

A Decision Procedure for *SHOIQ* with Transitive Closure of Roles

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Abstract. The Semantic Web makes an extensive use of the OWL DL ontology language, underlied by the *SHOIQ* description logic, to formalize its resources. In this paper, we propose a decision procedure for this logic extended with the transitive closure of roles in concept axioms, a feature needed in several application domains. The most challenging issue we have to deal with when designing such a decision procedure is to represent infinitely non-tree-shaped models, which are different from those of *SHOIQ* ontologies. To address this issue, we introduce a new blocking condition for characterizing models which may have an infinite non-tree-shaped part.

1 Introduction

The ontology language OWL-DL [1] is widely used to formalize data resources on the Semantic Web. This language is mainly based on the description logic *SHOIN* which is known to be decidable [2]. Although *SHOIN* provides *transitive roles* to model transitivity of relations, we can find several applications in which the *transitive closure of roles*, that is more expressive than transitive roles, is needed. For instance, we consider an ontology, namely \mathcal{O}_1 , that consists of the following axioms:

Human $\sqsubseteq \exists \text{hasAncestor}.\{\text{Eva}\}$, where *hasAncestor* is transitive
hasParent $\sqsubseteq \text{hasAncestor}$, $\{\text{Mike}\} \sqsubseteq \text{Human}$, $\{\text{Mike}\} \sqsubseteq \forall \text{hasParent}.\perp$

We can see that \mathcal{O}_1 is consistent. However, the last axiom in \mathcal{O}_1 would be considered as a design error which should lead to inconsistency. If the transitive role “*hasAncestor*” is replaced with the transitive closure “*hasParent*⁺” (and the second axiom is removed), the first axiom becomes:

$$\text{Human} \sqsubseteq \exists \text{hasParent}^+.\{\text{Eva}\}$$

It follows that the modified ontology is consistent. The point is that an instance of “*hasParent*⁺” represents exactly a sequence of instances of “*hasParent*” while an instance of “*hasAncestor*” corresponds to a sequence of instances of *itself*. In this paper, we consider an extension of *SHOIQ* by enabling transitive closure of roles in concept axioms. In the general case, transitive closure is not expressible in the first order logic [3], the logic from which DL is a sublanguage, while the second order logic is sufficiently expressive to do so.

In the DL literature ([4]; [5]), there have been works dealing with transitive closure of roles. Recently, Ortiz [5] has proposed an algorithm for deciding consistency in the logic \mathcal{ALCQIb}_{reg}^+ which allows for transitive closure of roles. However, nominals are disallowed in this logic. It is known that reasoning with a DL including number restrictions, inverse roles, nominals and transitive closure of roles is hard. The reason for this is that there exists an ontology in that DL whose models have an *infinite* non-tree-shaped part. Calvanese *et al.* [6] have presented an automata-based technique for dealing with the logic \mathcal{ZOIQ} that includes transitive closure of roles, and showed that the sublogics \mathcal{ZIQ} , \mathcal{ZOQ} and \mathcal{ZOI} are decidable. To obtain this result, the authors have introduced the *quasi-forest model property* to characterize models of ontologies in these sublogics. Although they are very expressive, none of these sublogics includes \mathcal{SHOIQ} with transitive closure of roles, namely $\mathcal{SHOIQ}_{(+)}$. The following example¹, noted \mathcal{K}_1 , shows that there is an ontology in $\mathcal{SHOIQ}_{(+)}$ which does not enjoy the quasi-forest model property. We consider the following axioms:

- (1) $\{o\} \sqsubseteq A$; $A \sqcap B \sqsubseteq \perp$; $A \sqsubseteq \exists R.A \sqcap \exists R'.B$; $B \sqsubseteq \exists S^+.\{o\}$
- (2) $\{o\} \sqsubseteq \forall X^-. \perp$; $\top \sqsubseteq \leq 1 X.\top$; $\top \sqsubseteq \leq 1 X^-. \top$ where $X \in \{R, R', S\}$

Figure 1 shows an infinite non-tree-shaped model of \mathcal{K}_1 . In fact, each individual x that satisfies $\exists S^+.\{o\}$ must have two distinct paths from x to the individual satisfying nominal o . Intuitively, we can see that (i) such a x must satisfy $\exists S^+.\{o\}$ and B , (ii) an individual satisfying B must connect to another individual satisfying A which must have a R -path to nominal o , and (iii) two concepts A and B are disjoint.

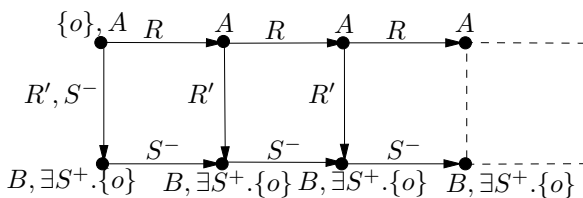


Fig. 1. An infinite non tree-shaped model of \mathcal{K}_1

This example shows that methods ([7], [8], [6]) based on the hypothesis which says that if an ontology is consistent it has a *quasi-forest model*, could fail to address the problem of consistency in a DL including simultaneously \mathcal{O} (nominals), \mathcal{I} (inverse roles), \mathcal{Q} (number restrictions) and transitive closure of roles.

In this paper, we propose a decision procedure for the problem of consistency in \mathcal{SHOIQ} with transitive closure of roles in concept axioms. The underlying idea of our algorithm is founded on the *star-type* and *frame* notions introduced by Pratt-Hartmann [9]. This technique uses star-types to represent individuals and “tiles” them together to form a frame for representing a model. For each star-type σ , we maintain a function $\delta(\sigma)$ which stores the number of individuals satisfying this star-type. To obtain termination, we introduce two additional structures for establishing a new blocking condition:

¹ This example is initially proposed by Sebastian Rudolph from an informal discussion.

(i) the first one, namely *cycles*, describes duplicate parts of a model resulting from interactions of logic constructors in \mathcal{SHOIQ} , (ii) the second one, namely *blocking-blocked cycles*, describes parts of a model bordered by cycles which allow a frame to satisfy transitive closure of roles occurring in concepts of the form $\exists R^+.C$.

2 The Description Logic $\mathcal{SHOIQ}_{(+)}$

In this section, we present the syntax, the semantics and main inference problems of $\mathcal{SHOIQ}_{(+)}$. In addition, we introduce a tableau structure for $\mathcal{SHOIQ}_{(+)}$, which allows us to represent a model of a $\mathcal{SHOIQ}_{(+)}$ knowledge base.

Definition 1. Let \mathbf{R} be a non-empty set of role names and $\mathbf{R}_+ \subseteq \mathbf{R}$ be a set of transitive role names. We use $\mathbf{R}_1 = \{P^- \mid P \in \mathbf{R}\}$ to denote a set of inverse roles, and $\mathbf{R}_\oplus = \{Q^+ \mid Q \in \mathbf{R} \cup \mathbf{R}_1\}$ to denote a set of transitive closure of roles. Each element of $\mathbf{R} \cup \mathbf{R}_1 \cup \mathbf{R}_\oplus$ is called a $\mathcal{SHOIQ}_{(+)}$ -role. A role inclusion axiom is of the form $R \sqsubseteq S$ for two $\mathcal{SHOIQ}_{(+)}$ -roles R and S such that $R \notin \mathbf{R}_\oplus$ and $S \notin \mathbf{R}_\oplus$. A role hierarchy \mathcal{R} is a finite set of role inclusion axioms. An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty set $\Delta^{\mathcal{I}}$ (domain) and a function $\cdot^{\mathcal{I}}$ which maps each role name to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ such that

$$\begin{aligned} R^{-\mathcal{I}} &= \{\langle x, y \rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \langle y, x \rangle \in R^{\mathcal{I}}\} \text{ for all } R \in \mathbf{R}, \\ \langle x, z \rangle \in S^{\mathcal{I}}, \langle z, y \rangle \in S^{\mathcal{I}} &\text{ implies } \langle x, y \rangle \in S^{\mathcal{I}} \text{ for each } S \in \mathbf{R}_+, \text{ and} \\ (Q^+)^{\mathcal{I}} &= \bigcup_{n>0} (Q^n)^{\mathcal{I}} \text{ with } (Q^1)^{\mathcal{I}} = Q^{\mathcal{I}}, \end{aligned}$$

$$(Q^n)^{\mathcal{I}} = \{\langle x, y \rangle \in (\Delta^{\mathcal{I}})^2 \mid \exists z \in \Delta^{\mathcal{I}}, \langle x, z \rangle \in (Q^{n-1})^{\mathcal{I}}, \langle z, y \rangle \in Q^{\mathcal{I}}\} \text{ for } Q^+ \in \mathbf{R}_\oplus$$

* An interpretation \mathcal{I} satisfies a role hierarchy \mathcal{R} if $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$ for each $R \sqsubseteq S \in \mathcal{R}$. Such an interpretation is called a model of \mathcal{R} , denoted by $\mathcal{I} \models \mathcal{R}$. To simplify notations for nested inverse roles and transitive closures of roles, we define two functions \cdot^\ominus and \cdot^\oplus as follows:

$$R^\ominus = \begin{cases} R^- & \text{if } R \in \mathbf{R}; \\ S & \text{if } R = S^- \text{ and } S \in \mathbf{R}; \\ (S^-)^+ & \text{if } R = S^+, S \in \mathbf{R}, \\ S^+ & \text{if } R = (S^-)^+, S \in \mathbf{R} \end{cases} \quad R^\oplus = \begin{cases} R^+ & \text{if } R \in \mathbf{R}; \\ S^+ & \text{if } R = (S^+)^+ \text{ and } S \in \mathbf{R}; \\ (S^-)^+ & \text{if } R = S^- \text{ and } S \in \mathbf{R}; \\ (S^-)^+ & \text{if } R = (S^+)^- \text{ and } S \in \mathbf{R} \end{cases}$$

* A relation $\underline{\boxplus}$ is defined as the transitive-reflexive closure \mathcal{R}^+ of \sqsubseteq on $\mathcal{R} \cup \{R^\ominus \sqsubseteq S^\ominus \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{R^\oplus \sqsubseteq S^\oplus \mid R \sqsubseteq S \in \mathcal{R}\} \cup \{Q \sqsubseteq Q^\oplus \mid Q \in \mathbf{R} \cup \mathbf{R}_1\}$. We define a function $\text{Trans}(R)$ which returns true iff there is some $Q \in \mathbf{R}_+ \cup \{P^\ominus \mid P \in \mathbf{R}_+\} \cup \{P^\oplus \mid P \in \mathbf{R} \cup \mathbf{R}_1\}$ such that $Q \underline{\boxplus} R \in \mathcal{R}^+$. A role R is called simple w.r.t. \mathcal{R} if $\text{Trans}(R) = \text{false}$.

