William Tutte: Flow Problems 17

William Tutte was one of the founders of modern graph theory. He left his mark on almost all aspects of graph theory, and both solved hard problems and created much of the foundations of graph theory. He was a Fellow of the Royal Society, and won many other awards for his seminal work.

I present a result of mine that is joint with Atish Das Sarma. It concerns a famous open problem of Tutte, and a new approach to the problem.

Peter Tait thought he had proved the Four-Color Theorem in 1880. He did prove that the theorem was equivalent to proving that every cubic bridgeless planar graph was three-edge-colorable. He then "proved" that there was always a three-coloring of the edges by asserting the following: every such graph has a Hamiltonian Cycle. If this were true—it is not—then there is an easy way to three-color the edges. Color the edges on the cycle alternately green and blue. Then, color all the remaining edges yellow. This is a legal three-coloring.

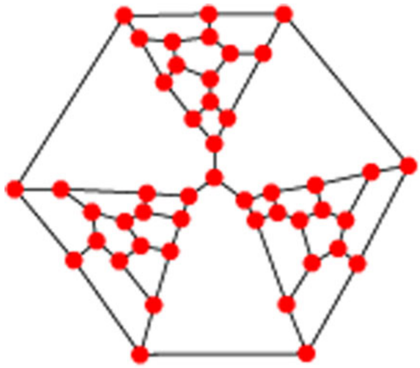
The cycle must have an even number of edges; otherwise, the alternation of colors will fail. I once got a negative referee report back on a paper from an unnamed conference, saying this was a problem—what if the Hamiltonian Cycle had an odd number of edges? Well, the number of edges on such a cycle is the same as the number of vertices. And, of course, every cubic graph has an even number of vertices. The proof of this is simple: the degree of each node is 3 so the n vertices have

$$3n/2$$

edges. Clearly, this forces n to be even, since we cannot have graphs with a fractional number of edges.

The statement of Tait stood for decades, until Tutte thought about it. In 1946 he discovered the following graph:

R.J. Lipton, K.W. Regan, *People, Problems, and Proofs*,
DOI 10.1007/978-3-642-41422-0_17,



This graph is easily seen to be cubic, planar, and bridgeless. Also a simple case argument shows there is no Hamiltonian Cycle. Tait was wrong, and his proof of the Four-Color Theorem was thus incorrect. It would be exactly thirty years until a proof of the Four-Color Theorem would be found. It was a heavily computer-dependent proof, but a proof nonetheless, by Kenneth Appel and Wolfgang Haken.

17.1 Flow Conjectures

Suppose that $G = (V, E)$ is an undirected graph. Direct the edges of the graph in any manner at all; it will turn out not to make any difference. Then a *flow* is a mapping ϕ from the edges E to the integers, so that *Kirchhoff's Rule* is satisfied: the amount flowing into each vertex is the same as the amount flowing out of the vertex. Formally, for each vertex v, let A_v^+ be the edges directed to v and A_v^- the edges directed from v. Then

$$\sum_{e \in A_v^+} \phi(e) = \sum_{e \in A_v^-} \phi(e).$$

A flow is a k-flow if the values $\phi(e)$ are all in the range

$$-(k-1), \ldots, 0, \ldots, k-1.$$

It is called a nowhere-zero k-flow provided $\phi(e)$ is never 0.

Tutte made a number of conjectures about nowhere-zero flows on graphs, motivated by the coloring theorems for planar graphs. His conjectures have been partially solved, but some are still open. In particular he conjectured:

Theorem 17.1 *Every bridgeless graph has a nowhere-zero 5-flow.*

The best known results to date is François Jaeger's 8-flow theorem and Paul Seymour's 6-flow theorem. In the end notes we list course notes by Emo Welzl that give a nice survey of what is known about flows.

17.2 Tensor Trick

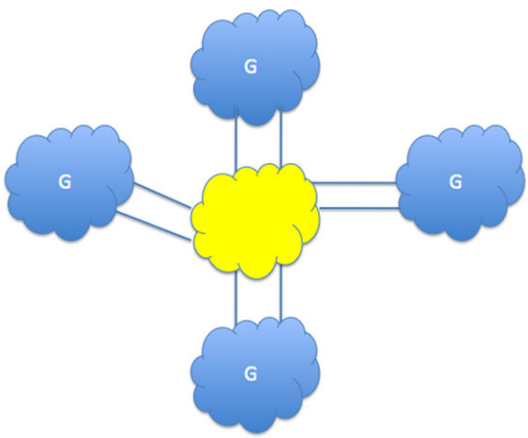
One of the powerful ideas in science and mathematics is the notion of amplification. I discussed this before in 2009. Sometimes amplification is called the **Tensor Trick**. It is a simple idea, but can yield surprisingly powerful consequences.

