

Raymond Smullyan is a great writer of popular books on logic—especially books on various forms of the [liar paradox](#). He is also a first-rate logician whose [book](#) on set theory with Melvin Fitting was just republished by Dover. This book gives a very readable and somewhat unusual approach to set theory, especially the independence results of Kurt Gödel and Paul Cohen.

We will use an example from their book to explain Georg Cantor’s famous diagonal argument. I have an idea why some people have difficulty with Cantor’s beautiful proof. If you know it well, you may wish to read on anyway; if you do not know it or do not believe it, then please read on and see if I have something to say. I have discussed the argument before, but still feel there is something to add to the discussion.

I have never had the pleasure of meeting Smullyan, but have heard a number of stories about him. One concerns the etiquette at Yale University, where I had my first job in the Computer Science department. One day I wrote a letter and gave it to a secretary to type up on our CS letterhead. My letter started

Dear Dr. X:
I am sending you my referee report on . . .

The letter came back slightly changed

Dear Mr. X:
I am sending you my referee report on . . .

I immediately asked why the change from “Dr.” to “Mr.”? The answer was: “This is Yale, we assume that all professionals have doctorates and call them ‘Mr.’ or ‘Ms.’ and so we never use any formal titles.” Okay I was new, I was young, so the letter went out as changed.

Later on I heard a story about Smullyan, who had given a talk at Yale, in the 1950s, before he had his PhD. After his talk there was tea in his honor, and his host introduced Smullyan around to all.

Mr. Blue, Mr. Green, Ms. Red, Mr. Yellow, and so on, I would like to introduce you to *Smullyan*.

Note, the introduction was to “Smullyan” not “Mr. Smullyan.” I do not know first-hand if the story is true or not, but I like it very much. Oh well.

Let’s turn to the classic argument of Cantor that proves the reals are uncountable. Somehow his argument is subtle, and many do not “get it.” There are some who are puzzled when they first see the proof, there are others who never understand it, and there are some who do not believe it at all. I would like to use Alice and Bob to help explain what the roadblock may be to understanding Cantor.

16.1 Alice and Bob Play Some Games

Consider the following simple game that Alice challenges Bob to play. Alice picks a natural number X , which she hides from Bob. Bob then must try and guess the number. This is a pretty tough game for Bob, but Alice is quite fair. She allows Bob to make an infinite number of guesses. It should be clear that Bob has a simple winning strategy. He just announces his strategy; he will say

$$1, 2, 3, 4, \dots,$$

Eventually he will say X and he will win the game. Alice agrees.

She then challenges Bob to another game. This time Alice picks a secret X that is an integer. Again Bob has an infinite number of guesses. Bob thinks for a moment and says his strategy now is to say

$$0, -1, 1, -2, 2, \dots,$$

They both agree that he wins again.

So far the games have been pretty easy, and Bob is feeling pretty good. Then Alice says she will pick a *pair* (X, Y) of natural numbers. She smiles and asks Bob what is his strategy now? This time Bob has to think a bit. He notices he cannot use a strategy like

$$(1, 1), (1, 2), \dots, (1, k), \dots$$

and then switch to the pairs

$$(2, 1), (2, 2), \dots, (2, k), \dots,$$

The problem is that if Alice picked $(2, 10)$ he will never get to it; all of his infinite number of guesses will be of pairs of the form $(1, Y)$. Then Bob sees the “trick”: guess first $(1, 1)$ and then $(1, 2)$ and $(2, 1)$. In general guess all the pairs that sum to k before all the pairs that sum to $k + 1$. If Alice selects (X, Y) , then Bob will guess it when $k = X + Y$. He wins again.

Alice has many games like this, but here is the last one she decides to ask Bob. She said now my secret is a *finite* set of natural numbers X . She again challenges Bob to guess her secret. Bob gets this one pretty fast—he is catching on to the

games. He realizes he can just generalize the pair idea. He breaks his guesses up into groups. In the k -th group he will guess all finite sets of natural numbers that sum to at most k .

Alice gives up, for now, since Bob has handled all her challenges so well.