The reason for the introduction of two functions \cdot^\ominus and \cdot^\oplus in Definition 1 is that they avoid using R^{--} and R^{++} . Moreover, it remains a unique nested case $(R^-)^+$. According to Definition 1, axiom $R \sqsubseteq Q^\oplus$ is not allowed in a role hierarchy \mathcal{R} since this may lead to undecidability [10] even if R is simple. Notice that the closure \mathcal{R}^+ may contain $R \sqsubseteq Q^\oplus$ if $R \sqsubseteq Q$ belongs to \mathcal{R} .

Definition 2 (terminology). Let \mathbf{C} be a non-empty set of concept names with a non-empty subset $\mathbf{C}_o \subseteq \mathbf{C}$ of nominals. The set of $\mathcal{SHOIQ}_{(+)}$ -concepts is inductively defined as the smallest set containing all C in \mathbf{C} , \top , $C \sqcap D$, $C \sqcup D$, $\neg C$, $\exists R.C$, $\forall R.C$, $(\leq n S.C)$ and $(\geq n S.C)$ where n is a positive integer, C and D are $\mathcal{SHOIQ}_{(+)}$ -concepts, R is an $\mathcal{SHOIQ}_{(+)}$ -role and S is a simple role w.r.t. a role hierarchy. We denote \perp for $\neg\top$. The interpretation function $\cdot^{\mathcal{I}}$ of an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ maps each concept name to a subset of $\Delta^{\mathcal{I}}$ such that $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$, $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$, $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$, $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$, $|\{o^{\mathcal{I}}\}| = 1$ for all $o \in \mathbf{C}_o$, $(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}}, \langle x, y \rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$, $(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}}, \langle x, y \rangle \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\}$, $(\geq n S.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid |\{y \in C^{\mathcal{I}} \mid \langle x, y \rangle \in S^{\mathcal{I}}\}| \geq n\}$, $(\leq n S.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid |\{y \in C^{\mathcal{I}} \mid \langle x, y \rangle \in S^{\mathcal{I}}\}| \leq n\}$ where $|S|$ is denoted for the cardinality of a set S . An axiom $C \sqsubseteq D$ is called a general concept inclusion (GCI) where C, D are $\mathcal{SHOIQ}_{(+)}$ -concepts (possibly complex), and a finite set of GCIs is called a terminology \mathcal{T} . An interpretation \mathcal{I} satisfies a GCI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and \mathcal{I} satisfies a terminology \mathcal{T} if \mathcal{I} satisfies each GCI in \mathcal{T} . Such an interpretation is called a model of \mathcal{T} , denoted by $\mathcal{I} \models \mathcal{T}$. A pair $(\mathcal{T}, \mathcal{R})$ is called a $\mathcal{SHOIQ}_{(+)}$ knowledge base where \mathcal{R} is a $\mathcal{SHOIQ}_{(+)}$ role hierarchy and \mathcal{T} is a $\mathcal{SHOIQ}_{(+)}$ terminology. A knowledge base $(\mathcal{T}, \mathcal{R})$ is said to be consistent if there is a model \mathcal{I} of both \mathcal{T} and \mathcal{R} , i.e., $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{R}$. A concept C is called satisfiable w.r.t. $(\mathcal{T}, \mathcal{R})$ iff there is some interpretation \mathcal{I} such that $\mathcal{I} \models \mathcal{R}$, $\mathcal{I} \models \mathcal{T}$ and $C^{\mathcal{I}} \neq \emptyset$. Such an interpretation is called a model of C w.r.t. $(\mathcal{T}, \mathcal{R})$. A concept D subsumes a concept C w.r.t. $(\mathcal{T}, \mathcal{R})$, denoted by $C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds in each model \mathcal{I} of $(\mathcal{T}, \mathcal{R})$. \triangleleft

Since unsatisfiability, subsumption and consistency w.r.t. a $\mathcal{SHOIQ}_{(+)}$ knowledge base can be reduced to each other, it suffices to study knowledge base consistency. For the ease of construction, we assume all concepts to be in *negation normal form* (NNF), i.e., negation occurs only in front of concept names. Any $\mathcal{SHOIQ}_{(+)}$ -concept can be transformed to an equivalent one in NNF by using DeMorgan's laws and some equivalences as presented in [11]. According to [12], $\text{nfn}(C)$ can be computed in polynomial time in the size of C . For a concept C , we denote the nfn of C by $\text{nfn}(C)$ and the nfn of $\neg C$ by $\dot{\neg}C$. Let D be a $\mathcal{SHOIQ}_{(+)}$ -concept in NNF. We define $\text{cl}(D)$ to be the smallest set that contains all sub-concepts of D including D . For a knowledge base $(\mathcal{T}, \mathcal{R})$, we reuse $\text{cl}(\mathcal{T}, \mathcal{R})$ introduced by Horrocks *et al.* [7] to denote all sub-concepts occurring in the axioms of $(\mathcal{T}, \mathcal{R})$ as follows:

$$\text{cl}(\mathcal{T}, \mathcal{R}) = \bigcup_{C \sqsubseteq D \in \mathcal{T}} \text{cl}(\text{nfn}(\neg C \sqcup D), \mathcal{R}) \text{ where}$$

$$\text{cl}(E, \mathcal{R}) = \text{cl}(E) \cup \{\dot{\neg}C \mid C \in \text{cl}(E)\} \cup \quad (1)$$

$$\{\forall S.C \mid (\forall R.C \in \text{cl}(E), S \sqsubseteq R) \text{ or } (\dot{\neg}\forall R.C \in \text{cl}(E), S \sqsubseteq R)\} \cup \quad (2)$$

$$\bigcup_{\exists Q^{\oplus}.C \text{ occurs in } \mathcal{T}} \text{cl}(\exists Q.C \sqcup \exists Q.\exists Q^{\oplus}.C) \quad (3)$$

Since (1) consists of sub-concepts from \mathcal{T} and (2) is formed from concepts in (1) by replacing a role or a logic constructor with respective another role occurring in \mathcal{R} or

another logic constructor, both of these sets are bounded by $\mathcal{O}(|(\mathcal{T}, \mathcal{R})|)$. Thus, $\text{cl}(\mathcal{T}, \mathcal{R})$ is bounded by $\mathcal{O}(|(\mathcal{T}, \mathcal{R})|)$.

We have $\text{cl}(\mathcal{T}, \mathcal{R})$ is bounded by $\mathcal{O}(|(\mathcal{T}, \mathcal{R})|)$ [7]. To translate *star-type* and *frame* structures presented by Pratt-Hartmann (2005) for \mathcal{C}^2 into those for \mathcal{SHOIQ} , we need to add new sets of concepts, denoted $\text{cl}_1(\mathcal{T}, \mathcal{R})$ and $\text{cl}_2(\mathcal{T}, \mathcal{R})$, to the signature of a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$.

$$\text{cl}_1(\mathcal{T}, \mathcal{R}) = \{\leq mS.C \mid \{(\leq nS.C), (\geq nS.C)\} \cap \text{cl}(\mathcal{T}, \mathcal{R}) \neq \emptyset, 1 \leq m \leq n\} \cup \\ \{\geq mS.C \mid \{(\leq nS.C), (\geq nS.C)\} \cap \text{cl}(\mathcal{T}, \mathcal{R}) \neq \emptyset, 1 \leq m \leq n\}$$

For a generating concept $(\geq nS.C)$ and a set $I \subseteq \{0, \dots, \lceil \log n + 1 \rceil\}$, we denote $\mathcal{C}_{(\geq nS.C)}^I = \prod_{i \in I} C_{(\geq nS.C)}^i \cap \prod_{j \notin I} \neg C_{(\geq nS.C)}^j$ where $C_{(\geq nS.C)}^i$ are new concept names

for $0 \leq i \leq \lceil \log n + 1 \rceil$. We define $\text{cl}_2(\mathcal{T}, \mathcal{R})$ as follows:

$$\text{cl}_2(\mathcal{T}, \mathcal{R}) = \{C_{(\geq nS.C)}^i \mid (\geq nS.C) \in \text{cl}(\mathcal{T}, \mathcal{R}) \cup \text{cl}_1(\mathcal{T}, \mathcal{R}), 0 \leq i \leq \lceil \log n + 1 \rceil\} \cup \\ \{\mathcal{C}_{(\geq nS.C)}^I \mid (\geq nS.C) \in \text{cl}(\mathcal{T}, \mathcal{R}) \cup \text{cl}_1(\mathcal{T}, \mathcal{R}), I \subseteq \{0, \dots, \lceil \log n + 1 \rceil\}\}$$

Remark 1. If numbers are encoded in binary then the number of new concept names $C_{(\geq nS.D)}^i$ for $0 \leq i \leq \lceil \log n + 1 \rceil$, is bounded by $\mathcal{O}(|(\mathcal{T}, \mathcal{R})|)$ since n is bounded by $\mathcal{O}(2^{|\mathcal{T}, \mathcal{R}|})$. This implies that $|\text{cl}_2(\mathcal{T}, \mathcal{R})|$ is bounded by $\mathcal{O}(|(\mathcal{T}, \mathcal{R})|)$. Note that two concepts $\mathcal{C}_{(\geq nS.C)}^I$ and $\mathcal{C}_{(\geq nS.C)}^J$ are disjoint for all $I, J \subseteq \{0, \dots, \lceil \log n + 1 \rceil\}$, $I \neq J$. The concepts $\mathcal{C}_{(\exists S.C)}$ and $\mathcal{C}_{(\geq nS.C)}^I$ will be used for building chromatic star-types. This notion will be clarified after introducing the frame structure (Definition 6).

