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Applications of an Idea of Voronoĭ, a Report

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In memoriam László Fejes Tóth (1915–2005)

The idea of Voronoĭ's proof of his well-known criterion that a positive definite quadratic form is extreme if and only if it is eutactic and perfect, is as follows: Identify positive definite quadratic forms on $\mathbb{E}^d$ with their coefficient vectors in $\mathbb{E}^{\frac{1}{d}d(d+1)}$. This translates certain problems on quadratic forms into more transparent geometric problems in $\mathbb{E}^{\frac{1}{2}d(d+1)}$ which, sometimes, are easier to solve. Since the 1960s this idea has been applied successfully to various problems of quadratic forms, lattice packing and covering of balls, the Epstein zeta function, closed geodesics on the Riemannian manifolds of a Teichmüller space, and other problems.

This report deals with recent applications of Voronoĭ's idea. It begins with geometric properties of the convex cone of positive definite quadratic forms and a finiteness theorem. Then we describe applications to lattice packings of balls and smooth convex bodies, to the Epstein zeta function and a generalization of it and, finally, to John type and minimum position problems.

1. Introduction

A classical criterion of Voronoĭ [80, 81, 82] says that a positive definite quadratic form on Euclidean d-space $\mathbb{E}^d$ is (locally) extreme if and only if it is eutactic and perfect. Equivalently, a lattice packing of balls has (locally) maximum density, if it is eutactic and perfect. To prove this result, Voronoĭ identified the positive definite quadratic forms on $\mathbb{E}^d$ with their coefficient vectors in $\mathbb{E}^{\frac{1}{2}d(d+1)}$. Slightly earlier Plücker [59] and Klein [51] used a similar idea in the context of line geometry. By Voronoĭ's method, certain problems

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on positive definite quadratic forms, resp. on lattice packing of balls in $\mathbb{E}^d$, are translated into more transparent geometric problems in $\mathbb{E}^{\frac{1}{2}d(d+1)}$.

While Voronoĭ's criterion won immediate recognition and was widely acclaimed, the idea of his proof was ignored for decades. It drew attention only since about 1960. It was applied systematically to the following areas: Barnes, Dickson and the Russian school of the geometry of numbers led by Delone and Ryshkov and their collaborators Stogrin and Dolbilin used it for lattice packing and covering problems. For coverings we add Bambah, Schürmann, Vallentin and the author and refer to [6, 7, 22, 41, 75, 77]. The minimization problem for the Epstein zeta function was at first investigated by the British school of the geometry of numbers (Rankin [62], Cassels [16], Ennola [24] and Montgomery [57]). Later, following a suggestion of Sobolev, who re-discovered the zeta function in the context of numerical integration, this problem was studied by the Russian school, see [21, 64]. Related recent results are due to Sarnak and Strömbergsson [68], Coulangeon [19] and the author [45]. General properties of the density of lattice packings of balls, considered as a function on the space of lattices, were studied by Ash [1], who showed that the density is a Morse function. His work was continued by Bergé and Martinet [12]. Extensions and refinements of Voronoĭ's results on extremum properties of quadratic forms are due to the school on quadratic forms in Bordeaux. It includes Martinet, Bergé, Bachoc, Nebe and Coulangeon, see the monograph [54] of Martinet. We mention also the contributions of Barnes, Sloane and Conway for which we refer to the comprehensive volume [18]. Extensions to periodic sets are due to Schürmann [75]. The kissing number of a lattice packing is related to the number of closed geodesics on the fundamental torus of the lattice. This observation led Bavard [10] and Schmutz Schaller [69, 70, 71] to investigate the closed geodesics on the Riemannian manifolds of a Teichmüller space.

This article gives an overview of the pertinent work of the author. See, in particular the papers [34, 38, 39, 44, 45] and the joint article with Schuster [47]. We have included also work of other authors. A few results are new. The section headings give a first idea of the results that will be presented:

The cone of positive definite quadratic forms,
Weakly eutactic lattices,
Extremum properties of the lattice packing density,
Extremum properties of the product of the lattice packing density and its polar,
Extremum properties of zeta functions,

Extremum properties of the product of zeta functions and their polars,
John type results and minimum ellipsoidal shells,
Minimum position problems.

These results belong to the geometry of numbers, to convex geometry and to the asymptotic theory of normed spaces. In many cases, similar results hold both for Euclidean balls and o-symmetric, smooth convex bodies. If so, the results for balls are presented in more detail since in some cases they are more far reaching and have classical arithmetic interpretations in terms of positive definite quadratic forms. While the results in different areas seem to be unrelated, they are bound together by their outlook and the method of proof. One may speculate, whether they are related in a deeper sense. For one such relation see Corollary 12. A few proofs have been included. This was done in case of new results, or to illustrate the technique of proof.

Since this is a report, the material is organized as follows: For each topic the definitions, the results and the comments are put together, while the proofs are presented later and may well be skipped.

For general information on the geometry of numbers, on positive quadratic forms and on convex geometry we refer to the author and Lekkerkerker [46], Conway and Sloane [18], Zong [84], Martinet [54], the author [37], and Schürmann [75].

Let the symbols tr, dim, bd, relint, $\mathrm{relint}_{\mathcal{S}}$, pos, lin, conv, $\|\cdot\|$, $\cdot$, V, B^d, S^{d-1}, ^{T}, ${}^\perp$ stand for trace, dimension, boundary, interior relative to the affine hull, interior relative to the linear subspace $\mathcal{S}$, positive(=non-negative), linear and convex hull, Euclidean norm, inner product, volume, unit ball and unit sphere in Euclidean d-space $\mathbb{E}^d$, transposition, and orthogonal complement.

2. The Cone of Positive Definite Quadratic Forms

Most results in this report deal in one way or another with positive definite quadratic forms. In some cases geometric properties of the cone $\mathcal{P}^d$ of positive definite quadratic forms or of certain subsets of it are indispensable tools for the proofs. It thus seems justified to begin this overview with an investigation of geometric properties of the cone $\mathcal{P}^d$.

A (real) quadratic form on $\mathbb{E}^d$,

$$q(x) = \sum a_{ik} x_i x_k, \quad x \in \mathbb{E}^d,$$

its (real) symmetric $d \times d$ coefficient matrix

$$A = (a_{ik})$$

and its (real) coefficient vector

$$(a_{11}, \dots, a_{1d}, a_{22}, \dots, a_{2d}, \dots, a_{dd})^T$$

in $\mathbb{E}^{\frac{1}{2}d(d+1)}$ may be identified. The family of all positive definite quadratic forms on $\mathbb{E}^d$ then corresponds to an open convex cone $\mathcal{P}^d$ in $\mathbb{E}^{\frac{1}{2}d(d+1)}$ with apex at the origin O, the *cone of positive definite quadratic forms.* The closure $\mathcal{Q}^d$ of $\mathcal{P}^d$ is the *cone of positive semi-definite quadratic forms* on $\mathbb{E}^d$. The cones $\mathcal{P}^d$ and $\mathcal{Q}^d$, certain polyhedra and unbounded convex bodies in $\mathcal{P}^d$, as well as polyhedral subdivisions of $\mathcal{P}^d$ play an important role in the geometric theory of positive definite quadratic forms, including reduction theory.

Thus, $\mathcal{P}^d$ and $\mathcal{Q}^d$ appear as natural objects of investigation. To our surprise, we found only a few pertinent results, due to Ryshkov and Baranovskiĭ [67], Ryshkov [66], Bertraneu and Fichet [14], Barvinok [8], Wickelgren [83] and the author [39]. Ryshkov and Baranovskiĭ showed that the group of linear automorphisms of $\mathcal{P}^d$ is transitive on $\mathcal{P}^d$. Ryshkov seems to have proved that each linear automorphism of $\mathcal{P}^d$ is of a particularly simple form. Wickelgren characterized the linear automorphisms of the Ryshkov polyhedron $\mathcal{R}^d$ in $\mathcal{P}^d$ and Bertraneu and Fichet gave a description of the extreme faces of $\mathcal{Q}^d$ and, as a consequence, showed that the lattice of extreme faces of $\mathcal{Q}^d$ is isomorphic to the lattice of linear subspaces of $\mathbb{E}^d$ and, thus, modular.

In this section, we report on the results of the author [39], beginning with an analog for exposed faces of the result of Bertraneu and Fichet. The next result says that the exposed and the extreme faces of $\mathcal{Q}^d$ coincide. These results then are used as tools for the proofs of all further results: First, extending well-known notions for polytopes, flag transitivity of the group of all orthogonal transformations and neighborliness properties of the convex cone $\mathcal{Q}^d$ are studied. Then we investigate singularity properties of boundary points and faces of $\mathcal{Q}^d$, and show the simple fact that $\mathcal{Q}^d$ is self-dual. Finally, the group of isometries of $\mathcal{Q}^d$ will be described. Each isometry is generated by an orthogonal transformation of $\mathbb{E}^d$.

Extreme and Exposed Faces of $\mathcal{Q}^d$

An *extreme face* or *face* $\mathcal{F}$ of the cone $\mathcal{Q}^d$ is a subset of $\mathcal{Q}^d$ with the following property: If a relative interior point of a line segment in $\mathcal{Q}^d$ is contained in $\mathcal{F}$, then the whole line segment is contained in $\mathcal{F}$. The empty set $\emptyset$ and the cone $\mathcal{Q}^d$ are faces of $\mathcal{Q}^d$. Each extreme face of $\mathcal{Q}^d$ is itself a closed convex cone. A special face is an *exposed face,* i.e. the intersection of $\mathcal{Q}^d$ with a support hyperplane. To simplify, also $\emptyset$ and $\mathcal{Q}^d$ are said to be exposed.

For $u \in \mathbb{E}^d$ define the *tensor product* $u \otimes u$ to be the symmetric $d \times d$ matrix $u\,u^T \in \mathbb{E}^{\frac{1}{2}d(d+1)}$. (The linear mapping $x \to u \otimes u\,x$ maps $x \in \mathbb{E}^d$ onto the point $(u \cdot x)u$ and, if u is a unit vector, this is the orthogonal projection of x onto the line $\operatorname{lin}\{u\}$.)

Theorem 1. *Let $\mathcal{F} \subseteq \mathcal{Q}^d$. Then the following properties* (i) *and* (ii) *of the set $\mathcal{F}$ are equivalent:*

(i) *$\mathcal{F}$ is an exposed face of $\mathcal{Q}^d$.*
(ii) *There is a linear subspace S of $\mathbb{E}^d$ such that $\mathcal{F} = \operatorname{pos}\{u \otimes u : u \in S\}$.*

Moreover,

(iii) *if* (ii) *holds, then $\dim \mathcal{F} = \frac{1}{2}c(c+1)$, where $c = \dim S$.*

Theorem 2. *Each extreme face of $\mathcal{Q}^d$ is exposed.*

Since by Theorem 2, extreme and exposed faces coincide, from now on we will speak simply of *faces* of $\mathcal{Q}^d$.

The Face Lattice of $\mathcal{Q}^d$

The above results, which show that the faces of $\mathcal{Q}^d$ can be represented in a particularly simple way, lead to a series of properties of $\mathcal{Q}^d$.

