

**Proposition 3.4.** *Let $A_1 \cup A_2 \cup A_3 = \mathbb{RP}^2$ be a closed cover and suppose that $A_1 \cap A_2 \cap A_3 = A_1 \cap A_2 = A_2 \cap A_3 = A_1 \cap A_3$. Then either $A_1 \cap A_2 \cap A_3$ is not null homotopic or there is $i \in \{1, 2, 3\}$ such that $A_i - (A_1 \cap A_2 \cap A_3)$ is not null homotopic.*

**Proof.** If $A_1 \cap A_2 \cap A_3$ is null homotopic, then by duality there is an essential curve $\alpha$ of $\mathbb{RP}^2$ contained in $\mathbb{RP}^2 - (A_1 \cap A_2 \cap A_3)$. This essential curve must lie in some connected component of $\mathbb{RP}^2 - (A_1 \cap A_2 \cap A_3)$. Therefore since $A_1 \cap A_2 \cap A_3 = A_1 \cap A_2 = A_2 \cap A_3 = A_1 \cap A_3$, there must be $i \in \{1, 2, 3\}$ such that $\alpha$ is contained in $A_i - (A_1 \cap A_2 \cap A_3)$. ■

For the following theorems, we need a definition. Let $F$ be a family of red, blue and green convex sets in $\mathbb{R}^3$. A transversal plane (resp. line) is a bicolor transversal plane (resp. line) if it cuts all convex sets of two different colors.

**Theorem 3.7.** *Let $F$ be a family of red, blue and green convex sets in $\mathbb{R}^3$. Suppose every pair of convex sets of $F$ with different color intersects and suppose that every bicolor transversal plane through the origin is a transversal plane to all convex sets of $F$. Then there is a transversal line to all convex sets of $F$.*

**Proof.** Let us begin by analyzing the situation in the line. Suppose we have a family of red, blue and green intervals in the line with the property that every pair of intervals of different color intersect. Then by Helly's Theorem in the line, either all intervals have a point in common or there is a pair of intervals of the same color that do not intersect. In the latter case, all the intervals of the other two colors have a point in common.

As always, let $\mathcal{R}_{RB} \subset \mathbb{R}P^2$ be the collection of directions orthogonal to transversal planes to the red and the blue convex sets of $F$. Similarly, we have $\mathcal{R}_{RG} \subset \mathbb{R}P^2$ and $\mathcal{R}_{GB} \subset \mathbb{R}P^2$ for the other two combinations of colors. Our first argument proves that $\mathcal{R}_{RB} \cup \mathcal{R}_{RG} \cup \mathcal{R}_{GB} = \mathbb{RP}^2$ is a closed cover. Now note that $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 = \mathcal{R}_1 \cap \mathcal{R}_2 = \mathcal{R}_1 \cap \mathcal{R}_3 = \mathcal{R}_2 \cap \mathcal{R}_3$. Furthermore, our hypothesis implies that for $i \in \{1, 2, 3\}$, $\mathcal{R}_i - (\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3)$ is null homotopic. Therefore by Proposition 3.10, and since the directions in which there are transversal planes to all convex sets of $F$ coincide with

$\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3$, there is an essential curve of transversal planes to all convex sets of $F$ and consequently there is a transversal line to all convex sets of $F$. ■

**Theorem 3.8.** *Let $F$ be a family of red, blue and green convex sets in $\mathbb{R}^3$. Suppose every non-monochromatic triple is intersecting and every bicolor transversal line parallel to the $xy$-axis is a transversal line to all convex sets of $F$. Then there is, parallel to every plane of $\mathbb{R}^3$, a transversal line to all convex sets of $F$.*

**Proof.** Let us begin by analyzing the situation in the plane. Suppose we have a family of red, blue and green convex sets in the plane with the property that every non-monochromatic triple is intersecting. Then the family $F$ is pairwise intersecting, and furthermore, by Helly's Theorem in the plane, either all convex sets have a point in common or there are three convex sets of the same color that do not intersect but which are pairwise intersecting. If this is so, then by Lemma 1 ($k = \lambda = 2$) of [10], all convex sets of the other two colors have a point in common.

As always, let $\mathcal{R}_1 \subset \mathbb{R}P^2$ be the collection of directions parallel to transversal lines to the red and the blue convex sets of $F$. Similarly, we have $\mathcal{R}_2 \subset \mathbb{R}P^2$ and $\mathcal{R}_3 \subset \mathbb{R}P^2$ for the other two combinations of colors. Our first argument proves that $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 = \mathbb{RP}^2$ is a closed cover. Now note that $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 = \mathcal{R}_1 \cap \mathcal{R}_2 = \mathcal{R}_1 \cap \mathcal{R}_3 = \mathcal{R}_2 \cap \mathcal{R}_3$. Furthermore, our hypothesis implies that for $i \in \{1, 2, 3\}$, $\mathcal{R}_i - (\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3)$ is null homotopic. Therefore by Proposition 3.10, and since the directions in which there are transversal lines to all convex sets of $F$ coincide with $\mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3$, there is an essential curve of transversal lines to all convex sets of $F$ and consequently given a plane $H \subset \mathbb{R}^3$, one of these transversal lines must be parallel to $H$. ■

A *system* $\Omega$ of $\lambda$-*planes* in $\mathbb{R}^d$ is a continuous selection of a unique $\lambda$-plane in every direction of $\mathbb{R}^d$. In [2], it is proved that $\lambda + 1$ systems of $\lambda$-planes in $\mathbb{R}^d$ coincide in some direction. We use this fact to prove the following theorem.

**Theorem 3.9.** *Let $F$ be a family of red, blue, white and green convex sets in $\mathbb{R}^3$. Suppose that every non-heterochromatic triple is intersecting. Then there is a transversal line to all convex sets.*

**Proof.** By Helly's Theorem in the plane, there is, parallel to every direction, a transversal line to the red and blue convex sets. The same is true for white and green. So we have two different systems of lines. Consequently by Theorem 2 of [2], they must coincide in some direction. ∎

A similar argument proves that a family of convex sets in $\mathbb{R}^4$ painted with six colors and with the property that every non-heterochromatic triple is intersecting has a transversal plane to all convex sets.

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