Finally, we denote $\mathbf{CL}(\mathcal{T}, \mathcal{R}) = \text{cl}(\mathcal{T}, \mathcal{R}) \cup \text{cl}_1(\mathcal{T}, \mathcal{R}) \cup \text{cl}_2(\mathcal{T}, \mathcal{R})$, and use $\mathbf{R}(\mathcal{T}, \mathcal{R})$ to denote the set of all role names occurring in \mathcal{T}, \mathcal{R} with their inverse. The definition of $\mathbf{CL}(\mathcal{T}, \mathcal{R})$ is inspired from the Fischer-Ladner closure that was introduced in [13]. The closure $\mathbf{CL}(\mathcal{T}, \mathcal{R})$ contains not only sub-concepts syntactically obtained from \mathcal{T} but also sub-concepts that are semantically derived from \mathcal{T} w.r.t. \mathcal{R} . For instance, if $\forall S.C$ is a sub-concept from \mathcal{T} and $R \sqsubseteq S \in \mathcal{R}$ then $\forall R.C \in \mathbf{CL}(\mathcal{T}, \mathcal{R})$.

To describe a model of a $\mathcal{SHOIQ}_{(+)}$ knowledge base in a more intuitive way, we use a tableau structure that expresses semantic constraints resulting directly from the logic constructors in $\mathcal{SHOIQ}_{(+)}$.

Definition 3. Let $(\mathcal{T}, \mathcal{R})$ be an $\mathcal{SHOIQ}_{(+)}$ knowledge base. A tableau T for $(\mathcal{T}, \mathcal{R})$ is defined to be a triplet $(\mathbf{S}, \mathcal{L}, \mathcal{E})$ such that \mathbf{S} is a set of individuals, $\mathcal{L}: \mathbf{S} \rightarrow 2^{\mathbf{CL}(\mathcal{T}, \mathcal{R})}$ and $\mathcal{E}: \mathbf{R}(\mathcal{T}, \mathcal{R}) \rightarrow 2^{\mathbf{S} \times \mathbf{S}}$. For all $s, t \in \mathbf{S}$, $C, C_1, C_2 \in \mathbf{CL}(\mathcal{T}, \mathcal{R})$, and $R, S, Q^\oplus \in \mathbf{R}(\mathcal{T}, \mathcal{R})$, T satisfies the following properties:

- P1 If $C_1 \sqsubseteq C_2 \in \mathcal{T}$ and $s \in \mathbf{S}$ then $\text{nnf}(\neg C_1 \sqcup C_2) \in \mathcal{L}(s)$;
- P2 If $C \in \mathcal{L}(s)$, then $\neg C \notin \mathcal{L}(s)$;
- P3 If $C_1 \sqcap C_2 \in \mathcal{L}(s)$, then $C_1 \in \mathcal{L}(s)$ and $C_2 \in \mathcal{L}(s)$;
- P4 If $C_1 \sqcup C_2 \in \mathcal{L}(s)$, then $C_1 \in \mathcal{L}(s)$ or $C_2 \in \mathcal{L}(s)$;
- P5 If $\forall S.C \in \mathcal{L}(s)$ and $\langle s, t \rangle \in \mathcal{E}(S)$, then $C \in \mathcal{L}(t)$;
- P6 If $\exists S.C \in \mathcal{L}(s)$ then there is some $t \in \mathbf{S}$ such that $\langle s, t \rangle \in \mathcal{E}(S)$ and $\{C, \mathcal{C}_{(\exists S.C)}\} \subseteq \mathcal{L}(t)$;
- P7 If $\forall S.C \in \mathcal{L}(s)$ and $\langle s, t \rangle \in \mathcal{E}(R)$ for $R \sqsubseteq S$ and $\text{Trans}(R)$ then $\forall R.C \in \mathcal{L}(t)$;

- P8** If $\exists Q^\oplus.C \in \mathcal{L}(s)$ then $(\exists Q.C \sqcup \exists Q.\exists Q^\oplus.C) \in \mathcal{L}(s)$ and there are $s_1, \dots, s_{n-1} \in \mathbf{S}$ such that $\exists Q.C \in \mathcal{L}(s_0) \cup \mathcal{L}(s_{n-1})$, $\langle s_i, s_{i+1} \rangle \in \mathcal{E}(Q)$ with $0 \leq i < n-1$, $s_0 = s$ and $\exists Q^\oplus.C \in \mathcal{L}(s_j)$ for all $0 \leq j < n-1$.
- P9** $\langle s, t \rangle \in \mathcal{E}(R)$ iff $\langle t, s \rangle \in \mathcal{E}(R^\ominus)$;
- P10** If $\langle s, t \rangle \in \mathcal{E}(R)$, $R \sqsubseteq S$ then $\langle s, t \rangle \in \mathcal{E}(S)$;
- P11** If $(\geq n S C) \in \mathcal{L}(s)$ then there are $t_1, \dots, t_n \in \mathbf{S}$ such that $\{C, \mathcal{C}_{(\geq n S C)}^{I_i}\} \subseteq \mathcal{L}(t_i)$ and $\langle s, t_i \rangle \in \mathcal{E}(S)$ for all $1 \leq i \leq n$, and $I_j, I_k \subseteq \{0, \dots, \lceil \log n + 1 \rceil\}$, $I_j \neq I_k$ for all $1 \leq j < k \leq n$;
- P12** If $(\leq n S C) \in \mathcal{L}(s)$ then $|S^T(s, C)| \leq n$;
- P13** If $(\leq n S C) \in \mathcal{L}(s)$ and $\langle s, t \rangle \in \mathcal{E}(S)$ then $\{C, \neg C\} \cap \mathcal{L}(t) \neq \emptyset$ where $S^T(s, C) := \{t \in \mathbf{S} \mid \langle s, t \rangle \in \mathcal{E}(S) \wedge C \in \mathcal{L}(t)\}$;
- P14** If $o \in \mathcal{L}(s) \cap \mathcal{L}(t)$ for some $o \in \mathbf{C}_o$ then $s = t$.
- P15** For each $o \in \mathbf{C}_o$, if o occurs in \mathcal{T} then there is $s \in \mathbf{S}$ such that $o \in \mathcal{L}(s)$.

Note that the property **P8** is added to deal with transitive closure of roles. The following lemma establishes the equivalence between a model of an ontology and a tableau.

Lemma 1. *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base. $(\mathcal{T}, \mathcal{R})$ is consistent iff there is a tableau for $(\mathcal{T}, \mathcal{R})$.*

A proof of Lemma 1 can be found in [14].

3 A Decision Procedure For $\mathcal{SHOIQ}_{(+)}$

This section starts by translating *star-type* and *frame* structures presented by Pratt-Hartmann (2005) for \mathcal{C}^2 into those for $\mathcal{SHOIQ}_{(+)}$.

Definition 4 (star-type). *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base. A star-type is a pair $\sigma = \langle \lambda(\sigma), \xi(\sigma) \rangle$, where $\lambda(\sigma) \in 2^{\mathbf{CL}(\mathcal{T}, \mathcal{R})}$ is called core label, $\xi(\sigma) = (\langle r_1, l_1 \rangle, \dots, \langle r_d, l_d \rangle)$ is a d -tuple over $2^{\mathbf{R}(\mathcal{T}, \mathcal{R})} \times 2^{\mathbf{CL}(\mathcal{T}, \mathcal{R})}$. A pair $\langle r, l \rangle$ is a ray of σ if $\langle r, l \rangle = \langle r_i, l_i \rangle$ for some $1 \leq i \leq d$. We use $\langle r(\rho), l(\rho) \rangle$ to denote a ray $\rho = \langle r, l \rangle$ where $r(\rho) = r$ and $l(\rho) = l$.*

- A star-type σ is *nominal* if $o \in \lambda(\sigma)$ for some $o \in \mathbf{C}_o$.
- A star-type σ is *chromatic* if $\rho \neq \rho'$ implies $l(\rho) \neq l(\rho')$ for two rays ρ, ρ' of σ . When a star-type σ is chromatic, $\xi(\sigma)$ can be considered as a set of rays.
- Two star-types σ, σ' are *equivalent* if $\lambda(\sigma) = \lambda(\sigma')$, and there is a bijection π between $\xi(\sigma)$ and $\xi(\sigma')$ such that $\pi(\rho) = \rho'$ implies $r(\rho') = r(\rho)$ and $l(\rho') = l(\rho)$.

We denote Σ for the set of all star-types for $(\mathcal{T}, \mathcal{R})$. ◁

Note that for a chromatic star-type σ , $\xi(\sigma)$ can be considered as a set of rays since rays are distinct and not ordered. We can think of a star-type σ as the set of individuals x satisfying all concepts in $\lambda(\sigma)$, and each ray ρ of σ corresponds to a “neighbor” individual x_i of x such that $r(\rho)$ is the label of the link between x and x_i ; and x_i satisfies all concepts in $l(\rho)$. In this case, we say that x satisfies σ .