An (algebraic) lattice $\langle L, \vee, \wedge \rangle$ is *modular* if it satisfies the *modular law,*

$$(l \wedge m) \vee n = l \wedge (m \vee n) \quad \text{for} \quad l, m, n \in L.$$

It is *orthomodular,* if it has 0 and 1 and for each $l \in L$ there is an *orthocomplement,* i.e. an element $l^{\perp} \in L$ such that

$$l \vee l^{\perp} = 1, \;\; l \wedge l^{\perp} = 0, \;\; (l^{\perp})^{\perp} = l \;\text{ and }\; l \leq m \;\Rightarrow\; l^{\perp} \geq m^{\perp},$$

and satisfies the *orthomodular law*,

$$l \le m \;\Rightarrow\; m = l \vee (m \wedge l^{\perp}) \quad \text{for} \quad l, m \in L.$$

It is well-known that the family of all linear subspaces of $\mathbb{E}^d$, including $\emptyset = 0$ and $\mathbb{E}^d = 1$, with the following definitions of $\wedge$, $\vee$ is a lattice with 0 and 1:

$$S \wedge T = S \cap T,$$

$$S \vee T = \bigcap \{U : U \text{ linear subspace of } \mathbb{E}^d \text{ with } S, T \subseteq U\} = S + T$$

$$\text{for linear subspaces } S, T \text{ of } \mathbb{E}^d.$$

This lattice is both modular and orthomodular. The family of all faces of $\mathcal{Q}^d$, including $\emptyset$ and $\mathcal{Q}^d$, is a lattice with respect to the following lattice operations $\wedge$, $\vee$:

$$\mathcal{F} \wedge \mathcal{G} = \mathcal{F} \cap \mathcal{G},$$

$$\mathcal{F} \vee \mathcal{G} = \bigcap \{\mathcal{H} : \mathcal{H} \text{ face of } \mathcal{Q}^d \text{ with } \mathcal{F}, \mathcal{G} \subseteq \mathcal{H}\}$$

$$\text{for faces } \mathcal{F},\ \mathcal{G} \text{ of } \mathcal{Q}^d.$$

Since by Theorem 1 and 2, these lattices are isomorphic, we get the following result:

Corollary 1. *The lattice of all faces of $\mathcal{Q}^d$ is modular and orthomodular.*

The Flag Transitivity of $\mathcal{Q}^d$

For $d \times d$ matrices $A = (a_{ik})$ and $B = (b_{ik})$ in $\mathbb{E}^{\frac{1}{2}d(d+1)}$ or E^{d^2} define an inner product and a norm by $A \cdot B = \sum a_{ik} b_{ik}$ and $\|A\| = \left(\sum a_{ik}^2\right)^{\frac{1}{2}}$. A group of transformations which map $\mathcal{P}^d$ or $\mathcal{Q}^d$ onto itself is called a *group of automorphisms* or *symmetries* of $\mathcal{P}^d$ or $\mathcal{Q}^d$. If the transformations are linear, orthogonal or isometric we speak of *linear* or *orthogonal automorphisms,* or, of a *group of isometries* with respect to the norm just defined. Then the following holds: Let U be an orthogonal transformation of $\mathbb{E}^d$. Then the transformation

$$\mathcal{U} : A \to UAU^T \text{ for } A \in \mathcal{P}^d \text{ or } \mathcal{Q}^d, \text{ respectively,}$$

is an orthogonal automorphism of $\mathcal{P}^d$ or $\mathcal{Q}^d$, respectively.

Extending the definition for convex polytopes, a sequence $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{d-1}$ of faces of $\mathcal{Q}^d$ is called a *flag* or a *tower* of $\mathcal{Q}^d$, if

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_{d-1} \quad \text{and} \quad \dim \mathcal{F}_c = \frac{1}{2}c(c+1) \quad \text{for} \quad c = 1, 2, \dots, d-1.$$

A group of automorphisms of $\mathcal{Q}^d$ is *flag transitive* if for any two flags

$$\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{d-1} \quad \text{and} \quad \mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_{d-1}$$

there is an automorphism $\mathcal{U}$ in the group such that $\mathcal{U}\mathcal{F}_i = \mathcal{G}_i$ for $i = 1, 2, \dots, d-1$.

Corollary 2. *The group of orthogonal automorphisms of $\mathcal{Q}^d$ is flag transitive.*

The Neighborliness of $\mathcal{Q}^d$

The notion of neighborliness for a convex polytope, see Grünbaum [48], Ch. 7, can be adapted to the present situation as follows: For $k = 1, 2, \dots,$ the convex cone $\mathcal{Q}^d$ is said to be *k-almost neighborly,* if the positive (=non-negative) hull of any k extreme rays of $\mathcal{Q}^d$ with endpoint O is contained in a proper face of $\mathcal{Q}^d$.

Corollary 3. $\mathcal{Q}^d$ *is* $(d-1)$-, *but not d-almost neighborly.*

Polarity and Self-Polarity of $\mathcal{Q}^d$

The *dual* or *polar cone* of the convex cone $\mathcal{Q}^d$ with apex O is the convex cone

$$\mathcal{Q}^{d^*} = \left\{N \in \mathbb{E}^{\frac{1}{2}d(d+1)} \,:\, A \cdot N \geq 0 \text{ for } A \in \mathcal{Q}^d\right\},$$

i.e. the (interior) normal cone of $\mathcal{Q}^d$ at its apex O.

Corollary 4. $\mathcal{Q}^d = \mathcal{Q}^{d^*}$, *i.e.,* $\mathcal{Q}^d$ *is self-polar.*

How Singular are the Faces of $\mathcal{Q}^d$?

A face $\mathcal{F}$ of $\mathcal{Q}^d$ is *k-singular,* if k is the dimension of the (interior) normal cone of the convex cone $\mathcal{Q}^d$ at (a relative interior point of) $\mathcal{F}$.

Corollary 5. *Let $\mathcal{F}$ be a non-empty proper face of $\mathcal{Q}^d$ with* $\dim \mathcal{F} = \frac{1}{2}c(c+1)$. *Then $\mathcal{F}$ is $\frac{1}{2}(d-c)(d-c+1)$-singular.*

The Isometries of $\mathcal{Q}^d$

The following result shows that the isometries of $\mathcal{Q}^d$ are orthogonal automorphisms and thus are determined by orthogonal transformations of the underlying space $\mathbb{E}^d$. This is a phenomenon which appears also in several other instances in convex geometry, see the survey [35], to which we add Böröczky and Schneider [15] and Schneider [74]. This result shows that, in particular, the space of isometries of $\mathcal{Q}^d$ is rather small.

Theorem 3. *Let $\mathcal{U}$ be a mapping of $\mathcal{Q}^d$ onto itself. Then the following properties are equivalent:*

(i) *$\mathcal{U}$ is an isometry.*
(ii) *There is an orthogonal $d \times d$ matrix $U \in \mathbb{E}^{d^2}$ such that $\mathcal{U}A = UAU^T$ for $A \in \mathcal{Q}^d$.*

Conclusion

Remark 1. While $\mathcal{Q}^d$ is far from being a polyhdral cone, it shares many properties with highly symmetric, neighborly, and self dual polyhedral convex cones.

3. Weakly Eutactic Lattices

In later sections we will frequently encounter (geometric) lattices which are eutactic, possibly in a weaker or stronger sense. Thus it is appropriate to give some information on such lattices. Bergé and Martinet [13] and

Bavard [11] gave descriptions of the weakly eutactic lattices in $\mathbb{E}^2$, $\mathbb{E}^3$, $\mathbb{E}^4$. We extract from their results the semi-eutactic lattices in $\mathbb{E}^2$, $\mathbb{E}^3$ and point out their relationship to the Bravais classification of lattices in crystallography. The aforementioned authors, with Ash [2] as a forerunner, showed that in general dimensions there are only finitely many similarity classes of weakly eutactic lattices. We outline a new geometric proof of this result, using the Ryshkov polyhedron.

Eutactic Lattices and the Bravais Classification

Let L be a (geometric) *lattice* in $\mathbb{E}^d$, that is the set of all integer linear combinations of d linearly independent vectors. The volume of the parallelepiped generated by these vectors is the *determinant* $d(L)$ of L. The *set* M_L *of minimum points,* or the *first layer* of L, consists of all points $l \in L\backslash\{o\}$ with minimum Euclidean norm. The lattice L is called *weakly eutactic, semi-eutactic, eutactic, strongly eutactic,* or *perfect* with respect to B^d, or $\|\cdot\|$, if

$$I = \sum_{l \in M_L} \lambda_l \, l \otimes l \text{ with suitable } \lambda_l \begin{Bmatrix} \text{real} \\ \geq 0 \\ > 0 \\ = \text{const} \end{Bmatrix}, \quad \text{resp.}$$

$$\mathbb{E}^{\frac{1}{2}d(d+1)} = \operatorname{lin}\{u \otimes u \,:\, u \in M_L\}.$$

Note that any perfect lattice is weakly eutactic.

The *Bravais classification* of lattices is used in crystallography and classifies lattices by their groups of orthogonal automorphisms which keep the origin o fixed. In dimensions 2 and 3 there are 5, resp. 14 Bravais classes of lattices. For more information see Erdös, Gruber and Hammer [25] and Engel [23]. The *kissing number* of L is the number of minimum points.

Theorem 4. *The following is a list of the similarity classes of the semi-eutactic lattices in* $\mathbb{E}^2$ *and* $\mathbb{E}^3$, *containing the symbols of their Bravais classes, their usual names, their eutaxy type, a remark whether they are* *perfect, and their kissing number.*

$d=2:$	*tp*	*square*	*strongly eutactic*	4
	hp	*hexagonal*	*strongly eutactic perfect*	6
$d=3:$	*cP*	*cubic primitive*	*strongly eutactic*	6
	hP	*special hexagonal primitive*	*eutactic*	8
	cF	*cubic face centered*	*strongly eutactic perfect*	12
	cI	*cubic body centered*	*strongly eutactic*	8
	tI	*special tetragonal body centered*	*eutactic*	8

These lattices in $\mathbb{E}^2$ and $\mathbb{E}^3$ make up certain Bravais types (tp, hp, cP, cF, cI), or form a subset of a Bravais type (hP, tI). In the latter case we have added the adjective 'special'.

Theorem 5. *There are only finitely many similarity classes of weakly eutactic, resp. perfect lattices in* $\mathbb{E}^d$.

Since each perfect lattice is weakly eutactic, the result for perfect lattices follows from that for weakly eutactic lattices.

Open Problems

In the later Corollaries 6 and 11 there are specified the families of Bravais classes corresponding to those lattices in $\mathbb{E}^2$ and $\mathbb{E}^3$ which have particular extremum properties. It would be of interest, to know whether there are other properties of lattices which lead to the same families of Bravais classes.

Problem 1. Specify geometric properties of lattices which single out the Bravais classes

$$\{hp\},\ \{tp,hp\},\ \{cF\},\ \{cP,cF,cI\},\ \{hP,cP,cI,cF,tI\}$$

among the 5 Bravais classes for $d=2$, and the 14 Bravais classes for $d=3$, respectively. Is there a connection between such properties and extremum properties of the density of lattice packings of balls or the Epstein zeta function?

Problem 2. What is the precise relation of similarity classes of weakly eutactic lattices and Bravais classes of lattices in $\mathbb{E}^d$ for general d?

Outline of the Proof of Theorem 5

We begin with some preparations. A lattice L may be represented in the form $L = B\mathbb{Z}^d$, where B is a non-singular $d \times d$-matrix and $\mathbb{Z}^d$ is the integer lattice. The columns of the matrix B then form a basis of L. The positive definite quadratic form $q(x) = (Bx)^2 = B^T B \cdot x \otimes x$ then is the *metric form* (of the basis matrix B) of L. The metric forms of L are unique up to equivalence. Conversely, a quadratic form q in $\mathcal{P}^d$ can be written in the form $q(x) = A \cdot x \otimes x$, where A is a symmetric $d \times d$ matrix. Then q is the metric form of all lattices of the form $RA^{\frac{1}{2}}\mathbb{Z}^d$, where R is orthogonal. The set M_q of *minimum points* of q consists of all $u \in \mathbb{Z}^d \backslash \{o\}$ such that $q(u)$ is minimum. We have the following dictionary:

$L = B\mathbb{Z}^d$	$q(x) = B^T B \cdot x \otimes x = A \cdot x \otimes x$
class of all lattices L	class of all quadratic forms in $\mathcal{P}^d$
similar to L	equivalent to a multiple of q
$M_L = BM_q$	$M_q = B^{-1}M_L$
$I = \sum_{l \in M_L} \lambda_l \, l \otimes l$	$A^{-1} = \sum_{u \in M_q} \lambda_u \, u \otimes u$
$\mathbb{E}^{\frac{1}{2}d(d+1)} = \text{lin}\,\{l \otimes l \,:\, l \in M_L\}$	$\mathbb{E}^{\frac{1}{2}d(d+1)} = \text{lin}\,\{u \otimes u \,:\, u \in M_q\}$

Call the positive definite quadratic form q *weakly eutactic,* resp. *perfect* if the corresponding lattice L satisfies this condition.

Let $m > 0$. The *Ryshkov polyhedron* $\mathcal{R}^d(m)$ is defined by

$$\mathcal{R}^d(m) = \left\{ A \in \mathcal{P}^d \,:\, A \cdot u \otimes u \geq m \text{ for } u \in \mathbb{Z}^d \backslash \{o\} \right\}$$

$$= \bigcap_{\substack{u \in \mathbb{Z}^d \backslash \{o\} \\ \text{primitive}}} \left\{ A \in \mathbb{E}^{\frac{1}{2}d(d+1)} \,:\, A \cdot u \otimes u \geq m \right\} \cap \mathcal{P}^d,$$

where u is *primitive* if the points o and u are the only points of $\mathbb{Z}^d$ on the line segment $[o, u]$. The following properties of $\mathcal{R}^d(m)$ can easily be verified, see [37]:

> $\mathcal{R}^d(m)$ is a generalized polyhedron, i.e. its intersection with any convex polytope, is a convex polytope.