16.2 Let's Look at Bob's Strategies

In set theory terminology Alice's games prove:

- (1) The natural numbers are countable.
- (2) The integers are countable.
- (3) The pairs of natural numbers are countable.
- (4) The finite sets of natural numbers are countable.

The (3) game can be viewed as proving the rational numbers are countable.

Note there was something very special about Bob's strategies. He did not need to keep them secret. Even if Alice knew his strategy for the second game, for example, she still would lose. No matter what number she selected, positive or negative, he would eventually guess the number. This is true of all of Bob's strategies. They win whether or not Alice knows them.

This is a very special situation in playing games, even playing infinite games. Since Bob's strategies are so powerful, I believe there should be no argument with the conclusions. For example, the pairs of natural numbers are countable. Do you agree?

16.3 Is Diagonalization Cheating?

Alice now challenges Bob to another game. Now Alice selects an infinite set of natural numbers X . Bob's task is the same as before: he must guess an infinite series of sets of natural numbers

$$S_1, S_2, \dots$$

Bob wins if, as before, one of his sets $S_k = X$. Otherwise, Alice wins.

The classic proof of Cantor shows the following: if Bob selects a strategy and Alice **knows his strategy**, then Alice can produce an X that makes Bob lose. Recall how she does this. She looks at each of his sets S_i in turn, and decides whether or not to put i into her set X so that $X \neq S_i$.

Perhaps the point that many find unfair is **Alice gets to see Bob's strategy**. This seems like a huge advantage. In the previous games, Bob's strategies were so strong that he could allow her to see his strategies. Now Alice can only be sure of winning if she knows exactly what Bob's strategy is. If Alice does not know his sequence of sets

$$S_1, S_2, \dots$$

then she **cannot** guarantee that she will always win. Whatever she picks for X it *might* just happen to be S_1 . If that is true, then Bob wins instantly.

16.4 Can We Fix This?

Cantor's theorem is correct, there are more infinite sets of natural numbers than natural numbers. But there is no strategy for Alice that always wins the game without "cheating"—without her seeing exactly what Bob's strategy is.

This seems unfair, it seems like Alice is cheating. Is this the reason you might have been puzzled the first time you saw Cantor's proof, or is this the reason you still are worried there is something wrong with it?

Alice has a fix. She tells Bob that she will let him keep his strategy secret, but she will now use a mixed strategy. That is Alice will pick her secret set X randomly. Now she claims she will win with probability 1. She asserts that this means that there is no strategy

$$S_1, S_2, \dots$$

such that Bob always wins. She further argues that this means the set of all sets of natural numbers is not countable. If it was, then Bob would have a winning strategy.

Bob thinks about this last statement. In all the previous games he won because the sets were countable, he agrees that if she can win even some of the time, then Alice is right. Alice says here is how she picks X : each i is in her set with probability exactly $1/2$.

The claim, from Alice, is that the probability $X = S$ for any set S Bob picks is 0. Thus, the probability that her secret set X is in his list is bounded above by the union bound by

$$0 + 0 + \dots$$

Alice thus wins with probability 1, and Bob agrees that he has no infinite list of **all** the sets of natural numbers. He sees that if there were a *complete* list of all infinite sets of natural numbers, then mixed strategy or not, Alice would always lose. Bob would eventually hit Alice's set. Do you agree?

16.5 Open Problems

This discussion is related directly to Chap. 15 on various proofs that the reals are uncountable. There was strong discussion at the time on whether or not the random proof is valid. I think the point here is that the randomization is needed to make the game between Alice and Bob "fair." What do you think?

16.6 Notes and Links

Original post:

<http://rjlipton.wordpress.com/2010/06/11/does-cantors-diagonalization-proof-cheat/>

Liar paradox:

http://en.wikipedia.org/wiki/Liar_paradox

Smullyan and Fitting book:

<http://www.amazon.com/Theory-Continuum-Problem-Dover-Mathematics/dp/0486474844>

<http://rjlipton.wordpress.com/2010/01/20/are-the-reals-really-uncountable/>