Definition 5 (valid star-type). Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base. Let σ be a star-type for $(\mathcal{T}, \mathcal{R})$ where $\sigma = \langle \lambda(\sigma), \xi(\sigma) \rangle$. The star-type σ is valid if σ is chromatic and the following conditions are satisfied:

1. If $C \sqsubseteq D \in \mathcal{T}$ then $\text{nnf}(\neg C \sqcup D) \in \lambda(\sigma)$;
2. $\{A, \neg A\} \not\subseteq \lambda$ for every concept name A where $\lambda = \lambda(\sigma)$ or $\lambda = l(\rho)$ for each $\rho \in \xi(\sigma)$;
3. If $C_1 \sqcap C_2 \in \lambda(\sigma)$ then $\{C_1, C_2\} \subseteq \lambda(\sigma)$;
4. If $C_1 \sqcup C_2 \in \lambda(\sigma)$ then $\{C_1, C_2\} \cap \lambda(\sigma) \neq \emptyset$;
5. If $\exists R.C \in \lambda(\sigma)$ then there is some ray $\rho \in \xi(\sigma)$ such that $C \in l(\rho)$ and $R \in r(\rho)$;
6. If $(\leq nS.C) \in \lambda(\sigma)$ and there is some ray $\rho \in \xi(\sigma)$ such that $S \in r(\rho)$ then $C \in l(\rho)$ or $\neg C \in l(\rho)$;
7. If $(\leq nS.C) \in \lambda(\sigma)$ and there is some ray $\rho \in \xi(\sigma)$ such that $C \in l(\rho)$ and $S \in r(\rho)$ then there is some $1 \leq m \leq n$ such that $\{(\leq mS.C), (\geq mS.C)\} \subseteq \lambda(\sigma)$;
8. For each ray $\rho \in \xi(\sigma)$, if $R \in r(\rho)$ and $R \underline{\mathbb{K}} S$ then $S \in r(\rho)$;
9. If $\forall R.C \in \lambda(\sigma)$ and $R \in r(\rho)$ for some ray $\rho \in \xi(\sigma)$ then $C \in l(\rho)$;
10. If $\forall R.D \in \lambda(\sigma)$, $S \underline{\mathbb{K}} R$, $\text{Trans}(S)$ and $R \in r(\rho)$ for some ray $\rho \in \xi(\sigma)$ then $\forall S.D \in l(\rho)$;
11. If $\exists Q^\oplus.C \in \lambda(\sigma)$ then $(\exists Q.C \sqcup \exists Q.\exists Q^\oplus.C) \in \lambda(\sigma)$;
12. If $(\geq nS.C) \in \lambda(\sigma)$ then there are n distinct rays $\rho_1, \dots, \rho_n \in \xi(\sigma)$ such that $\{C, \mathcal{C}_{(\geq nS.C)}^{I_i}\} \subseteq l(\rho_i)$, $S \in r(\rho_i)$ for all $1 \leq i \leq n$; and $I_j, I_k \subseteq \{0, \dots, \log n + 1\}$, $I_j \neq I_k$ for all $1 \leq j < k \leq n$;
13. If $(\leq nS.C) \in \lambda(\sigma)$ and there do not exist $n + 1$ rays $\rho_0, \dots, \rho_n \in \xi(\sigma)$ such that $C \in l(\rho_i)$ and $S \in r(\rho_i)$ for all $0 \leq i \leq n$. \triangleleft

Roughly speaking, a star-type σ is valid if each individual x satisfies *semantically* all concepts in $\lambda(\sigma)$. In fact, each condition in Definition 5 represents the semantics of a constructor in $\mathcal{SHOIQ}_{(+)}$ except for transitive closure of roles. From valid star-types, we can “tile” a model instead of using expansion rules for generating nodes as described in tableau algorithms. Before presenting how to “tile” a model from star-types, we need some notation that will be used in the remainder of the paper.

Notation 1. We call $\mathcal{P} = \langle (\sigma_1, \rho_1, d_1), \dots, (\sigma_k, \rho_k, d_k) \rangle$ a sequence where $\sigma_i \in \Sigma$, $\rho_i \in \xi(\sigma_i)$ and $d_i \in \mathbb{N}$ for $1 \leq i \leq k$.

- $\text{tail}(\mathcal{P}) = (\sigma_k, \rho_k, d_k)$, $\text{tail}_\sigma(\mathcal{P}) = \sigma_k$, $\text{tail}_\rho(\mathcal{P}) = \rho_k$, $\text{tail}_\delta(\mathcal{P}) = d_k$ and $|\mathcal{P}| = k$. We denote $\mathcal{L}(\mathcal{P}) = \lambda(\text{tail}_\sigma(\mathcal{P}))$.
- $\mathfrak{p}^i(\mathcal{P}) = (\sigma_i, \rho_i, d_i)$, $\mathfrak{p}_\sigma^i(\mathcal{P}) = \sigma_i$, $\mathfrak{p}_\rho^i(\mathcal{P}) = \rho_i$ and $\mathfrak{p}_\delta^i(\mathcal{P}) = d_i$ for each $1 \leq i \leq k$.
- an operation $\text{add}(\mathcal{P}, (\sigma, \rho, d))$ extends \mathcal{P} to a new sequence with $\text{add}(\mathcal{P}, (\sigma, \rho, d)) = \langle \mathcal{P}, (\sigma, \rho, d) \rangle$.

Definition 6 (frame). Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base. A frame for $(\mathcal{T}, \mathcal{R})$ is a tuple $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$, where

1. \mathcal{N} is a set of valid star-types such that σ is not equivalent to σ' for all $\sigma, \sigma' \in \mathcal{N}$;
2. $\mathcal{N}_o \subseteq \mathcal{N}$ is a set of nominal star-types;

3. Ω is a function that maps each pair (σ, ρ) with $\sigma \in \mathcal{N}$ and $\rho \in \xi(\sigma)$ to a sequence $\Omega(\sigma, \rho) = \langle (\sigma_1, \rho_1, d_1), \dots, (\sigma_m, \rho_m, d_m) \rangle$ with $\sigma_i \in \mathcal{N}$, $\rho_i \in \xi(\sigma_i)$, $d_i \in \mathbb{N}$ for $1 \leq i \leq m$ such that for each σ_i with $1 \leq i \leq m$, it holds that $l(\rho) = \lambda(\sigma_i)$, $l(\rho_i) = \lambda(\sigma)$ and $r(\rho_i) = r^-(\rho)$ where $r^-(\rho) = \{R^\ominus \mid R \in r(\rho)\}$.
4. δ is a function $\delta : \mathcal{N} \rightarrow \mathbb{N}$. By abuse of notation, we also use δ to denote a function which maps each pair (σ, ρ) with $\sigma \in \mathcal{N}$ and $\rho \in \xi(\sigma)$ into a number in \mathbb{N} , i.e., $\delta(\sigma, \rho) \in \mathbb{N}$. \triangleleft

Since a frame cannot contain two equivalent star-type (Condition 1 in Definition 6), the number of different star-types in a frame is bounded. The following lemma provides such a bound.

Lemma 2. *Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ be a frame for a $SHOIQ_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$. The number of different star-types is bounded by $\mathcal{O}(2^{2^{|\langle \mathcal{T}, \mathcal{R} \rangle|}})$.*

The lemma is a consequence of the following facts : (i) the number of different core labels of star-types is bounded by $\mathcal{O}(|\langle \mathcal{T}, \mathcal{R} \rangle|)$, (ii) the number of different ray labels of star-types is bounded by $\mathcal{O}(2^{|\langle \mathcal{T}, \mathcal{R} \rangle|})$, and (iii) the number of different rays of a star-type is bounded by $\mathcal{O}(2^{|\langle \mathcal{T}, \mathcal{R} \rangle|})$ due to binary coding of numbers.

The frame structure, as introduced in Definition 6, allows us to compress individuals of a model into star-types. For each star-type σ and each ray $\rho \in \xi(\sigma)$, a list $\Omega(\sigma, \rho)$ of triples (σ_i, ρ_i, d_i) with $\rho_i \in \xi(\sigma_i)$ is maintained where σ_i is a ‘‘neighbor’’ star-type of σ via $\rho \in \xi(\sigma)$, and d_i indicates the d_i -th ‘‘layer’’ of rays of σ_i . We can think a layer of rays of σ_i as an individual that connects to its neighbor individuals via the rays of σ_i . The following definition presents how to connect such layers to form paths in a frame.

Definition 7 (path). *Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ be a frame for a $SHOIQ_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$. A path is inductively defined as follows:*

1. A sequence $\langle \emptyset, (\sigma, \rho, 1) \rangle$ is a path, namely nominal path, if $\sigma \in \mathcal{N}_o$ and $\rho \in \xi(\sigma)$;
2. A sequence $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$ with $\mathcal{P} \neq \emptyset$ and $\text{tail}(\mathcal{P}) = (\sigma_0, \rho_0, d_0)$, is a path if $(\sigma, \rho, d) = \mathbf{p}^{d_0}(\Omega(\sigma_0, \rho'))$ for each $\rho' \neq \rho_0$. In this case, we say that $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$ is the ρ' -neighbor of \mathcal{P} , and two paths $\mathcal{P}, \langle \mathcal{P}, (\sigma, \rho, d) \rangle$ are neighbors. Additionally, if $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$ is a ρ' -neighbor of \mathcal{P} and $Q \in r(\rho')$ then $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$ is a Q -neighbor of \mathcal{P} . In this case, we say that $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$ is a Q -neighbor of \mathcal{P} , or \mathcal{P} is a Q^\ominus -neighbor of $\langle \mathcal{P}, (\sigma, \rho, d) \rangle$.