The facets of $\mathcal{R}^d(m)$ are precisely the sets

$$\mathcal{R}^d(m) \cap \left\{A \in \mathbb{E}^{\frac{1}{2}d(d+1)} \,:\, A \cdot u \otimes u = m\right\}, \quad u \in \mathbb{Z}^d \backslash \{o\} \text{ primitive.}$$

$\operatorname{bd} \mathcal{R}^d(m)$ is the set of all (coefficient matrices of positive definite quadratic forms) $q \in \mathcal{P}^d$ with (homogeneous) *minimum* $\min\left\{q(u) : u \in \mathbb{Z}^d \backslash \{o\}\right\} = m$.

$\operatorname{bd} \mathcal{R}^d(m)$ is the disjoint union of the relative interiors of its faces.

The mappings $\mathcal{U} : A \to U^T A U$ for $A \in \mathbb{E}^{\frac{1}{2}d(d+1)}$, where U is an integer $d \times d$ matrix with determinant ± 1, map $\mathcal{R}^d(m)$ onto itself. Two faces $\mathcal{F}, \mathcal{G}$ of $\mathcal{R}^d(m)$ are *equivalent* if there is such a mapping $\mathcal{U}$ with $\mathcal{G} = \mathcal{U}\mathcal{F}$.

There are pairwise non-equivalent vertices $V_1, \dots, V_k$ of $\mathcal{R}^d(m)$, such that any other vertex is equivalent to one of these.

The latter result can be used to show that

(1) there are pairwise non-equivalent faces $\mathcal{F}_1, \dots, \mathcal{F}_p$ of $\mathcal{R}^d(m)$, such that any other face is equivalent to one of these.

Finally, the following hold:

(2) Let $A \in \operatorname{relint} \mathcal{F}$, where $\mathcal{F}$ is a face of $\mathcal{R}^d(m)$. Then the (interior) normal cone of $\mathcal{R}^d(m)$ at A resp. $\mathcal{F}$, equals $\operatorname{pos}\{u_1 \otimes u_1, \dots, u_j \otimes u_j\}$ where $u_1 \otimes u_1, \dots, u_j \otimes u_j$ are (interior) normal vectors of those facets of $\mathcal{R}^d(m)$ which contain A and thus $\mathcal{F}$.

(3) $q = A \cdot x \otimes x$ is contained precisely in those facets of $\mathcal{R}^d(m)$ with normal vectors $u \otimes u : u \in M_q$.

Let $\delta > 0$. The *discriminant body* $\mathcal{D}^d(\delta)$, is given by

$$\mathcal{D}^d(\delta) = \{A \in \mathcal{P}^d \,:\, \det A \geq \delta\}.$$

It is well-known that

(4) $\mathcal{D}^d(\delta)$ is an unbounded, smooth, and strictly convex set in $\mathcal{P}^d$ with non-empty interior. For A in $\operatorname{bd} \mathcal{D}^d(\delta)$ the vector A^{-1} is an interior normal vector of $\operatorname{bd} \mathcal{D}^d(\delta)$ at A.

After these preparations, the first step of the proof is to show the following

(5) Let the positive definite quadratic form $q = A \cdot x \otimes x$ have minimum m and determinant δ. Assume that $A \in \operatorname{relint} \mathcal{F}$, where $\mathcal{F}$ is a face of $\mathcal{P}^d$, and that q is weakly eutactic. Then $\mathcal{D}(\delta)$ touches $\mathcal{F}$ at A.

The normal cone of $\mathcal{R}^d(m)$ at q (or $\mathcal{F}$) is $\operatorname{pos}\{u_1 \otimes u_1, \dots, u_j \otimes u_j\}$, where $u_1 \otimes u_1, \dots, u_j \otimes u_j$ are normal vectors of those facets of $\mathcal{R}^d(m)$ which contain q (or $\mathcal{F}$), see (2). Hence,

$$\mathcal{F} - A \subseteq \operatorname{lin}\{u_1 \otimes u_1, \dots, u_j \otimes u_j\}^{\perp}.$$

By (3), the vectors $u_1, \dots, u_j$ are the minimum vectors of the quadratic form $q = A \cdot x \otimes x$. The weak eutaxy of q then shows that

$$A^{-1} \in \operatorname{lin}\{u_1 \otimes u_1, \dots, u_j \otimes u_j\}, \quad \text{or} \quad A^{-1\perp} \supseteq \operatorname{lin}\{u_1 \otimes u_1, \dots, u_j \otimes u_j\}^{\perp}.$$

By (4), the matrix A^{-1} is an interior normal vector of $\operatorname{bd} \mathcal{D}^d(\delta)$ at A. Hence,

$$A + A^{-1\perp}$$

is the tangent hyperplane of $\mathcal{D}^d(\delta)$ at A (or q). This together with the earlier inclusions finally yields,

$$\mathcal{F} \subseteq A + \operatorname{lin}\{u_1 \otimes u_1, \dots, u_j \otimes u_j\}^{\perp} \subseteq A + A^{-1\perp},$$

concluding the proof of (5).

Since, by (4), $\mathcal{D}^d(\delta)$ is smooth and strictly convex and the surfaces $\operatorname{bd} \mathcal{D}^d(\delta)$, $\delta > 0$, are strictly convex and their union is $\mathcal{P}^d$,

> there is for each face $\mathcal{F}$ of $\mathcal{R}^d(m)$ at most one value of $\delta > 0$ such that $\mathcal{D}(\delta)$ touches $\mathcal{F}$ at a relative interior point.

Thus, by (5), there are at most n weakly eutactic forms contained in the facets $\mathcal{F}_1, \dots, \mathcal{F}_p$, say $q_1, \dots, q_n$. Since weak eutaxy is invariant with respect to equivalence and multiplication with positive integers, we see, by (1), that

> the weakly eutactic forms are precisely the forms in $\mathcal{P}^d$ which are equivalent to positive multiples of the forms $q_1, \dots, q_n$.

Taking into account the above dictionary, the proof is complete. ■

4. Maximum Properties of the Lattice Packing Density

In this section refined maximum properties of the density of lattice packings of balls and convex bodies are studied. The results obtained are due to the author [44] and refine and extend the classical criterion of Voronoĭ. The notions of semi-eutaxy, eutaxy and perfection are used to characterize lattices which provide lattice packings of balls, resp. of smooth convex bodies with semi-stationary, maximum and ultra-maximum density. It is surprising to observe that maximum and ultra-maximum lattice packings of balls coincide, and that the proof is simple. Relations to the Bravais classification of lattices are specified.

Let C be a *convex body,* i.e. a compact convex subset of $\mathbb{E}^d$ with non-empty interior. We assume that C is o-symmetric and smooth that is, the boundary is of class $\mathcal{C}^1$. Note that for lattice packing problems the assumption of central symmetry of C is not an essential restriction. Let $\|l\|_C$ be the norm on $\mathbb{E}^d$ for which C is the unit ball. Let L be a lattice. The *homogeneous* or *first successive minimum* of L with respect to C is defined by

$$\lambda = \lambda(C, L) = \min\left\{ \|l\|_C \,:\, l \in L\backslash\{o\}\right\}.$$

Then the convex bodies $\frac{\lambda}{2}C + l \,:\, l \in L$ do not overlap, and thus form a *lattice packing* with *packing lattice* L. This lattice packing is said to be *provided* by L. In the following the *density*

$$\delta(C, L) = \frac{V\left(\frac{\lambda}{2}C\right)}{d(L)} = \frac{\lambda^d V(C)}{2^d d(L)}$$

of this lattice packing will be investigated for given C, as L ranges over the space of all lattices in $\mathbb{E}^d$. Let $M_L = \left\{ l \in L \,:\, \|l\|_C = \lambda(C, L)\right\}$ be the set of minimum points or the first layer of L with respect to C.

The connection between Voronoĭ type and maximum properties of a lattice L can roughly be described as follows: The different Voronoĭ type properties of L are equivalent to different positions of a certain convex polytope in $\mathbb{E}^{d^2}$ relative to the origin. (The origin is an exterior point, a point on the relative boundary, or in the relative interior.) These simple geometric properties turn out to be equivalent to different maximum properties of L.

Extremum and Voronoĭ Type Properties

The lattice L is (upper) *semi-stationary, stationary, maximum* or *ultra-maximum* with respect to $\delta(B^d,\cdot)$, if

$$\frac{\delta\big(B^d,(I+A)L\big)}{\delta(B^d,L)}\begin{Bmatrix}\leq 1+o\big(\|A\|\big)\\ =1+o\big(\|A\|\big)\\ \leq 1\\ \leq 1-\mathrm{const}\|A\|\end{Bmatrix}\quad \text{as}\ \ A\to O,\ A\in\mathcal{T},$$

where $\mathcal{T}$ is the subspace

$$\mathcal{T}=\Big\{A\in\mathbb{E}^{\frac{1}{2}d(d+1)}\ :\ \mathrm{tr}\,A=A\cdot I=0\Big\}=I^{\perp}$$

of $\mathbb{E}^{\frac{1}{2}d(d+1)}$ of codimension 1 with normal vector I. The restriction of A to $\mathcal{T}$ is not essential. It helps to avoid clumsy formulations of our results. An inequality holds as $A\to O$, $A\in\mathcal{T}$, if it holds for all $A\in\mathcal{T}$ with sufficiently small norm. The symbols $o(\cdot)$ and const >0 depend only on B^d and L.

In order to characterize these maximum properties, we need the Voronoĭ type notions of *semi-eutactic, eutactic,* and *perfect* lattice (or first layer) with respect to B^d, see Sect. 3.

Characterization of Semi-Stationary and Ultra-Extreme Lattices

Theorem 6. *The following properties* (i) *and* (ii) *of* $\delta(B^d,\cdot)$ *and* L *are equivalent:*

(i) *L is semi-stationary.*
(ii) *L is semi-eutactic.*

Moreover,

(iii) *there is no stationary lattice.*

This result implies, in particular, that $\delta(B^d,\cdot)$, considered as a function on the space of lattices in $\mathbb{E}^d$, is ‘not differentiable’. More surprising is the next result, the main result of this section. It shows that maximality and ultra-maximality with respect to $\delta(B^d,\cdot)$ coincide.

Theorem 7. *The following properties of $\delta(B^d,\cdot)$ and L are equivalent:*

(i) *L is ultra-maximum.*
(ii) *L is maximum.*
(iii) *L is perfect and eutactic.*

The equivalence of (ii) and (iii) is Voronoĭ's criterion.

Bravais Types of Lattices with Maximum Properties

The following result is a consequence of Theorems 4, 6 and 7:

Corollary 6. *In $\mathbb{E}^2$ and $\mathbb{E}^3$ it is the following lattices of determinant* 1 *which are semi-stationary, resp. ultra-maximum with respect to $\delta(B^d,\cdot)$:*

$d=2:$	*tp*	*square*	$d=3:$	*cP*	*cubic primitive*
	hp	*hexagonal*		*cF*	*cubic face centered*
				cI	*cubic body centered*
				hP	*special hexagonal primitive*
				tI	*special tetragonal body centered*
$d=2:$	*hp*	*hexagonal*	$d=3:$	*cF*	*cubic face centered*

For general d, there are, up to orthogonal transformations, only finitely many lattices of determinant 1 *which are semi-stationary, resp. ultra-maximum.*

Extremum and Voronoĭ Type Properties

In the following, the results for balls will be extended to convex bodies. Let C be an o-symmetric, smooth convex body. Replace $\mathbb{E}^{\frac{1}{2}d(d+1)}$, $\mathcal{T}$, B^d and $\|\cdot\|$ by $\mathbb{E}^{d^2}$, $\mathcal{S}=\{A\in\mathbb{E}^{d^2},\ \mathrm{tr}\,A=A\cdot I=0\}$, C and $\|\cdot\|_C$. The density $\delta(C,L)$ and the notions of semi-stationary, etc. lattice with respect to $\delta(C,\cdot)$ are defined as earlier.