Theorem 17.2 *Suppose all bridgeless graphs on n vertices have 5-flows with $o(n)$ zeroes. Then the nowhere-zero 5-flow conjecture is true.*

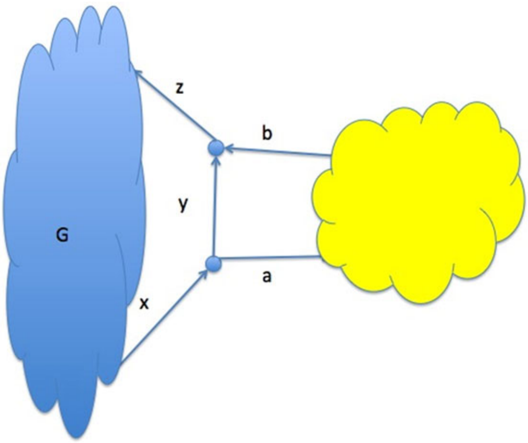
17.3 Sketch of Proof

The proof is based on the notion of amplification. Suppose that G is a counter-example graph. That is, suppose every 5-flow on G has at least one zero. Then we construct another graph H that contains many copies of G. Moreover, if there is a flow on H that has $o(n)$ zeroes, then there is a way to recover a perfect nowhere-zero flow for G.

More precisely, assume that G is a bridgeless graph without a nowhere-zero 5-flow. Create many copies of G and attach them via two edges to a cycle. In the picture below, the blue graphs are copies of G, and the yellow cloud is a cycle.

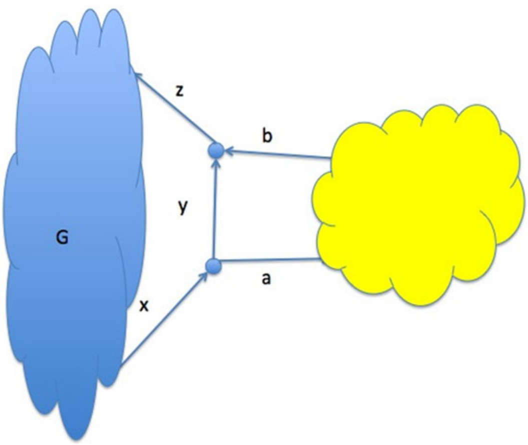


Call this resulting graph H. It is important to see how we do this attachment. We take an edge e of G and split it into three edges x, y, z and add edges a, b to the cycle. See the picture below.



For enough copies of G there must be a 5-flow of H with so few zeroes that one copy of G and its extra edges are all non-zero. The last step of the proof is to argue that we can use these values on G and the extra edges to create a valid nowhere-zero flow for G.

Recall we added the extra two edges to G as in the following figure. We are using the edge labels to also stand for their values, to avoid excessive notation.



The flow must satisfy

$$x = y + a,$$
$$z = y + b,$$
$$a = b.$$

The last equality holds since the net flow into a subgraph must be 0. Then it is easy to see that $x = z$. This shows we can collapse and join the edge x and z into one, since they have the same value. This collapse forms the original graph G, since this replaces the edge we split into three pieces. Note that the resulting flow is non-zero, and this yields a nowhere-zero 5-flow for G.

17.4 Open Problems

Can these tensor results actually be used to prove the nowhere-zero 5-flow conjecture?

Can the same tensor ideas be used on other open questions concerning graphs? We believe it can also work on the Tutte 4-flow conjecture, but there are some details to be checked. The above method does not depend on the flow being a 5-flow. Tutte's nowhere-zero 4-flow conjecture has an additional constraint on the family of graphs, since some bridgeless graphs do not have 4-flows.

The tensor method may apply, we think, to other graph conjectures such as the *Double Cover Conjecture*, which is covered in a paper by Melody Chan.

17.5 Notes and Links

Original post:
http://rjlipton.wordpress.com/2010/06/02/the-tensor-trick-and-tuttles-flow-conjectures/
Previous post on four-color theorem:
http://rjlipton.wordpress.com/2009/04/24/the-four-color-theorem/
Tait's conjecture:
http://en.wikipedia.org/wiki/Tait's_conjecture
Nowhere-zero flow:
http://en.wikipedia.org/wiki/Nowhere-zero_flow
Notes on flows by Emo Welzl:
http://www.ti.inf.ethz.ch/ew/courses/GA09/NZF.pdf
Referenced post on amplifiers:
http://rjlipton.wordpress.com/2009/08/10/the-role-of-amplifiers-in-science/
Paper by Chan:
http://math.berkeley.edu/~mtchan/cdc.pdf
We note that Kenneth Appel passed away as this book was entering final production. We wrote a tribute to him and his daughter at
http://rjlipton.wordpress.com/2013/05/28/kenneth-and-laurel-appel-in-memoriam/

Picture credits:
Tutte Graph:
http://en.wikipedia.org/wiki/Tait's_conjecture
except Wikipedia has changed its figure.
The other three figures are original.