We define $\mathcal{P} \sim \mathcal{P}'$ if $\text{tail}_\sigma(\mathcal{P}) = \text{tail}_\sigma(\mathcal{P}')$ and $\text{tail}_\delta(\mathcal{P}) = \text{tail}_\delta(\mathcal{P}')$. Since \sim is an equivalence relation over the set of all paths, we use \mathcal{P} to denote the set of all equivalence classes $[\mathcal{P}]$ of paths in \mathcal{F} . For $[\mathcal{P}], [\mathcal{Q}] \in \mathcal{P}$, we define:

1. $[\mathcal{P}]$ is a neighbor (ρ' -neighbor) of $[\mathcal{Q}]$ if there are $\mathcal{P}' \in [\mathcal{P}]$ and $\mathcal{Q}' \in [\mathcal{Q}]$ such that \mathcal{Q}' is a neighbor (ρ' -neighbor) of \mathcal{P}' ;
2. $[\mathcal{Q}]$ is a reachable path of $[\mathcal{P}]$ via a ray $\rho \in \xi(\text{tail}_\sigma(\mathcal{P}))$ if there are $[\mathcal{P}_1], \dots, [\mathcal{P}_n] \in \mathcal{P}$ such that $[\mathcal{P}_i] \neq [\mathcal{P}_j]$ for $1 \leq i < j \leq n$, $[\mathcal{P}] = [\mathcal{P}_1]$, $[\mathcal{Q}] = [\mathcal{P}_n]$, $[\mathcal{P}_2]$ is the ρ -neighbor of $[\mathcal{P}_1]$, $[\mathcal{P}_{i+1}]$ is a neighbor of $[\mathcal{P}_i]$ for all $1 \leq i < n - 1$.
3. $[\mathcal{Q}]$ is a Q -neighbor of $[\mathcal{P}]$ if there are $\mathcal{P}' \in [\mathcal{P}]$ and $\mathcal{Q}' \in [\mathcal{Q}]$ such that \mathcal{Q}' is a Q -neighbor of \mathcal{P}' , or \mathcal{P}' is a Q^\ominus -neighbor of \mathcal{Q}' ;

4. $[\mathcal{Q}]$ is a Q -reachable path of $[\mathcal{P}]$ if there are $[\mathcal{P}_1], \dots, [\mathcal{P}_n] \in \mathcal{P}$ such that $[\mathcal{P}_i] \neq [\mathcal{P}_j]$ for $1 \leq i < j \leq n$, $[\mathcal{P}] = [\mathcal{P}_1]$, $[\mathcal{Q}] = [\mathcal{P}_n]$, $[\mathcal{P}_2]$ is the ρ -neighbor of $[\mathcal{P}_1]$, and $[\mathcal{P}_{i+1}]$ is a Q -neighbor of $[\mathcal{P}_i]$ for all $1 \leq i < n$. \triangleleft

Since two paths \mathcal{P} and \mathcal{P}' meet at the same star-type (i.e. $\text{tail}_\sigma(\mathcal{P}) = \text{tail}_\sigma(\mathcal{P}')$) and the same layer (i.e. $\text{tail}_\delta(\mathcal{P}) = \text{tail}_\delta(\mathcal{P}')$) should be considered as identical, we define the equivalence relation \sim in Definition 7 to formalize this idea. Note that for two paths $\mathcal{P}, \mathcal{P}'$ with $\text{tail}_\rho(\mathcal{P}) \neq \text{tail}_\rho(\mathcal{P}')$, we have $\mathcal{P} \sim \mathcal{P}'$ if $\text{tail}_\sigma(\mathcal{P}) = \text{tail}_\sigma(\mathcal{P}')$ and $\text{tail}_\delta(\mathcal{P}) = \text{tail}_\delta(\mathcal{P}')$. This does not allow for extending $\text{tail}_\rho(\mathcal{P})$ to $\text{tail}_\rho([\mathcal{P}])$. As a consequence, there may be several “predecessors” of an equivalence class $[\mathcal{P}]$. However, we can define $\text{tail}_\sigma([\mathcal{P}]) = \text{tail}_\sigma(\mathcal{P})$, $\text{tail}_\delta([\mathcal{P}]) = \text{tail}_\delta(\mathcal{P})$ and $\mathcal{L}([\mathcal{P}]) = \mathcal{L}(\mathcal{P})$. In the sequel, we use \mathcal{P} instead of $[\mathcal{P}]$ whenever it is clear from the context.

In a tree-shaped structure where each node has a unique predecessor, each path \mathcal{P} is identical to its equivalence class $[\mathcal{P}]$. This no longer holds for the general graph structure. The notion of paths in a frame is needed to define cycles which are crucial to establish termination condition when building a frame.

Definition 8 (cycle). Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ be a frame for a $SHOI\mathcal{Q}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$ with a set \mathcal{P} of paths in \mathcal{F} . Let \mathcal{R} be a set of pairs (\mathcal{P}_r, ξ_r) , called root paths, where $\mathcal{P}_r \in \mathcal{P}$ and $\xi_r \subseteq \xi(\text{tail}_\sigma(\mathcal{P}_r))$. Let Θ be a set of quadruples $(\mathcal{P}, \rho, \mathcal{Q}, \nu)$ where $\mathcal{P}, \mathcal{Q} \in \mathcal{P}$ ($\mathcal{P} \neq \mathcal{Q}$), respectively called cycled and cycling paths of Θ , $\rho \in \xi(\text{tail}_\sigma(\mathcal{P}))$, $\nu \in \xi(\text{tail}_\sigma(\mathcal{Q}))$, respectively called cycled and cycling rays of Θ . A ρ -neighbor of a cycled (resp. cycling) path \mathcal{P} is a cycled (resp. cycling) neighbor of \mathcal{P} if ρ is a cycled (resp. cycling) ray of \mathcal{P} . We say that Θ is a cycle w.r.t. a set \mathcal{R} of root paths if for each quadruple $(\mathcal{P}, \rho, \mathcal{Q}, \nu) \in \Theta$ the following conditions are satisfied:

1. $o \notin \mathcal{L}(\mathcal{P}) \cup \mathcal{L}(\mathcal{Q}) \cup \bigcup_{\rho \in \xi(\text{tail}_\sigma(\mathcal{P})) \cup \xi(\text{tail}_\sigma(\mathcal{Q}))} l(\rho)$ for all $o \in \mathbf{C}_o$;
2. $\mathcal{L}(\mathcal{P}) = l(\nu)$, $\mathcal{L}(\mathcal{Q}) = l(\rho)$ and $r(\rho) = r^-(\nu)$.
3. for each ray $\rho' \in \xi(\text{tail}_\sigma(\mathcal{P}))$ that is not cycled, there are a sequence $\mathcal{P}_1, \dots, \mathcal{P}_n \in \mathcal{P}$, some $(\mathcal{P}_0, \rho_0, \mathcal{Q}_0, \nu_0) \in \Theta$ and a root path $(\mathcal{P}_r, \xi_r) \in \mathcal{R}$ such that $\mathcal{P}_i \neq \mathcal{P}_j$ for $1 \leq i < j \leq n$, $\mathcal{P}_1 = \mathcal{P}$, \mathcal{P}_2 is the ρ' -neighbor of \mathcal{P}_1 , \mathcal{P}_{i+1} is a neighbor of \mathcal{P}_i for $1 \leq i < n$, $\mathcal{P}_k = \mathcal{Q}_0$ for some $1 < k < n - 1$, and $\mathcal{P}_n = \mathcal{P}_r$, \mathcal{P}_{n-1} is a ρ_r -neighbor of \mathcal{P}_n with $\rho_r \in \xi_r$.
4. for each ray $\nu' \in \xi(\text{tail}_\sigma(\mathcal{Q}))$ that is not cycling and each sequence $\mathcal{P}_1, \dots, \mathcal{P}_n \in \mathcal{P}$ such that $\mathcal{P}_i \neq \mathcal{P}_j$ for $1 \leq i < j \leq n$, $\mathcal{P}_1 = \mathcal{Q}$, \mathcal{P}_2 is the ν' -neighbor of \mathcal{Q} , and \mathcal{P}_{i+1} is a neighbor of \mathcal{P}_i for $1 \leq i < n$, there is some $(\mathcal{P}_0, \rho_0, \mathcal{Q}_0, \nu_0) \in \Theta$ such that one of the following conditions is satisfied:
 - (a) there is some $1 < k \leq n$ with $\mathcal{P}_k = \mathcal{Q}_0$ or $\mathcal{P}_k = \mathcal{P}_0$, and \mathcal{P}_i is not a cycling and cycled neighbor for all $1 \leq i \leq k$;
 - (b) there are $\mathcal{P}_{n+1}, \dots, \mathcal{P}_{n+m} \in \mathcal{P}$ with $\mathcal{P}_0 = \mathcal{P}_{n+m}$ or $\mathcal{Q}_0 = \mathcal{P}_{n+m}$ such that $\mathcal{P}_i \neq \mathcal{P}_j$ for $1 \leq i < j \leq n + m$, \mathcal{P}_{i+1} is a neighbor of \mathcal{P}_i for all $n \leq i < n + m$, and \mathcal{P}_i is not a cycling and cycled neighbor for all $1 \leq i \leq n + m$;

We use \mathcal{R}_0 to denote the set of all pairs $(\mathcal{P}_r, \xi(\text{tail}_\sigma(\mathcal{P}_r)))$ where \mathcal{P}_r is a nominal path. A primary cycle Θ_0 is a cycle w.r.t. \mathcal{R}_0 . Furthermore, we define a reachable cycle Θ' of a cycle of Θ if Θ' is a cycle w.r.t. the set of all pairs (\mathcal{P}_r, ξ_r) where \mathcal{P}_r is a cycled path of Θ and ξ_r is the set of all cycled rays of \mathcal{P}_r .