In order to specify the versions of eutaxy and perfection which are needed to characterize the maximum properties of $\delta(C,\cdot)$, we proceed as follows: For $l\in\mathbb{E}^d\backslash\{o\}$ let u be the exterior unit normal vector of the smooth convex body $\|l\|_C C$ at its boundary point l, and put $n=l/l\cdot u$. Then the lattice L, or the set M_L of its minimum points with respect to C, is *semi-eutactic, eutactic, strongly eutactic,* or *perfect* with respect to C, if

$$I = \sum_{l \in M_L} \lambda_l \, u \otimes n \quad \text{with suitable} \quad \lambda_l \begin{Bmatrix} \geq 0 \\ > 0 \\ = \text{const} \end{Bmatrix}, \quad \text{resp.}$$

$$\mathbb{E}^{d^2} = \text{lin}\,\{u \otimes n \, : \, l \in M_L\}.$$

Characterization of Semi-Stationary and Ultra-Maximum Lattices

Theorems 6 and 7 now assume the following form.

Theorem 8. *The following properties* (i) *and* (ii) *of* $\delta(C,\cdot)$ *and* L *are equivalent:*

(i) *L is semi-stationary.*
(ii) *L is semi-eutactic.*

Moreover,

(iii) *there is no stationary lattice.*

Theorem 9. *The following properties of* $\delta(C,\cdot)$ *and* L *are equivalent:*

(i) *L is ultra-maximum.*
(ii) *L is perfect and eutactic.*

While there are many semi-stationary lattices for $\delta(C,\cdot)$, for example all lattices which provide lattice packings of C of maximum density, this is not clear for ultra-maximum lattices. See the later discussion.

The *kissing number* $k(C,L)$ of the lattice L with respect to the convex body C or the norm $\|\cdot\|_C$ is $\#M_L$, the number of minimum points. Equivalently, let $\lambda = \lambda(C,L)$. Then $k(C,L)$ is the number of bodies of the lattice packing $\left\{\frac{\lambda}{2}C + l \, : \, l \in L\right\}$, which touch the body $\frac{\lambda}{2}C$. The next estimate is an immediate consequence of Theorem 9.

Corollary 7. *Let L be an ultra-maximum lattice for $\delta(C,\cdot)$. Then the kissing number satisfies the inequality $k(C,L) \geq 2d^2$.*

Remark 2. If L is ultra-maximum then $k(C,L) \geq 2d^2$ by Corollary 7. A theorem of Minkowski says, if C is strictly convex, then holds $k(C,L) \leq 2^{d+1} - 2$. Since $2^{d+1} - 2 < 2d^2$ for $d = 2, 3, 4$, a strictly convex smooth o-symmetric body can have an ultra-maximum lattice only if $d \geq 5$ – if at all. This explains why it is difficult to specify examples.

The proofs of Theorems 8 and 9 are more complicated than those of Theorems 6 and 7 yet, in essence, follow the same line.

Baire Categories

In the following Baire categories will be used several times. A topological space is *Baire* if any of its meager subsets has dense complement, where a set is *meager* or *of first Baire category,* if it is a countable union of nowhere dense sets. A version of the Baire category theorem says that each locally compact or metrically complete space is Baire. When speaking of *most* or of *typical* elements of a Baire space, we mean all elements, with a meager set of exceptions. The space of all o-symmetric convex bodies endowed with its natural topology is locally compact according to a version of Blaschke's selection theorem and, thus, Baire. See the author [36, 37] for information on Baire type results in convex geometry.

A result of Klee [50] and the author [32] says that most o-symmetric convex bodies are smooth and strictly convex.

Open Problems

Problem 3. Is it true that in all sufficiently high dimensions, for most o-symmetric convex bodies

(i) the maximum and the ultra-maximum lattices coincide,

(ii) the kissing number of each maximum or ultra-maximum lattice equals $2d^2$?

Problem 4. If there are convex bodies with maximum lattices which are not ultra-maximum, characterize the maximum lattices.

What is the situation in the special case of lattices which provide lattice packings of maximum density? A result of the author [33] says that for most o-symmetric convex bodies the kissing number of any lattice packing of maximum density is at most $2d^2$. If, in addition, the lattice is ultra-maximum, then, by Theorem 9, the kissing number is at least $2d^2$ and, thus, equals $2d^2$. An estimate of Swinnerton-Dyer [78] implies that a lattice which provides a packing of maximum density, has kissing number at least $d(d+1)$. For many years I thought that for most o-symmetric convex bodies the kissing number of lattice packings of maximum density is $d(d+1)$. Recently I have changed the opinion:

Problem 5. Show that in all sufficiently high dimensions and for most o-symmetric convex bodies C, the lattice which provides a lattice packing of C of maximum density, has the following properties:

(i) L is unique up to dilatations.
(ii) L is eutactic and perfect and, thus, ultra-maximum.
(iii) L has kissing number $2d^2$.
(iv) the packing $\left\{\frac{\lambda}{2}C + l \,:\, l \in L\right\}$, $\lambda = \lambda(C, L)$ is (perhaps?) connected.

Proof of Theorem 7

In order to show the reader the simple yet effective idea underlying the proofs of Theorems 6–9, we present the proof of Theorem 7. We begin with some remarks. Since $\delta(B^d, L)$ does not change if L is replaced by a multiple of it, we may assume that $\lambda(B^d, L) = 1$ and thus, $l = n = u$, $\|l\| = 1$ for $l \in M_L$. Trivially,

$$\lambda(B^d, L) = \min\left\{\|l\| \,:\, l \in M_L\right\} < \min\left\{\|l\| \,:\, l \in L\backslash\left(M_L \cup \{o\}\right)\right\}.$$

Note that

$$\|l + Al\|^2 = \|l\|^2\left(1 + 2A\cdot n\otimes n + A^2\cdot n\otimes n\right) = 1 + 2A\cdot l\otimes l + O\left(\|A\|^2\right)$$

$$\text{as}\quad A \to O,\quad A \in \mathcal{T},\quad l \in M_L,$$

$$\det(I + A) = 1 - \frac{1}{2}\|A\|^2 + O\left(\|A\|^2\right)\quad \text{as}\quad A \to O,\quad A \in \mathcal{T}$$

and, since L is discrete,

$$\begin{aligned}\lambda\left(B^d, (I + A)l\right) &= \min\left\{\|l + Al\| \,:\, l \in M_L\right\}\\ &< \min\left\{\|l + Al\| \,:\, l \in L\backslash M_L \cup \{o\}\right\}\\ &\text{if}\ \ A \in \mathcal{T}\ \ \text{has sufficiently small norm.}\end{aligned}$$

Thus we get,

$$\begin{aligned}
\delta\big(B^d,(I+A)L\big) &= \frac{\lambda\big(B^d,(I+A)L\big)^d V(B^d)}{2^d d\big((I+A)L\big)} \\
&= \frac{V(B^d)}{2^d d(L)}\,\frac{\min\big\{\|l+Al\|^2 \,:\, l\in M_L\big\}^{\frac{d}{2}}}{\det(I+A)} \\
&= \frac{V(B^d)}{2^d d(L)}\min\big\{1+2A\cdot n\otimes n + A^2\cdot n\otimes n \,:\, l\in M_L\big\}^{\frac{d}{2}} \\
&\quad\times\Big(1-\frac{1}{2}\|A\|^2+O\big(\|A\|^3\big)\Big)^{-1} \\
&= \frac{\lambda(B^d,L)^d V(B^d)}{2^d d(L)}\Big(\min\{1+dA\cdot l\otimes l \,:\, l\in M_L\}+O\big(\|A\|^2\big)\Big) \\
&= \delta(B^d,L)\Big(1+d\min\{A\cdot l\otimes l \,:\, l\in M_L\}+O\big(\|A\|^2\big)\Big) \\
&\quad \text{as } A\to O,\quad A\in\mathcal{T}.
\end{aligned}$$

(i) $\Leftrightarrow$ (iii):

L is ultra-maximum

$$\Leftrightarrow \delta\big(B^d,(I+A)L\big) = \delta(B^d,L)\Big(1+\min\{dA\cdot l\otimes l \,:\, l\in M_L\}+O\big(\|A\|^2\big)\Big)$$

$$\leq \delta(B^d,L)\big(1-\text{const}\|A\|\big) \text{ as } A\to O, A\in\mathcal{T}$$

$$\Leftrightarrow 1+d\min\{A\cdot l\otimes l\}\leq 1-\text{const}\|A\| \text{ as } A\to O,\ A\in\mathcal{T}$$

$$\Leftrightarrow \min\{A\cdot l\otimes l \,:\, l\in M_L\}\leq -\text{const}\|A\| \text{ for } A\in\mathcal{T}$$

$$\Leftrightarrow \min\big\{A\cdot(l\otimes l)^{\mathcal{T}} \,:\, l\in M_L\big\}<0 \text{ for all } A\in\mathcal{T}\backslash\{O\}$$

$$\Leftrightarrow O=I^{\mathcal{T}}\in\operatorname{relint}_{\mathcal{T}}\operatorname{conv}\big\{(l\otimes l)^{\mathcal{T}} \,:\, l\in M_L\big\}$$

$$\Leftrightarrow I\in\operatorname{pos}\{l\otimes l \,:\, l\in M_L\} \text{ since } I\cdot l\otimes l = l\cdot l>0,$$

$$\mathbb{E}^{\frac{1}{2}d(d+1)} = \operatorname{lin}\{l \otimes l \,:\, l \in M_L\}$$

$\Leftrightarrow L$ is eutactic and perfect.

(ii) $\Leftrightarrow$ (iii):

L is maximum

$$\Leftrightarrow \delta\big(B^d, (I+A)L\big) \le \delta(B^d, L) \text{ for } A \to O,\ A \in \mathcal{T}$$

$$\Leftrightarrow \min\big\{1 + 2A \cdot l \otimes l + A^2 \cdot l \otimes l \,:\, l \in M_L\big\}$$

$$\le \left(1 - \frac{1}{2}\|A\|^2 + O\big(\|A\|^3\big)\right)^{\frac{2}{d}} \quad \text{as } A \to O,\ A \in \mathcal{T}$$

$$\Leftrightarrow \min\left\{A \cdot l \otimes l + \frac{1}{2} A^2 \cdot l \otimes l \,:\, l \in M_L\right\} \le 1 - \frac{1}{d}\|A\|^2 + O\big(\|A\|^3\big)$$

as $A \to O,\ A \in \mathcal{T}$

$$\Rightarrow \min\{A \cdot l \otimes l \,:\, l \in M_L\} \le -\frac{1}{d}\|A\|^2 \quad \text{as } A \to O,\ A \in \mathcal{T}$$

$$\Rightarrow \min\{A \cdot l \otimes l \,:\, l \in M_L\} < 0 \quad \text{for } A \in \mathcal{T} \backslash \{O\}$$

$\dots$

$\Rightarrow L$ is eutactic and perfect

$\Rightarrow L$ is ultra-maximum. ■

5. Maximum Properties of the Product of the Lattice Packing Density and Its Polar

This section deals with refined maximum properties of the expressions

$$\delta(C, \cdot)\,\delta(C^*, \cdot^*)$$

in a neighborhood of a lattice L, where * indicates polarity. In particular, we consider the case when $C = B^d$ and thus $C^* = B^d$, which has been studied before by Bergé and Martinet [12]. There are many results of a related type in the geometry of numbers, see [46], Sect. 14, in convex geometry, see [37], and in the asymptotic theory of normed spaces, see Gruber [38], and Sects. 8 and 9. Related results hold for

$$\frac{\delta(C,\cdot)}{\delta(C,L)}+\frac{\delta(C^*,\cdot^*)}{\delta(C^*,L^*)}.$$

Since the results for the weighted sum are very similar to those for the product of the densities, only results for the latter will be presented.

Let C be an o-symmetric, smooth and strictly convex body and L a lattice in $\mathbb{E}^d$. The *polar body* C^* and the *polar lattice* L^* are defined by

$$C^*=\{y\in\mathbb{E}^d\,:\,x\cdot y\le 1 \text{ for } x\in C\},\quad L^*=\{m\in\mathbb{E}^d\,:\,l\cdot m\in\mathbb{Z}\text{ for } l\in L\}.$$

Then $d(L)\,d(L^*)=1$ and $(BL)^*=B^{-T}L^*$ for non-singular $B\in\mathbb{E}^{d^2}$.