Note that a cycle Θ may encapsulate a *loop* if it includes two quadruples $(\mathcal{P}, \rho, \mathcal{Q}, \nu)$, $(\mathcal{P}', \rho', \mathcal{Q}', \nu')$ such that \mathcal{Q}' is a reachable path of \mathcal{Q} via ρ . A loop can be formed from a sequence $\mathcal{P}_1, \dots, \mathcal{P}_n \in \mathcal{P}$ ($n > 3$) such that $\mathcal{P}_1 = \mathcal{P}_n$, $\mathcal{P}_i \neq \mathcal{P}_j$ for $1 \leq i < j < n$ and \mathcal{P}_{i+1} is a neighbor of \mathcal{P}_i for $1 \leq i < n$. Moreover, it is possible that there are two quadruples $(\mathcal{P}, \rho, \mathcal{Q}, \nu)$, $(\mathcal{P}', \rho', \mathcal{Q}', \nu') \in \Theta$ such that $\mathcal{Q}' = \mathcal{Q}$, $\nu = \nu'$ and $\mathcal{P}' \neq \mathcal{P}$, $\rho \neq \rho'$, or $\mathcal{P}' = \mathcal{P}$, $\rho = \rho'$ and $\mathcal{Q}' \neq \mathcal{Q}$, $\nu \neq \nu'$.

Intuitively, a (primary) cycle allows one to “cut” all paths started from nominal paths of a frame into two parts : the first path which is connected to nominal paths is not replicated while the second part can be infinitely lengthened. Condition 1, Definition 8 says that a cycle should not include nominal star-types which must not replicated. Condition 2 says that a cycled path “matches” its cycling path via a ray with the same label. Condition 3 not only provides the relationship between two paths \mathcal{P}, \mathcal{Q} for each $(\mathcal{P}, \rho, \mathcal{Q}, \nu) \in \Theta$ but also ensures that all *non-cycled* neighbors of each \mathcal{P} are filled in a cycle. Condition 4 ensures that an extension of cycled paths \mathcal{P} via their cycled neighbors is possible by replicating paths from its cycling path \mathcal{Q} via cycling rays.

As a consequence, the existence of a cycle allows one to “unravel” a set \mathcal{P} of paths in a frame to obtain a possibly infinite set $\widehat{\mathcal{P}}$ of paths. The following lemma characterizes this crucial property and provides a bound on the size of a cycle.

Lemma 3. *Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ be a frame for a $SHOIQ_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$. Let Θ be a cycle in \mathcal{F} .*

1. *There exists an extension $\widehat{\mathcal{P}}_\Theta$ of paths between cycled and cycling paths such that each path $\mathcal{P}_0 \in \widehat{\mathcal{P}}_\Theta$ has exactly $|\xi(\text{tail}_\sigma(\mathcal{P}_0))|$ neighbors.*
2. *If Θ' is a reachable cycle of Θ then $|\Theta'| \leq |\Theta| \times |\xi|^{2^\ell}$ where $|\xi|$ is the maximal number of rays of a star-type, and $\ell = 2^{2 \times |\mathbf{CL}(\mathcal{T}, \mathcal{R})| \times |\mathbf{R}(\mathcal{T}, \mathcal{R})|}$.*

A proof of Lemma 3 can be based on the fact that all paths between cycling and cycled paths of a cycle do not cross the borders defined by the cycle. Therefore, these paths can be replicated and pasted to cycled paths. With regard to the size of a cycle, we can use the following construction: each path starts from a nominal star-type in \mathcal{N}_o and is lengthened through star-types (more precisely, through layers of rays of star-types). We define inductively a level n of a path \mathcal{P} as follows: (i) all nominal paths are at level 0, (ii) a path \mathcal{P}' is at level $i + 1$ if it has a neighbor at level i , and all neighbors of \mathcal{P}' are at a level which are equal or greater than i . This implies that there are no two neighbor paths which are located on two levels whose difference is greater than 1.

Assume that there is a pair of paths $(\mathcal{Q}, \mathcal{Q}')$ such that \mathcal{Q} is at level $i > 1$ and \mathcal{Q}' is a ν -neighbor of \mathcal{Q} at level $i - 1$ iff there is a pair of paths $(\mathcal{P}, \mathcal{P}')$ such that \mathcal{P} is at level $j > i$, \mathcal{P}' is a ρ -neighbor of \mathcal{P} at level $j + 1$, and $\mathcal{L}(\mathcal{Q}) = \mathcal{L}(\mathcal{P}')$, $\mathcal{L}(\mathcal{Q}') = \mathcal{L}(\mathcal{P})$, $r(\nu) = r^-(\rho)$. This implies that all such quadruples $(\mathcal{P}, \rho, \mathcal{Q}, \nu)$ can form a cycle. Moreover, there are at most ℓ different labels of pairs (\mathcal{Q}, ν) . This implies that one cycle can be detected after creating at most 2^ℓ levels. Thus, we have $|\Theta'| \leq |\Theta| \times |\xi|^{2^\ell}$ where $|\xi|$ is the maximal number of rays of star-type. A more complete proof of Lemma 3 can be found in [14].

Let Θ be a cycle in a frame. Definition 8 ensures that each reachable path of some path \mathcal{Q} with $(\mathcal{P}, \rho, \mathcal{Q}, \nu) \in \Theta$ goes through a star-type $\sigma = \text{tail}_\sigma(\mathcal{P}')$ with some

$(\mathcal{P}', \rho', \mathcal{Q}', \nu') \in \Theta$. As mentioned in Lemma 3, such a cycle allows one to “unravel” infinitely the frame to obtain a model of a KB in \mathcal{SHOIQ} (without transitive closure of roles). However, such a cycle structure is not sufficient to represent models of a KB with transitive closure of roles since a concept such as $\exists Q^\oplus.D \in \mathcal{L}(\mathcal{P})$ can be satisfied by a Q -reachable path \mathcal{P}' of \mathcal{P} which is arbitrarily far from \mathcal{P} . There are the following possibilities for an algorithm which builds a frame: (i) the algorithm stops building the frame as soon as a cycle Θ is detected such that each concept of the form $\exists Q^\oplus.D$ occurring in $\mathcal{L}(\mathcal{P})$ is satisfied for each cycled path \mathcal{P} of Θ , i.e., \mathcal{P} has a Q -reachable path \mathcal{P}' with $\exists Q.D \in \mathcal{L}(\mathcal{P})$, (ii) despite of several detected cycles, the algorithm continues building the frame until each concept of the form $\exists Q^\oplus.D$ occurring in $\mathcal{L}(\mathcal{P})$ is satisfied for each cycled path \mathcal{P} of Θ . If we adopt the first possibility, the completeness of such an algorithm cannot be established since there are models in which paths satisfying concepts of the form $\exists Q^\oplus.D$ can spread over several “iterative structures” such as cycles. For this reason, we adopt the second possibility by introducing into frames an additional structure, namely *blocking-blocked cycles*, which determines a sequence of cycles $\Theta_1, \dots, \Theta_k$ such that Θ_{i+1} is a reachable cycle of Θ_i for satisfying concepts of the form $\exists Q^\oplus.D$.

Definition 9 (blocking). Let $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ be a frame for a $\mathcal{SHOIQ}_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$ with a set \mathcal{P} of paths in \mathcal{F} . A cycle Θ' is blocked by a cycle Θ if there are cycles $\Theta_1, \dots, \Theta_k$ with $\Theta = \Theta_1$, $\Theta' = \Theta_k$ such that Θ_{i+1} is a reachable cycle of Θ_i for $1 \leq i < k$, and the following conditions are satisfied:

1. For each $1 \leq i < k$, there is no cycle Θ'' such that
 - (a) Θ'' is a reachable cycle of Θ_i and Θ_{i+1} is a reachable cycle of Θ'' , and
 - (b) For each $(\mathcal{P}, \rho, \mathcal{Q}, \nu) \in \Theta''$ and each concept $\exists Q^\oplus.D \in \mathcal{L}(\mathcal{P})$, \mathcal{P} has a Q -reachable path \mathcal{P}' via a non cycled ray with $\exists Q.D \in \mathcal{L}(\mathcal{P}')$ iff the ν -neighbor \mathcal{Q}' of \mathcal{Q} has a Q -reachable path \mathcal{Q}'' via a non cycling ray with $\exists Q.D \in \mathcal{L}(\mathcal{Q}'')$.
2. For each $(\mathcal{P}_k, \rho_k, \mathcal{Q}_k, \nu_k) \in \Theta_k$, there is some $(\mathcal{P}_1, \rho_1, \mathcal{Q}_1, \nu_1) \in \Theta_1$ such that
 - (a) $\mathcal{L}(\mathcal{P}_1) = \mathcal{L}(\mathcal{P}_k)$, $\mathcal{L}(\mathcal{Q}_1) = \mathcal{L}(\mathcal{Q}_k)$, $r(\rho_1) = r(\rho_k)$, and
 - (b) If there is a concept $\exists Q^\oplus.D \in \mathcal{L}(\mathcal{P}_k)$ such that the path \mathcal{P}_k has no Q -reachable path \mathcal{P}' with $\exists Q.D \in \mathcal{L}(\mathcal{Q}')$ then the path \mathcal{Q}_1 has a Q -reachable path \mathcal{Q} such that the two following conditions are satisfied:
 - i. $\exists Q.D \in \mathcal{L}(\mathcal{Q})$, or \mathcal{Q} has a Q -reachable path \mathcal{Q}' with $\exists Q.D \in \mathcal{L}(\mathcal{Q}')$,
 - ii. there are $(\mathcal{P}_j, \rho_j, \mathcal{Q}_j, \nu_j) \in \Theta_j$, $(\mathcal{P}_{j+1}, \rho_{j+1}, \mathcal{Q}_{j+1}, \nu_{j+1}) \in \Theta_{j+1}$ with some $1 \leq j < k$ such that \mathcal{Q}' is a reachable path of \mathcal{Q}_j and \mathcal{Q}_{j+1} is a reachable path of \mathcal{Q}' .

In this case, we say that the path \mathcal{P}_k is blocked by the path \mathcal{Q}_1 via the ray ρ_k . \triangleleft

Definition 9 provides an exact structure of a frame in which blocked paths can be detected. Such a frame contains sequentially reachable cycles between a blocking cycle Θ_1 and its blocked cycle Θ_k , which allows for unravelling the frame between Θ_k and Θ_1 , and satisfying all concepts of the form $\exists Q^\oplus.D$ in the labels of paths in Θ_1 . Condition 1 ensures that there is no useless cycle for the satisfaction of concepts $\exists Q^\oplus.D$ which is located between two cycles Θ_i and Θ_i with $i < k$. For a concept $\exists Q^\oplus.D \in \mathcal{L}(\mathcal{P}_k)$ that is not satisfied from the path \mathcal{P}_k to all existing paths (i.e. it is

not satisfied in the “past”), it must be satisfied from \mathcal{P}_k to paths that are devised by unravelling (i.e. it is satisfied in “the future”). Therefore, it is required that such concepts $\exists Q^\oplus.D$ are satisfied in the “future” from the blocking path \mathcal{P}_1 of \mathcal{P}_k (Condition 2, Definition 9). Moreover, for a concept $\exists Q^\oplus.D \in \mathcal{L}(\mathcal{P})$ that is not satisfied in the “past”, either it is satisfied from \mathcal{P} to some paths that are explicitly added to the frame, or it is propagated to a some blocked path thanks to Property 11, Definition 4.

Remark 2. The constant k mentioned in Definition 9 depends to the number of distinct ray labels (i.e. the triple $\langle \mathcal{L}(\mathcal{P}), r(\rho), l(\rho) \rangle$ for each ray $\rho \in \text{tail}_\sigma(\mathcal{P})$) occurring a blocking cycle Θ_1 and the number of concepts $\exists Q^\oplus.D$ occurring in each cycling path label in Θ_1 . Since the number of distinct ray labels is bounded by ℓ (Lemma 3) and the number of concepts $\exists Q^\oplus.D$ occurring in each cycling path label is bounded by $\text{CL}(\mathcal{T}, \mathcal{R})$, we have k is bounded by $2 \times \ell$ where $\ell = 2^{2 \times |\text{CL}(\mathcal{T}, \mathcal{R})| \times |\mathbf{R}(\mathcal{T}, \mathcal{R})|}$.

Definition 10 (valid frame). Let $(\mathcal{T}, \mathcal{R})$ be a \mathcal{SHOIQ} knowledge base. A frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ with a set \mathcal{P} of paths is valid if the following conditions are satisfied:

1. For each nominal $o \in \mathbf{C}_o$, there is a unique $\sigma_o \in \mathcal{N}_o$ such that $o \in \lambda(\sigma_o)$ and $\delta(\sigma_o) = 1$;
2. For each star-type $\sigma \in \mathcal{N}$, σ is valid.
3. If $\exists Q^\oplus.C \in \mathcal{L}(\mathcal{P}_0)$ for some $\mathcal{P}_0 \in \mathcal{P}$ then there are $\mathcal{P}, \mathcal{P}' \in \mathcal{P}$ such that one of the following conditions is satisfied:
 - (a) $\mathcal{P}_0 = \mathcal{P} = \mathcal{P}'$ and $\exists Q.C \in \mathcal{L}(\mathcal{P}_0)$;
 - (b) \mathcal{P}' is a Q -reachable of \mathcal{P} , and $\exists Q.C \in \mathcal{L}(\mathcal{P}')$ where $\mathcal{P} = \mathcal{P}_0$ or \mathcal{P} blocks \mathcal{P}_0 ;
 - (c) \mathcal{P} is a Q^\ominus -reachable of \mathcal{P}' , and $\exists Q.C \in \mathcal{L}(\mathcal{P}')$ where $\mathcal{P} = \mathcal{P}_0$ or \mathcal{P} blocks \mathcal{P}_0 . ◁

Conditions 1-3 in Definition 10 ensure the satisfaction of tableau properties in Definition 3. Note that Condition 1 is compatible with the fact that cycles in a frame never consist of nominal star-types (Definition 8). In particular, Condition 3 provides the satisfaction of concepts $\exists Q^\oplus.D$ occurring in the labels of paths thanks to the blocking condition introduced in Definition 9.

We now present Algorithm 1 for building a valid frame. This algorithm starts by adding nominal star-types to the frame. For each non blocked path \mathcal{P} with a ray $\rho \in \xi(\text{tail}_\sigma(\mathcal{P}))$ such that $\delta(\text{tail}_\sigma(\mathcal{P}), \rho) = \delta(\text{tail}_\sigma(\mathcal{P})) + 1$, the algorithm picks in a non-deterministic way a valid star-type ω that matches $\text{tail}_\sigma(\mathcal{P})$ via ρ , and updates the values $\Omega(\text{tail}_\sigma(\mathcal{P}), \rho)$, $\Omega(\omega, \rho')$, $\delta(\text{tail}_\sigma(\mathcal{P}), \rho)$, $\delta(\omega, \rho')$, eventually, $\delta(\text{tail}_\sigma(\mathcal{P}))$ and $\delta(\omega)$ by calling $\text{updateFrame}(\dots)$. The algorithm terminates when a blocked cycle is detected. To check the blocking condition, the algorithm can compare \mathcal{R}_i for each new level i of rays with each \mathcal{R}_j for all $j < i$ (the notion of levels of rays in a frame is given in the proof of Lemma 3) where \mathcal{R}_j is denoted for the set of different ray labels at level i . If $\mathcal{R}_j = \mathcal{R}_i$ and the last cycle that was detected located at some level $l < j$, then a new (reachable) cycle from level j to i is formed.

Figure 2 depicts a frame when executing Algorithm 1 for \mathcal{K}_1 in the example presented in Section 1. The algorithm builds a frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ where $\mathcal{N} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ and $\mathcal{N}_o = \{\sigma_0\}$. The dashed arrows indicate how the function $\Omega(\sigma, \rho)$ can be built. For example, $\Omega(\sigma_0, \rho_0) = \{(\sigma_1, \nu_0, 1)\}$, $\Omega(\sigma_0, \rho_1) = \{(\sigma_2, \rho'_0, 1)\}$

Require: A $SHOIQ_{(+)}$ knowledge base $(\mathcal{T}, \mathcal{R})$
Ensure: A frame $\langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ for $(\mathcal{T}, \mathcal{R})$

- 1: Let Σ be the set of all star-types for $(\mathcal{T}, \mathcal{R})$
- 2: **for all** $o \in \mathbf{C}_o$ **do**
- 3: **if** there is no $\sigma \in \mathcal{N}$ such that $o \in \lambda(\sigma)$ **then**
- 4: Choose a star-type $\sigma_o \in \Sigma$ such that $o \in \lambda(\sigma_o)$
- 5: Set $\delta(\sigma_o) = 1$, $\mathcal{N} = \mathcal{N} \cup \{\sigma_o\}$ and $\mathcal{N}_o = \mathcal{N}_o \cup \{\sigma_o\}$
- 6: Set $\delta(\sigma_o, \rho) = 0$, $\Omega(\sigma_o, \rho) = \emptyset$ for all $\rho \in \xi(\sigma_o)$
- 7: **end if**
- 8: **end for**
- 9: **while** there is a path \mathcal{P} that is not blocked and a ray $\rho \in \xi(\text{tail}_\sigma(\mathcal{P}))$ such that $\text{tail}_\delta(\mathcal{P}) = \delta(\text{tail}_\sigma(\mathcal{P}), \rho) + 1$ **do**
- 10: Choose a star-type $\sigma' \in \Sigma$ such that there is a ray $\rho' \in \xi(\sigma')$ satisfying $l(\rho) = \lambda(\sigma')$, $l(\rho') = \lambda(\sigma)$, $r(\rho') = r^-(\rho)$, and $\sigma' \in \mathcal{N}$ implies $\delta(\sigma') = \delta(\sigma', \rho') + 1$ or $\delta(\sigma') = \delta(\sigma', \rho'')$ for all $\rho'' \in \xi(\sigma')$
- 11: updateFrame($\sigma, \rho, \sigma', \rho'$)
- 12: **end while**

Algorithm 1. An algorithm for building a frame

Require: A star-type $\sigma \in \mathcal{N}$ in a frame $\mathcal{F} = \langle \mathcal{N}, \mathcal{N}_o, \Omega, \delta \rangle$ with a ray $\rho \in \xi(\sigma)$, and a new star-type σ' with a ray $\rho' \in \xi(\sigma')$ such that $l(\rho) = \lambda(\sigma')$, $l(\rho') = \lambda(\sigma)$, $r(\rho') = r^-(\rho)$
Ensure: updateFrame($\sigma, \rho, \sigma', \rho'$)