Dual Maximum and Voronoĭ Type Properties

The lattice L is *dual semi-stationary, dual stationary, dual maximum,* or *dual ultra-maximum* with respect to the product $\delta(B^d,\cdot)\,\delta(B^d,\cdot^*)$, if

$$\frac{\delta\big(B^d,(I+A)L\big)\,\delta\Big(B^d,\big((I+A)L\big)^*\Big)}{\delta(B^d,L)\,\delta(B^d,L^*)}\left\{\begin{array}{l}\le 1+o\big(\|A\|\big)\\ =1+o\big(\|A\|\big)\\ \le 1\\ \le 1-\text{const}\|A\|\end{array}\right\}$$

$$\text{as}\quad A\to O,\quad A\in\mathcal{S}.$$

The lattice L or its first layer M_L is *dual semi-eutactic, dual eutactic, dual strongly eutactic* or *dual perfect* with respect to B^d, if

$$\sum_{l\in M_L}\lambda_l\,l\otimes l=\sum_{m\in M_{L^*}}\mu_m\,m\otimes m\ne O \text{ with suitable } \lambda_l,$$

$$\mu_m\left\{\begin{array}{l}\ge 0\\ >0\\ =\text{const}\end{array}\right\},\quad\text{resp.}$$

$$\mathbb{E}^{d^2}=\text{lin}\left(\{l\otimes l\,:\,l\in M_L\}\cup\{m\otimes m\,:\,m\in M_{L^*}\}\right),$$

where const for λ_l may be different from const for μ_m.

Characterization of Dual Semi-stationary and Dual Ultra-Extreme Lattices

In analogy to Theorems 6 and 7, we have the following results:

Theorem 10. *The following properties* (i) *and* (ii) *of* $\delta(B^d, \cdot)\,\delta(B^d, \cdot^*)$ *and* L *are equivalent:*

(i) *L is dual semi-stationary.*
(ii) *L is dual semi-eutactic.*

Theorem 11. *The following properties of* $\delta(B^d, \cdot)\,\delta(B^d, \cdot^*)$ *and* L *are equivalent:*

(i) *L is dual ultra-maximum.*
(ii) *L is dual maximum.*
(iii) *L is dual perfect and dual eutactic.*

The equivalence of (ii) and (ii i) is due to Bergé and Martinet [12].

Extension to Smooth Convex Bodies

Theorems 10 and 11 continue to hold with C instead of B^d, omitting statement (ii) in Theorem 11.

Proof of Theorem 11

To show the additional arguments needed for the proofs of these results, we present the proof of Theorem 11.

First, some tools are put together. Since $A = A^T$ for $A \in \mathcal{T}$, we have

$$\big((I+A)L\big)^* = (I+A)^{-T}L^* = \big(I - A + A^2 - + \cdots\big)\,L^* \quad \text{for} \quad A \in \mathcal{T},\ \|A\| < 1.$$

Thus,

$$(l + Al)^2 = \|l\|^2\big(1 + 2A\cdot n\otimes n + (An)^2\big),$$

$$\big(m - Am + A^2m - + \cdots\big)^2$$

$$= \|m\|^2\big(1 - 2A\cdot p\otimes p + 3A^2\cdot p\otimes p + O\big(\|A^3\|\big)\big),$$

where $n = l/\|l\|$, $p = m/\|m\|$. This, together with the definitions of λ and δ, yields the following equalities,

$$\begin{aligned}
&\delta\big(B^d, (I+A)L\big)\\
&\quad = \delta(B^d, L) \min\big\{ (l+Al)^2 \,:\, l \in M_L \big\}^{\frac{d}{2}} \det\,(I+A)^{-1}\\
&\quad = \delta(B^d, L) \min\Big\{ 1 + 2dA \cdot n \otimes n + O\big(\|A\|^2\big) \,:\, l \in M_L \Big\}\\
&\qquad \left(1 - \frac{1}{2}\|A\|^2 + O\big(\|A\|^3\big)\right)^{-1}\\
&\quad = \delta(B^d, L)\big(1 + d \min\{A \cdot n \otimes n \,:\, l \in M_L\} + O\big(\|A\|^2\big)\big)\\
&\quad \text{as } A \to O,\ A \in \mathcal{T},\ \text{where } O\big(\|A\|^2\big) \geq 0,\\
&\delta\big(B^d, \big((I+A)L\big)^*\big)\\
&\quad = \delta(B^d, L^*) \min\Big\{ 1 - dA \cdot p \otimes p + O\big(\|A\|^2\big) \,:\, m \in M_{L^*} \Big\}\\
&\qquad \big(\det\,(I+A)^{-T}\big)^{-1}\\
&\quad = \delta(B^d, L^*)\big(1 + d \min\{-A \cdot p \otimes p \,:\, m \in M_{L^*}\} + O\big(\|A\|^2\big)\big)\\
&\quad \text{as } A \to O,\ A \in \mathcal{T},\ \text{where } O\big(\|A\|^2\big) \geq 0.
\end{aligned}$$

(i) $\Leftrightarrow$ (iii):

$$\begin{aligned}
&\quad L \text{ is dual ultra-maximum}\\
&\Leftrightarrow\ \delta\big(B^d, (I+A)L\big)\, \delta\big(B^d\big((I+A)L\big)^*\big)\\
&\quad \leq \delta(B^d, L)\, \delta(B^d, L^*)\big(1 - \text{const}\|A\|\big)\\
&\quad \text{as }\ A \to O,\quad A \in \mathcal{T}\\
&\Leftrightarrow\ \big(1 + d \min\{\ \} + O\big(\|A\|^2\big)\big)\big(1 - d \max\{\ \} + O\big(\|A\|^2\big)\big)\\
&\quad \leq 1 - \text{const}\|A\|\\
&\quad \text{as }\ A \to O,\quad A \in \mathcal{T}\\
&\Leftrightarrow\ \min\{A \cdot n \otimes n \,:\, l \in M_L\} - \max\{A \cdot p \otimes p \,:\, m \in M_{L^*}\}\\
&\quad \leq -\text{const}\|A\|
\end{aligned}$$

$$
\begin{aligned}
&\text{as}\quad A\to O,\quad A\in\mathcal{T}\\
\Leftrightarrow\ &\min\left\{A\cdot(n\otimes n)^{\mathcal{T}}\ :\ l\in M_L\right\}\\
&<\max\left\{A\cdot(p\otimes p)^{\mathcal{T}}\ :\ m\in M_{L^*}\right\}\ \text{for } A\in\mathcal{T}\backslash\{O\}\\
\Leftrightarrow\ &\operatorname{relint}\operatorname{conv}\left\{(n\otimes n)^{\mathcal{T}},\ l\in M_L\right\}\\
&\cap\operatorname{relint}\operatorname{conv}\left\{(p\otimes p)^{\mathcal{T}}\ :\ m\in M_{L^*}\right\}\neq\emptyset\\
\Leftrightarrow\ &\sum_{l\in M_L}\lambda_l\, n\otimes n=\sum_{m\in M_{L^*}}\mu_m\, p\otimes p\neq O\ \text{with suitable } \lambda_l,\ \mu_m>O,\\
&\mathbb{E}^{\frac{1}{2}d(d+1)}=\operatorname{lin}\big(\{n\otimes n\ :\ l\in M_L\}\cup\{p\otimes p\ :\ m\in M_{L^*}\}\big)\\
\Leftrightarrow\ &L \text{ is dual eutactic and dual perfect.}
\end{aligned}
$$

(ii) $\Leftrightarrow$ (iii): See [12]. ■

6. Minimum Properties of Zeta Functions

Let L be a lattice in $\mathbb{E}^d$ with $d(L)=1$. The *Epstein zeta function* of L then is defined by

$$\zeta(L,s)=\sum_{l\in L\backslash\{o\}}\frac{1}{\|l\|^s}\quad\text{for}\quad s>d.$$

It plays an important role in crystal physics, hydrodynamics, numerical integration and other areas. It has been investigated ever since its discovery by Epstein and its re-discovery by Sobolev in his work on numerical integration. For several applications and in the context of the geometry of numbers a major problem on the zeta function is to study for a fixed $s>d$, for all sufficiently large s, or for all $s>d$ the lattices L with $d(L)=1$ for which $\zeta(\cdot,s)$ is (locally) minimum.

A *layer* of L consists of all vectors of $L\backslash\{o\}$ with the same norm. Order the layers by the norm of their vectors. The first layer then coincides with the set of minimum points of L with respect to B^d. Delone and Ryshkov [21] showed that a lattice L is minimum with respect to $\zeta(\cdot,s)$ for all sufficiently large s if and only if L is perfect and each layer is strongly eutactic or, in a different terminology, a spherical 2-design. If each layer of L is a spherical 4-design, then L minimizes $\zeta(\cdot, s)$ for each $s > d$, as shown by Coulangeon [19]. A different sufficient condition is due to Sarnak and Strömbergsson [68]. These authors show that many important lattices are minimum for each $s > d$, one example is the Leech lattice.

We characterize the lattices which, for given $s > d$ are stationary and quadratic minimum with respect to $\zeta(\cdot, s)$. This yields characterizations in other cases. Perhaps more important for applications are simple sufficient conditions. We state several such conditions, including one using automorphism groups. Finally, a relation to lattice packing of balls is mentioned. Most of these results can be extended to general zeta functions $\zeta_C(\cdot, s)$.

Minimum and Voronoĭ Type Properties, Spherical Designs and Automorphism Groups

Remark 3. Since $\zeta(\cdot, s)$ and $\zeta_C(\cdot, s)$ have the additional parameter s, it is not surprising that there are more properties needed than mere eutaxy, strong eutaxy, or perfection, to characterize the lattices L which are stationary, minimum, or quadratic minimum with respect to $\zeta(\cdot, s)$ or $\zeta_C(\cdot, s)$. The following stronger forms of eutaxy and perfection, together with automorphism groups, seem to be appropriate tools for such characterizations.

Let $s > d$. Then L is said to be *stationary, minimum,* or *quadratic minimum* with respect to $\zeta(\cdot, s)$, if

$$\frac{\zeta\left(\dfrac{I+A}{\det\,(I+A)^{\frac{1}{d}}}L, s\right)}{\zeta(L, s)} \left\{\begin{aligned} &= 1 + o\big(\|A\|\big) \\ &\geq 1 \\ &\geq 1 + \mathrm{const}\|A\|^2 \end{aligned}\right\} \quad \text{as} \quad A \to O, \quad A \in \mathcal{T}.$$

Let M be a finite, o-symmetric subset of S^{d-1} and put $\zeta = \zeta(L, s)$. The set M is a *spherical n-design* if the following identity holds for any polynomial $p : \mathbb{E}^d \to \mathbb{R}$ of degree at most n:

$$\int_{S^{d-1}} p(u)\, d\sigma(u) = \frac{1}{\# M} \sum_{l \in M} p(l).$$

Here σ is the usual rotation invariant area measure on S^{d-1}, normalized so that $\sigma\big(S^{d-1}\big) = 1$ and # stands for cardinal number. Venkov [79] showed that

the set M is a spherical n-design if and only if

$$\sum_{l\in M} (l\cdot x)^n = \text{const}\|x\|^n \quad \text{for} \quad x \in \mathbb{E}^d.$$

Let M be a layer of L. Then M is *strongly eutactic* or, after a suitable normalization, a *spherical* 2-*design*, if it satisfies one of the following equivalent conditions:

$$\sum_{l\in M} \frac{l\otimes l}{\|l\|^2} = \lambda I, \quad \text{or}$$

$$\sum_{l\in M} \frac{(l\cdot x)^2}{\|l\|^2} = \lambda \|x\|^2 \ \text{ for } \ x\in\mathbb{E}^d, \ \text{ where } \ \lambda = \frac{\#M}{d}, \ \text{ or}$$

$$\sum_{l\in M} A\cdot l\otimes l = 0 \ \text{ for } \ A\in\mathcal{T}.$$

The lattice L is *strongly eutactic,* if its first layer is. It is *fully eutactic* with respect to $\zeta(\cdot, s)$, if one of the following equivalent conditions holds:

$$\sum_{l\in L\backslash\{o\}} \frac{l\otimes l}{\|l\|^{s+2}} = \frac{\zeta}{d} I, \quad \text{or} \quad \sum_{l\in L\backslash\{o\}} \frac{(l\cdot x)^2}{\|l\|^{s+2}} = \frac{\zeta}{d}\|x\|^2 \ \text{ for } \ x\in\mathbb{E}^d, \quad \text{or}$$

$$\sum_{l\in L\backslash\{o\}} \frac{A\cdot l\otimes l}{\|l\|^{s+2}} = 0 \ \text{ for } \ A\in\mathcal{T}.$$

Refined versions of these notions are the following: a layer M is *ultra-eutactic* or a *spherical* 4-*design,* if one of the following conditions holds:

$$\sum_{l\in M} \frac{(l\cdot x)^4}{\|l\|^4} = \mu\|x\|^4 \ \text{ for } \ x\in\mathbb{E}^d, \ \text{ or, equivalently,}$$

$$\sum_{l\in M} \frac{(A\cdot l\otimes l)^2}{\|l\|^4} = \frac{2\mu}{3}\|A\|^2 + \frac{\mu}{3}(\operatorname{tr} A)^2 \ \text{ for } \ A\in\mathbb{E}^{\frac12 d(d+1)},$$

$$\text{where } \ \mu = \frac{3\#M}{d(d+2)}.$$

The lattice L is *completely eutactic* with respect to $\zeta(\cdot, s)$, if the one of the following properties holds:

$$\sum_{l \in L \backslash \{o\}} \frac{(l \cdot x)^4}{\|l\|^{s+4}} = \nu \, \|x\|^4 \text{ for } x \in \mathbb{E}^d, \quad \text{or, equivalently,}$$

$$\sum_{l \in L \backslash \{o\}} \frac{(A \cdot l \otimes l)^2}{\|l\|^{s+4}} = \frac{2\nu}{3} \, \|A\|^2 + \frac{\nu}{3} (\operatorname{tr} A)^2 \quad \text{for} \quad A \in \mathbb{E}^{\frac{1}{2}d(d+1)},$$

$$\text{where} \quad \nu = \frac{3\zeta}{d(d+2)}.$$

Finally, the layer M is *perfect,* if

$$\mathbb{E}^{\frac{1}{2}d(d+1)} = \operatorname{lin}\{l \otimes l \, : \, l \in M\}.$$

If the first layer of L is perfect, then L is *perfect.* The *automorphism* or *symmetry group* $\mathcal{A} = \mathcal{A}(L)$ of L is the group of all orthogonal transformations of $\mathbb{E}^d$ which map L onto itself.

Characterization of Stationary and Quadratic Minimum Lattices

Note that in contrast to the situation for densities, for zeta functions a semi-stationary lattice is already stationary.

Theorem 12. *Let $s > d$. Then the following properties of $\zeta(\cdot, s)$ and L are equivalent:*

(i) *L is stationary for s.*
(ii) *L is fully eutactic for s.*

Corollary 8. *The following properties of $\zeta(\cdot, \cdot)$ and L are equivalent:*

(i) *L is stationary for each $s > d$.*
(ii) *Each layer of L is strongly eutactic.*

Theorem 13. *Let $s > d$. Then the following properties of $\zeta(\cdot, s)$ and L are equivalent:*

(i) *L is quadratic minimum for s.*
(ii) *L is fully eutactic for s and satisfies the inequality*

$$\sum_{l \in L \backslash \{o\}} \frac{(A \cdot l \otimes l)^2}{\|l\|^{s+4}} > \frac{2\zeta(L, s)}{d(s+2)} \|A\|^2 \quad \text{for} \quad A \in \mathcal{T} \backslash \{O\}.$$

Corollary 9. *The following properties of $\zeta(\cdot,\cdot)$ and L are equivalent:*

(i) *L is quadratic minimum for each $s > d$.*

(ii) *Each layer of L is strongly eutactic and*

$$\sum_{l \in L\setminus\{o\}} \frac{(A \cdot l \otimes l)^2}{\|l\|^{s+4}} > \frac{2\zeta(L,s)}{d(s+2)}\|A\|^2 \text{ for each } s > d \text{ and } A \in \mathcal{T}\setminus\{O\}.$$

This yields, in particular, Coulangeon's criterion.

While Theorem 13 yields a characterization of the lattices, which are quadratic minimum for arbitrarily large s, in several cases the following sufficient conditions are more convenient to apply.

Corollary 10. *Each of the following two conditions is sufficient for L to be quadratic minimum with respect to $\zeta(\cdot, s)$ for all sufficiently large s:*

(i) *L is perfect and the automorphism group $\mathcal{A}(L)$ is transitive on the first layer of L.*

(ii) *L is perfect and each layer is strongly eutactic.*

Similarly, each of the following two conditions is sufficient for L to be quadratic minimum for each $s > d$:

(iii) *Each layer of L is ultra-eutactic.*

(iv) *L is completely eutactic for each $s > d$.*

Bravais Types of Lattices with Minimum Properties

The next result is a consequence of Theorem 4 and Corollaries 8 and 10:

Corollary 11. *In $\mathbb{E}^2$ and $\mathbb{E}^3$, it is precisely the following lattices of determinant 1 which are stationary, resp. quadratic minimum with respect to $\zeta(\cdot, s)$ for all $s > d$.*

$d = 2:$	*tp square*	$d = 3:$	*cP*	*cubic primitive*
	hp hexagonal		*cF*	*cubic face centered*
			cI	*cubic body centered,*

resp.

$d = 2:$	*hp hexagonal*	$d = 3:$	*cF*	*cubic face centered.*

For general d, there are, up to orthogonal transformations, only finitely many lattices of determinant 1 which are stationary, resp. quadratic minimum with respect to $\zeta(\cdot, s)$ for all $s > d$.

Zeta Functions and Ball Packing

A lattice which is quadratic minimum with respect to $\zeta(\cdot, s)$ for all sufficiently large s is perfect and each layer is strongly eutactic as can be shown by means of Theorem 13. Hence, by Theorem 7, the following remark holds:

Corollary 12. *Each lattice which is quadratic minimum with respect to $\zeta(\cdot, s)$ for all sufficiently large s, is ultra-maximum with respect to $\delta(B^d, \cdot)$.*

General Lattice Zeta Functions

Our next aim is to extend the above results to a more general type of zeta functions on lattices. Let C be a smooth, o-symmetric convex body, $\|\cdot\|_C$ the corresponding norm on $\mathbb{E}^d$ and L a lattice with $d(L) = 1$. The function ζ_C, defined by

$$\zeta_C(L, s) = \sum_{l \in L \setminus \{o\}} \frac{1}{\|l\|_C^s} \text{ for } s > d,$$

is called a *lattice zeta function* on $\mathbb{E}^d$.

Minimum and Voronoĭ Type Properties

Let $s > d$. The concepts of *stationary, minimum* or *quadratic minimum lattice* with respect to $\zeta_C(\cdot, s)$ are defined as earlier for B^d with $\mathcal{T}$ replaced by the subspace

$$\mathcal{S} = \left\{ A \in \mathbb{E}^{d^2} : \operatorname{tr} A = A \cdot I = 0 \right\}.$$

Similarly, the notions of *layer* and *eutactic, strongly eutactic, fully eutactic* and *perfect lattice* with respect to C or $\zeta_C(\cdot, s)$ are defined as before, but with C, $\|\cdot\|_C$, M_L, $u \otimes n$, ζ_C, $\mathcal{S}$, $\mathbb{E}^{d^2}$ instead of B^d, $\|\cdot\|$, M_L, $n \otimes n$, ζ, $\mathcal{T}$ and $\mathbb{E}^{\frac{1}{2}d(d+1)}$, respectively.

Characterization of Stationary and Quadratic Minimum Lattices

The extension of Theorem 12 is as follows:

Theorem 14. *Let $s > d$. Then the following properties of $\zeta_C(\cdot, s)$ and L are equivalent:*

(i) *L is stationary for s,*
(ii) *L is fully eutactic for s.*

Also Theorem 13 can be extended, but the corresponding necessary and sufficient condition for L to be quadratic minimum with respect to $\zeta_C(\cdot, s)$ for given $s > d$ is difficult to check. Thus we prefer to state the following result.

Theorem 15. *Let C be of class $\mathcal{C}^2$. Then the following properties of $\zeta_C(\cdot, \cdot)$ and L are equivalent:*

(i) *L is quadratic minimum for all sufficiently large s.*
(ii) *L is perfect and each layer is strongly eutactic.*

Corollary 10, with suitable modifications, holds also for C instead of B^d.

Open Problems

The earlier characterizations and sufficient conditions guarantee in a series of cases that L is stationary, minimum, or quadratic minimum with respect to $\zeta(\cdot, s)$, or $\zeta_C(\cdot, s)$ for a given $s > d$, for all sufficiently large s, or for all $s > d$. The problem arises, to make this family of results complete. We state one particular problem.

Problem 6. If there are lattices, which are minimum (but not quadratic minimum) with respect to $\zeta(\cdot, s)$ or $\zeta_C(\cdot, s)$ for a given $s > d$, for all sufficiently large s, and for all $s > d$, characterize the minimum lattices.

The next problem is related to Problem 5.

Problem 7. Show that in all sufficiently high dimensions, for most o-symmetric convex bodies C there are lattices which are quadratic minimum with respect to $\zeta(\cdot, s)$ and $\zeta_C(\cdot, s)$ respectively, for a given s, for all sufficiently large s, or for all $s > d$.

A positive answer to this problem would settle also the question of the existence of convex bodies with eutactic and perfect lattices, see Theorem 15.

Problem 8. Is it true, that in all sufficiently high dimensions and for most o-symmetric convex bodies C, the lattice L with $d(L) = 1$, for which $\zeta_C(\cdot, s)$ attains its absolute minimum for a given s, for all sufficiently large s, or for all $s > d$, has the following properties:

(i) L is unique,

(ii) L is quadratic minimum?

Proof of Theorem 13

To show the idea of the proofs, we present the following proof of Theorem 13. The equality

$$\|l + Al\|^2 = \|l\|^2(1 + 2A \cdot n \otimes n + A^2 \cdot n \otimes n)$$

$$\text{for} \quad A \in \mathbb{E}^{\frac{1}{2}d(d+1)}, \quad l \in \mathbb{E}^d \backslash \{o\}, \quad n = \frac{l}{\|l\|}$$

implies the formula

$$\begin{aligned}\frac{1}{\|l + Al\|^s} &= \frac{1}{\|l\|^s}\big(1 + 2A \cdot n \otimes n + A^2 \cdot n \otimes n\big)^{-\frac{s}{2}}\\ &= \frac{1}{\|l\|^s}\bigg(1 - sA \cdot n \otimes n - \frac{s}{2}A^2 \cdot n \otimes n\\ &\quad + \frac{s(s+2)}{2}(A \cdot n \otimes n)^2 + O\big(\|A\|^3\big)\bigg) \quad \text{as} \quad A \to O, \quad A \in \mathcal{T},\end{aligned}$$

which, in turn, yields the following identity, where the summation is over $l \in L\backslash\{o\}$ and ζ stands for $\zeta(L, s)$:

$$\begin{aligned}&\zeta\big((I + A)L, s\big)\\ &\quad = \zeta - sA \cdot \sum \frac{l \otimes l}{\|l\|^{s+2}} - \frac{s}{2}A^2 \cdot \sum \frac{l \otimes l}{\|l\|^{s+2}} + \frac{s(s+2)}{2} \sum \frac{(A \cdot l \otimes l)^2}{\|l\|^{s+4}}\\ &\qquad + O\big(\|A\|^3\big)\end{aligned}$$

$$= \zeta - sA \cdot \frac{\zeta}{d} I - \frac{s}{2} A^2 \cdot \frac{\zeta}{d} I + \frac{s(s+2)}{2} \sum \frac{(A \cdot l \otimes l)^2}{\|l\|^{s+4}} + O\big(\|A\|^3\big)$$

$$= \zeta - \frac{s\zeta}{2d} \|A\|^2 + \frac{s(s+2)}{2} \sum \frac{(A \cdot l \otimes l)^2}{\|l\|^{s+4}} + O\big(\|A\|^3\big)$$

as $A \to O,\ A \in \mathcal{T},$ if L is fully eutactic for s.

Note that

$$\zeta\left(\frac{I+A}{\det\,(I+A)^{\frac{1}{d}}} L, s\right) = \zeta\big((I+A)L, s\big) \det\,(I+A)^{\frac{s}{d}}.$$

(i) $\Leftrightarrow$ (ii): Since s is fixed, it is incorporated into const.