- 1: **if** there exists a star-type $\omega \in \mathcal{N}$ such that ω is equivalent to σ' **then**
- 2: Set $\delta(\sigma, \rho) = \delta(\sigma, \rho) + 1$
- 3: Let $\nu \in \xi(\omega)$ such that $r(\nu) = r(\rho')$ and $l(\nu) = l(\rho')$
- 4: **if** $\delta(\omega, \nu) = \delta(\omega)$ **then**
- 5: Set $\delta(\omega) = \delta(\omega) + 1$
- 6: **end if**
- 7: Set $\delta(\omega, \nu) = \delta(\omega, \nu) + 1$
- 8: add($\Omega(\omega, \nu), (\sigma, \rho, \delta(\sigma, \rho))$)
- 9: add($\Omega(\sigma, \rho), (\omega, \nu, \delta(\omega, \nu))$)
- 10: **else**
- 11: Add σ' to \mathcal{N}
- 12: Set $\delta(\sigma, \rho) = \delta(\sigma, \rho) + 1$
- 13: Set $\delta(\sigma') = 1$, $\delta(\sigma', \rho') = 1$ and $\Omega(\sigma', \rho') = \{(\sigma, \rho, \delta(\sigma, \rho))\}$
- 14: Set $\delta(\sigma', \rho'') = 0$ and $\Omega(\sigma', \rho'') = \emptyset$ for all $\rho'' \neq \rho'$
- 15: add($\Omega(\sigma, \rho), (\sigma', \rho', 1)$)
- 16: **end if**

Algorithm 2. updateFrame($\sigma, \rho, \sigma', \rho'$) updates \mathcal{F} when adding σ' to \mathcal{N}

where ρ_0 and ρ_1 are the respective horizontal and vertical rays of σ_0 ; ν_0 is the left ray of σ_1 ; ρ'_0 is the vertical ray of σ_2 . Moreover, the directed dashed arrow from σ_0 to σ_1 indicates that the ray ρ_0 of σ_0 can match the ray ν_0 on the left ray of σ_1 since $l(\rho_0) = \lambda(\sigma_1)$, $r(\nu_0) = \lambda(\sigma_0)$, $r(\nu_0) = r^-(\rho_0)$.

The algorithm generates $\delta(\sigma_0) = 1$, $\delta(\sigma_1) = 1$, $\delta(\sigma_2) = 1$ and forms a cycle Θ consisting of the following quadruples : $((\sigma_3, 3), \rho_1, (\sigma_3, 2), \rho_2)$ (ρ_1 and ρ_2 are the right and left rays of σ_3 , respectively) and $((\sigma_4, 2), \rho_3, (\sigma_4, 1), \rho_4)$ (ρ_3 and ρ_4 are the right and left rays of σ_4 respectively). Note that for the sake of brevity, we use just $\text{tail}_\sigma(\mathcal{P})$ and $\text{tail}_\delta(\mathcal{P})$ to denote a path in the quadruples.

The cycle Θ is blocked since all concepts $\exists S^+.\{o\}$ occurring in cycled paths are satisfied. A model of the ontology can be built by starting from σ_0 and getting (i) σ_4 via σ_1 , (ii) σ_3 via σ_1 , and (iii) σ_3 via σ_2 . From σ_3 and σ_4 , the model goes through σ_3 and σ_4 infinitely. Note that from any individual x satisfying σ_3 (or σ_4), i.e. the ‘‘label’’ of x contains $\exists Q^+.\{o\}$, there is a path containing S which goes back the individual satisfying σ_0 . Thus, the concept $\exists Q^+.\{o\}$ is satisfied for each individual whose label contains $\exists Q^+.\{o\}$.

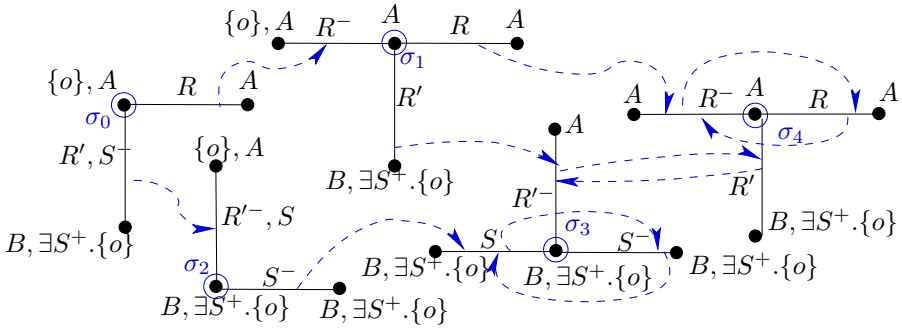


Fig. 2. A frame obtained by Algorithm 1 for \mathcal{K}_1 in the example in Section 1

Lemma 4. *Let $(\mathcal{T}, \mathcal{R})$ be a $\mathcal{SHOIQ}_{(+)}$ knowledge base.*

1. *Algorithm 1 terminates.*
2. *If Algorithm 1 can build a valid frame for $(\mathcal{T}, \mathcal{R})$ then there is a tableau for $(\mathcal{T}, \mathcal{R})$.*
3. *If there is a tableau for $(\mathcal{T}, \mathcal{R})$ then Algorithm 1 can build a valid frame \mathcal{F} for $(\mathcal{T}, \mathcal{R})$.*

Proof (sketch). Let Θ_k be a blocked cycle by Θ_1 . According to Remark 2, k is bounded by $\mathcal{O}(2^{|\langle \mathcal{T}, \mathcal{R} \rangle|})$. Moreover, after eliminating ‘‘useless cycles’’ between two cycles Θ_i and Θ_{i+1} for $1 \leq i < k$ according to Condition 1, Definition 9 the number of useful cycles between Θ_i and Θ_{i+1} is bounded by $\mathcal{O}(2^{|\langle \mathcal{T}, \mathcal{R} \rangle|})$. This implies that Algorithm 1 can add at most a triple exponential number of paths to the frame to form a blocked cycle. For the soundness of Algorithm 1, we can extend the set \mathcal{P} of paths to a set $\widehat{\mathcal{P}}$ of extended paths by ‘‘unravelling’’ the frame between blocking-blocked cycles. The set $\widehat{\mathcal{P}}$ allows one to satisfy concepts $\exists Q^\oplus.D$ in blocked paths which are not satisfied in the

“past”. Moreover, a concept $\exists Q^\oplus.D$ of a path that is not satisfied in the “past” will be propagated to a blocked path via a Q -path. Therefore, it will be satisfied in $\widehat{\mathcal{P}}$. Unlike the unravelling of a completion graph for \mathcal{SHOIQ} where there is no loop in the model, the unravelling of a frame may yield an infinite number of loops in the model. Note that the unravelling of a frame replicates cycles which may encapsulate loops.

Regarding completeness, we first reduce a tableau to a frame that does not contain any useless cycle. Then, we use the obtained frame to guide the algorithm (i) to choose valid star-types, (ii) to ensure that $\delta(\sigma) = 1$ for each nominal star-type σ , and (iii) to detect a pair (Θ_1, Θ_k) of blocking and blocked cycles as soon as some “representative” concepts of the form $\exists Q^\oplus.D$ in Θ_1 are satisfied. We refer the readers to [14] for a complete proof of Lemma 4. \square

The following theorem is a consequence of Lemma 4.

Theorem 1. *The problem of consistency for $\mathcal{SHOIQ}_{(+)}$ can be decided in non-deterministic triply exponential time in the size of a $\mathcal{SHOIQ}_{(+)}$ knowledge base.*

4 Optimizing The Algorithm

The algorithm for deciding the consistency of a $\mathcal{SHOIQ}_{(+)}$ knowledge base (Algorithm 1) uses at most a doubly exponential number of star-types to build a frame. This is due to the fact that numbers are encoded in binary, that is, a star-type may have an exponential number of rays. Pratt-Hartmann [9] has shown that it is possible to use an exponential number of star-types to represent a model of a KB in \mathcal{C}^2 which is slightly different from \mathcal{SHOIQ} in terms of expressiveness. If we can transfer this method to \mathcal{SHOIQ} for compressing star-types, it would be applied to $\mathcal{SHOIQ}_{(+)}$ since the number of star-types in a frame does not depend on the presence of transitive closure of roles.

Another technique presented in [15] can be used to reduce non-determinisms due to the choice of valid star-types. Instead of guessing a valid star-type from a set of valid star-types, this technique allows one to build a star-type σ by applying expansion rules to concepts in the core label of σ . Hence, when a star-type σ is transformed into σ' by an expansion rule, an algorithm that implements this technique has to update not only $\Omega(\sigma', \rho')$ and $\delta(\sigma')$ but also $\Omega(\sigma'', \rho'')$ and $\delta(\sigma'')$ for each neighbor σ'' of σ and σ' (σ'' is a neighbor of σ' if there is some $(\sigma'', \rho'', d'') \in \Omega(\sigma', \rho')$). These updates must ensure that each path which has got through σ can now get through σ' . This process of changes can spread over neighbors of σ'' and so on.

With regard to blocking, the technique presented in [15] can take advantage of a specific structure of frames for \mathcal{SHOIQ} to design an efficient algorithm for checking blocking condition. This structure consists of partitioning star-types into layers. Although such a structure of frames cannot be maintained for $\mathcal{SHOIQ}_{(+)}$, paths in a frame for $\mathcal{SHOIQ}_{(+)}$ would allow us to achieve the same behavior.

5 Conclusion

In this paper, we have presented a decision procedure for the description logic \mathcal{SHOIQ} with transitive closure of roles in concept axioms, whose decidability was not known.

The most significant feature of our contribution is to introduce a structure based on a new blocking condition for characterizing models which have an infinite non-tree-shaped part. This structure would provide an insight into regularity of such models which would be enjoyed by a more expressive DL, such as $ZOIQ$ [6], whose decidability remains open. In future work, we aim to improve the algorithm by making it more goal-directed and aim to investigate another open question about the hardness of $SHOIQ_{(+)}$.

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