L is quadratic minimum for s

$\Leftrightarrow$ L is stationary for s and

$$\zeta\left(\frac{I+A}{\det\,(I+A)^{\frac{1}{d}}} L, s\right) = \zeta\big((I+A)L, s\big) \det\,(I+A)^{\frac{s}{d}}$$
$$\geq \zeta\big(1 + \text{const}\|A\|^2\big)$$

$\Leftrightarrow$ L is fully eutactic with respect to $\zeta(\cdot, s)$ (by Theorem 12) and

$$\zeta - \frac{s\zeta}{2d}\|A\|^2 + \frac{s(s+2)}{2} \sum \frac{(A \cdot l \otimes l)^2}{\|l\|^{s+4}}$$
$$\geq \zeta(1 + \text{const}\|A\|^2) \left(1 - \frac{1}{2}\|A\|^2 + O\big(\|A\|^3\big)\right)^{-\frac{s}{d}} + O\big(\|A\|^3\big)$$

$\Leftrightarrow$ L is fully eutactic for s and

$$\zeta - \frac{s\zeta}{2d}\|A\|^2 + \frac{s(s+2)}{2} \sum \frac{(A \cdot l \otimes l)^2}{\|l\|^{s+4}}$$
$$\geq \zeta + \frac{s\zeta}{2d}\|A\|^2 + \zeta\text{const}\|A\|^2 + \zeta O\big(\|A\|^3\big)$$

$\Leftrightarrow$ L is fully eutactic for s and

$$\sum \frac{(A \cdot l \otimes l)^2}{\|l\|^{s+4}} > \frac{2\zeta}{d(s+2)} \|A\|^2 \text{ as } A \to O,\ A \in \mathcal{T} \backslash \{O\}.$$

In the last equivalence the implication $\Rightarrow$ is clear. To see the reverse implication $\Leftarrow$, note that the expression

$$\sum \frac{(A \cdot l \otimes l)^2}{\|l\|^{s+4}} - \frac{2\zeta}{d(s+2)} \|A\|^2$$

may be considered to be a quadratic form in the variable $A \in \mathcal{T}$. It is, obviously, positive definite and thus bounded below by $\text{const}\|A\|^2$ for a suitable constant. ■

7. Minimum Properties of the Product of Zeta Functions and Their Polars

Let L be a lattice with $d(L) = 1$. This section deals with minimum properties of the quantity

$$\zeta(\cdot, s)\, \zeta(\cdot^*, s)$$

on the space of lattices of determinant 1. We characterize dual stationary and dual quadratic minimum lattices, both, for B^d and C. Similar results hold for

$$\zeta_C(\cdot, s) + \zeta_{C^*}(\cdot^*, s).$$

Minimum and Voronoĭ Type Properties

Let $s > d$. The lattice L is *dual stationary, dual minimum,* or *dual quadratic minimum* with respect to $\zeta(\cdot, s)\, \zeta(\cdot^*, s)$, if

$$\frac{\zeta\left(\dfrac{I+A}{\det\,(I+A)^{\frac{1}{d}}} L, s\right) \zeta\left(\left(\dfrac{I+A}{\det\,(I+A)^{\frac{1}{d}}} L\right)^*, s\right)}{\zeta(L, s)\, \zeta(L^*, s)} \left\{ \begin{matrix} = 1 + o\big(\|A\|\big) \\ \geq 1 \\ \geq 1 + \text{const}\, s^2 \|A\|^2 \end{matrix} \right\}$$

$$\text{as} \quad A \to O, \quad A \in \mathcal{T}.$$

Let L_i, L_i^* be the layers of L, resp. L^*, $i = 1, 2, \dots$. We use the abbreviations $\zeta = \zeta(L, s)$ and $\zeta^* = \zeta(L^*, s)$. Call L_i *dual strongly eutactic* with respect to C, C^*, if

$$\frac{1}{\# L_i} \sum_{l \in L_i} \frac{l \otimes l}{\|l\|^2} = \frac{1}{\# L_i^*} \sum_{m \in L_i^*} \frac{m \otimes m}{\|m\|^2}.$$

The lattice L is *dual fully eutactic* with respect to $\zeta\zeta^*$ for s, if

$$\frac{1}{\zeta} \sum_{l \in L \setminus \{o\}} \frac{l \otimes l}{\|l\|^{s+2}} = \frac{1}{\zeta^*} \sum_{m \in L^* \setminus \{o\}} \frac{m \otimes m}{\|m\|^{s+2}}, \quad \text{or, equivalently,}$$

$$\frac{1}{\zeta} \sum_{l \in L \setminus \{o\}} \frac{A \cdot l \otimes l}{\|l\|^{s+2}} = \frac{1}{\zeta^*} \sum_{m \in L^* \setminus \{o\}} \frac{A \cdot m \otimes m}{\|m\|^{s+2}} \quad \text{for} \quad A \in \mathcal{T}.$$

The layer L_i, is *dual ultra-eutactic* with respect to C, C^*, if

$$\frac{1}{\# L_i} \sum_{l \in L_i} \frac{(A \cdot l \otimes l)^2}{\|l\|^4} = \frac{1}{\# L_i^*} \sum_{m \in L_i^*} \frac{(A \cdot m \otimes m)^2}{\|m\|^4} \quad \text{for} \quad A \in \mathcal{T}.$$

Characterization of Dual Stationary and Dual Quadratic Minimum Lattices

From a series of results we select from each of our two extremality types a characterization result and a sufficient condition.

Theorem 16. *Let $s > d$. Then the following properties of $\zeta(\cdot, s)\,\zeta(\cdot^*, s)$ and L are equivalent:*

(i) *L is dual stationary.*
(ii) *L is dual fully eutactic.*

Corollary 13. *Each of the following conditions is sufficient for L to be dual stationary with respect to $\zeta(\cdot, s)\,\zeta(\cdot^*, s)$ for each $s > d$.*

(i) *The first layer of L is perfect and $\mathcal{A}$ operates transitively on it.*
(ii) *L is dual fully eutactic for any $s > d$.*

For quadratic minimality the result is rather lengthy.

Theorem 17. *Let $s > d$. Then the following properties of $\zeta(\cdot, s)\,\zeta(\cdot^*, s)$ and L are equivalent:*

(i) *L is dual quadratic minimum for s.*

(ii) *L is dual fully eutactic for s and satisfies the inequality,*

$$(s+2)\left(\frac{1}{\zeta}\sum_{l\in L\setminus\{o\}}\frac{(A\cdot l\otimes l)^2}{\|l\|^{s+4}}+\frac{1}{\zeta^*}\sum_{m\in L^*\setminus\{o\}}\frac{(A\cdot m\otimes n)^2}{\|m\|^{s+4}}\right)$$

$$>\frac{4}{\zeta}A^2\sum_{l\in L\setminus\{o\}}\frac{l\otimes l}{\|l\|^{s+2}}+\frac{2s}{\zeta}\left(A\cdot\sum_{l\in L\setminus\{o\}}\frac{l\otimes l}{\|l\|^{s+2}}\right)^2$$

$$\text{as}\quad A\to O,\quad A\in\mathcal{T}\setminus\{O\}.$$

Corollary 14. *The following condition is sufficient that L be dual quadratic minimum with respect to $\zeta(\cdot, s)\,\zeta(\cdot^*, s)$ for each $s > d$: Each layer of L is ultra-eutactic and dual ultra-eutactic.*

Zeta Functions and Ball Packing

Also in the duality case, there is a relation between products of zeta functions and densities of ball packings, see [45].

Extension to General Zeta Functions

Finally, we mention that a good many of the duality results for the Epstein zeta function can be extended to the more general lattice zeta functions ζ_C defined by means of a smooth and strictly convex o-symmetric convex body C.

8. John Type Results and Minimum Ellipsoidal Shells

This and the next section contain results of John type and minimum position results from the asymptotic theory of normed spaces. If not stated otherwise, the results are from the article [45] of the author. Let C be a convex body. Then there is an inscribed ellipsoid of maximum volume and a circumscribed ellipsoid of minimum volume. The uniqueness of both ellipsoids was proved by Danzer, Laugwitz and Lenz [20]. John [49] specified conditions which an inscribed ellipsoid of maximum volume must satisfy. That these conditions are sufficient was shown by Pełczyński [58] and Ball [5].

We state and prove a precise version of John's theorem, specify for typical convex bodies the number of contact points between (the boundaries of) the body and the unique inscribed ellipsoid of maximum volume. Analogous results are considered for minimal ellipsoidal shells. Minimal ellipsoidal shells are unique for typical, but not for all convex bodies.

John's Ellipsoid Theorem

In the case when C is o-symmetric, the result is as follows:

Theorem 18. *Let $B^d \subseteq C$. Then the following properties are equivalent:*

(i) *B^d is the unique ellipsoid of maximum volume contained in C.*

(ii) *There is a finite set $M = \{\pm u_1, \dots, \pm u_k\}$ of common boundary points of B^d and C – such points are called* contact points *of B^d, C – such that*

$$I = \sum_{u \in M} \lambda_u \, u \otimes u \quad \textit{with suitable} \quad \lambda_u > 0 \quad \textit{and} \quad k \leq \frac{d(d+1)}{2}.$$

This result, or versions of it, was proved and refined many times. We mention Bastero and Romance [9], Giannopoulos, Peressinaki and Tsolomitis [29], Gordon, Litvak, Meyer and Pajor [31] and the author and Schuster [47]. The later proof is taken from [47] and fits into the present context.

John's theorem and its dual counterpart, the characterization of the unique circumscribed ellipsoid of minimum volume, has generated a voluminous literature both in convex geometry and the asymptotic theory of normed spaces. It includes various versions, extensions and new proofs of these characterizations, and applications to normed spaces, in particular, the following one, where the *Banach-Mazur distance* between two norms $\|\cdot\|_C$, $\|\cdot\|_D$ on $\mathbb{E}^d$ with unit balls C, D, is defined by

$$\begin{aligned}\delta^{BM}\big(\|\cdot\|_C, \|\cdot\|_D\big) &= \delta^{BM}(C, D)\\ &= \inf\big\{\lambda \geq 1 \,:\, C \subseteq AD \subseteq \lambda C,\ A \in \mathbb{E}^{d^2}\big\}.\end{aligned}$$

Corollary 15. *Let $\|\cdot\|_C$ be an arbitrary norm and $\|\cdot\|$ the usual Euclidean norm on $\mathbb{E}^d$. Then*

$$\delta^{BM}\big(\|\cdot\|, \|\cdot\|_C\big) \leq \sqrt{d}.$$

This result and its proof based on John's theorem are well known. For a reason which will be explained later, it is a bit surprising that John's theorem yields this estimate, see Corollary 16.

The Contact Number of Typical Convex Bodies

Given a convex body, the question arises, how many contact points are there between the convex body and its volume maximizing inscribed, resp. its volume minimizing circumscribed ellipsoid. A result of the author [34] gives the following answer:

Theorem 19. *For most o-symmetric convex bodies C the unique o-symmetric inscribed ellipsoid of maximum volume and the unique o-symmetric circumscribed ellipsoid of minimum volume, both have precisely $\frac{1}{2}d(d+1)$ pairs $\pm u$ of contact points with C.*

For an alternative proof of this result see Rudelson [63].

Minimum Ellipsoidal Shells

A pair of solid o-symmetric ellipsoids $\langle E, \varrho E\rangle$ is called a *minimal ellipsoidal shell* of C, if $E \subseteq C \subseteq \varrho E$, where $\varrho \geq 1$ is minimal. It is easy to see that $\varrho = \delta^{BM}\big(\|\cdot\|, \|\cdot\|_C\big)$. Maurey [55] (unpublished) showed that a minimal ellipsoidal shell need not be unique, see Lindenstrauss and Milman [53] and Praetorius [60].

In analogy to John's theorem and its dual, we have the following results due to Gruber [38]:

Theorem 20. *Let $B^d \subseteq C \subseteq \varrho B^d$. Then the following properties are equivalent:*

(i) $\big\langle B^d, \varrho B^d\big\rangle$ *is a (not necessarily unique) minimal ellipsoidal shell of C.*

(ii) *There are contact points $\pm u_1, \dots, \pm u_k \in \operatorname{bd} B^d \cap \operatorname{bd} C$ and $\pm v_1, \dots, \pm v_l \in \operatorname{bd} C \cap \operatorname{bd} \varrho B^d$ and reals $\lambda_1, \dots, \lambda_k$, $\mu_1, \dots, \mu_l > 0$, such that*

 (a) $2 \leq k, l$ *and* $k + l \leq \frac{1}{2}d(d+1) + 1$,

(b) $\sum_{i=1}^{k} \lambda_i \, u_i \otimes u_i = \sum_{j=1}^{l} \mu_j \, v_j \otimes v_j \neq O$,
(c) $\mathrm{lin}\,\{u_1, \dots, u_k\} = \mathrm{lin}\,\{v_1, \dots, v_l\}$.

While there are examples of convex bodies with more than one minimal ellipsoidal shell, this is a rare event, as the next result shows.

Theorem 21. *Most o-symmetric convex bodies C have a unique minimal ellipsoidal shell $\langle E, \varrho E\rangle$. The contact sets $\mathrm{bd}\, E \cap \mathrm{bd}\, C$ and $\mathrm{bd}\, C \cap \mathrm{bd}\, \varrho E$, each consist of at least 2 and at most $\frac{1}{2}d(d+1)-1$, together of $\frac{1}{2}d(d+1)+1$ pairs of points $\pm u$.*

Theorems 19 and 21 yield the following proposition:

Corollary 16. *For most o-symmetric convex bodies C neither the inscribed ellipsoid of maximum, nor the circumscribed ellipsoid of minimum volume, give rise to a minimum ellipsoidal shell.*

Remark 4. By Corollary 16, it is a happy, rather unexpected event, that John's theorem leads to a proof of Corollary 15. Being a characterization of minimal ellipsoidal shells, Theorem 20 should readily imply Corollary 15. This is, in fact, the case as the later proof of Corollary 15 shows.

Proofs of Theorem 18 and Corollary 15

Theorem 18: Let $h_C(v) = \max\{v \cdot x \,:\, x \in C\}$, $v \in \mathbb{E}^d$, be the *support function* of C. Then

$$C = \left\{x \in \mathbb{E}^d \,:\, v \cdot x \leq h_C(v) \text{ for } v \in S^{d-1}\right\}.$$

The set

$$\begin{aligned}\mathcal{E} &= \left\{A \in \mathbb{E}^{\frac{1}{2}d(d+1)} \,:\, AB^d \subseteq C\right\} \cap \mathcal{P}^d \\ &= \bigcap_{\substack{u \in \mathrm{bd}\, B^d \\ v \in S^{d-1}}} \left\{A \in \mathbb{E}^{\frac{1}{2}d(d+1)} \,:\, Au \cdot v = A \cdot v \otimes u \leq h_C(v)\right\} \cap \mathcal{P}^d\end{aligned}$$

represents the set all o-symmetric ellipsoids contained in C. Since $\mathcal{E}$ is the intersection of a family of closed halfspaces and the open convex cone $\mathcal{P}^d$, the set $\mathcal{E}$ is a convex subset of $\mathcal{P}^d$, which is closed in $\mathcal{P}^d$. To the ellipsoid B^d corresponds the matrix $I \in \mathcal{E}$. Let $\mathcal{D}^d = \mathcal{D}^d(1) = \{A \in \mathcal{P}^d \,:\, \det A \geq 1\}$.

(i) ⇔ (ii):

B^d is the unique o-symmetric ellipsoid of maximum volume in C

⇔ $\mathcal{E} \cap \mathcal{D}^d = \{I\}$, i.e. the convex body $\mathcal{E}$ touches the smooth and strictly convex body $\mathcal{D}^d$ only at I

⇔ the interior normal vector I of $\mathcal{D}^d$ at I is contained in the (exterior) normal cone $\mathcal{N}$ of $\mathcal{E}$ at I. Note that $\mathcal{N}$ is generated by the exterior normal vectors of those of the defining halfspaces $\{A : Au \cdot v = A \cdot v \otimes u \le h_C(v)\}$ of $\mathcal{E}$ which contain I as a boundary point. Thus $1 \ge u \cdot v = I \cdot v \otimes u = h_C(v) \ge 1$, and therefore $u \cdot v = h_C(v) = 1$, or $u = v$, $h_C(u) = 1$, or $u \otimes u = v \otimes u$, $u \in \operatorname{bd} B^d \cap \operatorname{bd} C = B^d \cap \operatorname{bd} C$. Thus $\mathcal{N} = \operatorname{pos}\{u \otimes u : u \in B^d \cap \operatorname{bd} C\}$

⇔ by Carathéodory's theorem for cones, we may choose a set of contact points $M = \{\pm u_1, \dots, \pm u_k\} \subseteq B^d \cap \operatorname{bd} C$ such that

$$I = \sum_{u \in M} \lambda_u \, u \otimes u \quad \text{with suitable} \quad \lambda_u > 0 \quad \text{and} \quad k \le \frac{1}{2} d(d+1). \qquad \blacksquare$$

Corollary 15: We may assume that $B^d \subseteq C \subseteq \varrho B^d$, where

$$\varrho = \delta^{BM}\big(\|\cdot\|, \|\cdot\|_C\big).$$

It is sufficient to show that $\varrho \le \sqrt{d}$. Equating the traces of the two sides of the equality in Theorem 20(iib) implies that

$$\sum_i \lambda_i = \sum_i \lambda_i \, u_i \cdot u_i = \sum_j \mu_j \, v_j \cdot v_j = \varrho^2 \sum_j \mu_j .$$

Noting that $\frac{1}{\sqrt{d}} I$ has norm 1 and that $(u_i \cdot v_j)^2 \le 1$, this yields that

$$\Big\| \sum_i \lambda_i \, u_i \otimes u_i \Big\| \ge \sum_i \lambda_i \, u_i \otimes u_i \cdot \frac{1}{\sqrt{d}} I = \frac{1}{\sqrt{d}} \sum \lambda_i \, u_i \cdot u_i = \frac{1}{\sqrt{d}} \sum \lambda_i ,$$

$$\Big(\sum_i \lambda_i \, u_i \otimes u_i \Big)^2 = \sum_i \lambda_i \, u_i \otimes u_i \cdot \sum_j \mu_j \, v_j \otimes v_j = \sum_{i,j} \lambda_i \mu_j \, (u_i \cdot v_j)^2$$

$$\le \sum_i \lambda_i \sum_j \mu_j = \frac{1}{\varrho^2} \Big(\sum_i \lambda_i \Big)^2 ,$$

$$\text{or} \quad \frac{1}{\varrho^2} \ge \frac{1}{d}, \quad \text{or} \quad \varrho \le \sqrt{d}. \qquad \blacksquare$$

9. Minimum Position Problems

Related to John's theorem is the following question: Consider a real function F on the space of all convex bodies or on a suitable subspace of it, for example on the space of all o-symmetric convex bodies, and a group $\mathcal{G}$ of affinities. Assume that this subspace is invariant under the affinities of $\mathcal{G}$. Characterize for a given convex body C in this subspace those among its images under affinities from $\mathcal{G}$, for which F is minimum, the *minimum F-positions of C* with respect to the group $\mathcal{G}$. For numerous pertinent results and applications see Milman and Pajor [56], Giannopoulos and Milman [26, 27], Gordon, Litvak, Meyer and Pajor [31] and the author [38] and the references there.

In the following we state minimum position results of the author [38] which were proved using ideas in the sense of Voronoĭ, while the classical proofs rely on a variational argument, see Giannopoulos and Milman [26]. We characterize circumscribed ellipsoids of minimum surface area and minimum positions for polar moments, mean width and surface area. Let C be an o-symmetric convex body.

Circumscribed Ellipsoids of Minimum Surface Area

In the light of John's theorem and its dual, the following question arises naturally: Given a convex body C, characterize the inscribed and circumscribed ellipsoids of maximum, resp. minimum surface area. Are these unique? Moreover, what are the corresponding minimum positions with respect to the group of volume preserving linear transformations? Can the surface area be replaced by general quermassintegrals?

Theorem 22. *There is a unique ellipsoid of minimum surface area containing C.*

The original proof of the author [38] was rather complicated and made use of projection bodies and Alexandrov's projection theorem. Much easier is the recent proof by Schröcker [76].

Assign to a convex body C the minimum surface area $S_m(C)$ of a circumscribed ellipsoid.

Theorem 23. *Up to rotations, C has a unique minimum S_m-position with respect to the group of volume-preserving linear transformations and the following properties are equivalent:*

(i) *C is in minimum S_m-position and B^d is the circumscribed ellipsoid of minimum surface area.*

(ii) *There are contact points $\pm u_1, \dots, \pm u_k \in \operatorname{bd} B^d \cap \operatorname{bd} C$ and $\lambda_1, \dots, \lambda_k > 0$ such that*

  (a) $d \leq k \leq \frac{1}{2}d(d+1)$,
  (b) $I = \sum \lambda_i \, u_i \otimes u_i$,
  (c) $\mathbb{E}^d = \operatorname{lin}\{u_1, \dots, u_k\}$.

Comparing the dual counterpart of Theorem 18 together with some addenda (see the author and Schuster [47]) and Theorem 23 yields the next result.

Corollary 17. *Let $C \subseteq B^d$. Then the following properties are equivalent:*

(i) *B^d is the unique circumscribed ellipsoid of C of minimum volume.*

(ii) *C is in minimum S_m-position with respect to volume-preserving linear transformations and B^d is the unique circumscribed ellipsoid of C with minimum surface area.*

These results can be extended to general quermassintegrals. Then we see that the minimum positions for all quermassintegrals – except for the volume – coincide.

Polar f-Moments

Let $f : [0, +\infty) \to [0, \infty)$ be a non-decreasing function. Then

$$M(C, f) = \int_C f\big(\|x\|\big)\, dx$$

is the *polar f-moment* of C. If $f(t) = t^2$, then $M(C, t^2)$ is the *polar moment of inertia.*

Theorem 24. *Let f be convex and assume that $f(t) = 0$ only for $t = 0$. Then C has, up to rotations, a unique minimum polar f-moment position with respect to volume-preserving linear transformations and the following properties are equivalent:*

(i) *C is in minimum polar f-moment position.*

(ii) $I = \lambda \int_C \frac{f'(\|x\|)}{\|x\|} x \otimes x \, dx$ *for suitable* $\lambda > 0$.

The integral here is to be understood entry-wise. We now minimize the product $M(AC, t^2)\, M\big((AC)^*, t^2\big)$, where A ranges over all non-singular linear transformations.

Theorem 25. *Up to similarities which keep o fixed, C has a unique minimum $M\big(\cdot\, C, t^2\big)\, M\big((\cdot\, C)^*, t^2\big)$-position with respect to non-singular linear transformations and the following properties are equivalent:*

(i) *C is in minimum $M(\cdot\, C, t^2)\, M\big((\cdot\, C)^*, t^2\big)$-position.*

(ii) $\int_C x \otimes x \, dx = \lambda \int_{C^*} x \otimes x \, dx \in \mathcal{P}^d$ *for a suitable* $\lambda > 0$.

Mean Width and Surface Area

The *mean width* of a convex body C is defined by

$$W(C) = \frac{2}{S(B^{d-1})} \int_{S^{d-1}} h_C(u) \, d\sigma(u),$$

where $S(\cdot)$ and $\sigma(u)$ denote the usual surface area measure on S^{d-1}.

Theorem 26. *Up to rigid motions, C has a unique minimum mean width position with respect to volume preserving affinities and the following properties are equivalent:*

(i) *C is in minimum mean width position.*

(ii) $I = \lambda \int_{S^{d-1}} \big\{ \operatorname{grad} h_C(u) \otimes u + u \otimes \operatorname{grad} h_C(u) \big\} \, d\sigma(u)$ *for a suitable* $\lambda > 0$.

A first characterization of the minimum surface area position of C with respect to volume-preserving affinities is due to Giannopoulos and Papadimitrakis [28]. A different result can be described as follows: The *projection body* ΠC of C is the o-symmetric convex body with support function

$$h_{\Pi C}(u) = v(C \mid u^\perp),$$

where $C \mid u^\perp$ is the orthogonal projection of C onto the subspace $u^\perp$ orthogonal to u of codimension 1 and $v(\cdot)$ the volume in $d-1$ dimensions. Since by Cauchy's surface area formula the mean width of the projection body is, up to a multiplicative constant, the surface area of the original body, Theorem 26 implies the following result:

Corollary 18. *Up to rigid motions, C has a unique minimum surface area position with respect to volume-preserving affinities and the following properties are equivalent:*

(i) *C is in minimum surface area position.*

(ii) $I = \lambda \int_{S^{d-1}} \{\operatorname{grad} h_{\Pi C}(u) \otimes u + u \otimes \operatorname{grad} h_{\Pi C}(u)\}\, d\sigma(u)$ *for a suitable* $\lambda > 0$.

There are similar results for $W(C)\, W(C^*)$ and $S(C)\, S(C^*)$.

Remark 5. The above characterizations of convex bodies in minimum position and similar results in the literature should permit one to prove all possible properties of the minimizing bodies. This seems to have been one of the objectives at the beginning of the development. So far, these expectations have not materialized, a minor exception being the proof of Corollary 15.

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