Generalizations of Approximations

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Abstract. In this paper we consider a generalization of the indiscernibility relation, i.e., a relation R that is not necessarily reflexive, symmetric, or transitive. There exist 36 basic definitions of lower and upper approximations based on such relation R. Additionally, there are six probabilistic approximations, generalizations of 12 corresponding lower and upper approximations. How to convert remaining 24 lower and upper approximations to 12 respective probabilistic approximations is an open problem.

1 Introduction

Rough set theory is based on ideas of lower and upper approximations. For completely defined data sets such approximations are defined using an *indiscerniblity* relation R [25, 26], an equivalence relation. A probabilistic approximation, a generalization of lower and upper approximations, was introduced in [36] and then studied in many papers, e.g., [19, 27–29, 34, 40, 42–45]. Probabilistic approximations are defined using an additional parameter, interpreted as probability, and denoted by α. Lower and upper approximations are special cases of the probability approximation, if $\alpha = 1$, the probabilistic approximation becomes the lower approximation; if α is quite small, the probabilistic approximation is equal to the upper approximation.

Some data sets, e.g., incomplete data sets, are described by relations that are not equivalence relations [8, 9]. Lower and upper approximations for such a relation R that does not need to be reflexive, symmetric or transitive were studied in many papers as well. Corresponding definitions were summarized in [14, 16], where also basic properties were studied. There exist 36 basic definitions of lower and upper approximations based on such general relation R. These lower and upper approximations were generalized to probabilistic approximations in [11]. There are six such probabilistic approximations, generalizations of 12 corresponding lower and upper approximations, since a probabilistic approximation, with α between 0 and 1, represents the entire spectrum of approximations, including lower and upper approximations. How to convert remaining 24 lower and upper approximations to 12 respective probabilistic approximations is an open problem.

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2 Equivalence Relations

First we will quote some definitions for complete data sets that are characterized by an equivalence relation, namely, by the indiscernibility relation [25, 26].

2.1 Lower and Upper Approximations

The set of all cases of a data set is denoted by U. Independent variables are called *attributes* and a dependent variable is called a *decision* and is denoted by d. The set of all attributes will be denoted by A. For a case x, the value of an attribute a will be denoted by $a(x)$. If for any $a \in A$ and $x \in U$ the value $a(x)$ is specified, the data set is called *completely specified*, or *complete*.

Rough set theory, see, e.g., [25] and [26], is based on the idea of an indiscernibility relation, defined for complete data sets. Let B be a nonempty subset of the set A of all attributes. The indiscernibility relation $IND(B)$ is a relation on U defined for $x, y \in U$ by

$$(x, y) \in IND(B) \;\; if\ and\ only\ if\ a(x) = a(y)\ for\ all\ a \in B.$$

A complete data set may be described by an (U, R) called an *approximation space*, where R is an indiscernibility relation $IND(B)$ on U.

The indiscernibility relation $IND(B)$ is an equivalence relation. Equivalence classes of $IND(B)$ are called *elementary sets* of B and are denoted by $[x]_B$. For completely specified data sets lower and upper approximations are defined on the basis of the indiscernibility relation. Any finite union of elementary sets, associated with B, will be called a *B-definable set*. Let X be any subset of the set U of all cases. The set X is called a *concept* and is usually defined as the set of all cases defined by a specific value of the decision. In general, X is not a B-definable set. However, set X may be approximated by two B-definable sets, the first one is called a *B-lower approximation* of X, denoted by $\underline{B}X$ and defined by

$$\{x \in U | [x]_B \subseteq X\}.$$

The second set is called a B-upper approximation of X, denoted by $\overline{B}X$ and defined by

$$\{x \in U | [x]_B \cap X \neq \emptyset\}.$$

The above shown way of computing lower and upper approximations, by constructing these approximations from singletons x, will be called the *first method*. The B-lower approximation of X is the greatest B-definable set, contained in X. The B-upper approximation of X is the smallest B-definable set containing X.

As it was observed in [26], for complete data sets we may use a *second method* to define the B-lower approximation of X, by the following formula

$$\cup\{[x]_B | x \in U, [x]_B \subseteq X\},$$

and the B-upper approximation of x may be defined, using the second method, by

$$\cup\{[x]_B | x \in U, [x]_B \cap X \neq \emptyset).$$

Note that for a binary relation R that is not an equivalence relation these two methods lead, in general, to different results.

2.2 Probabilistic Approximations

Let (U, R) be an approximation space, where R is an equivalence relation on U. A probabilistic approximation of the set X with the threshold α, $0 < \alpha \leq 1$, is denoted by $appr_\alpha(X)$ and defined by

$$\cup\{[x] \mid x \in U,\ Pr(X|[x]) \geq \alpha\},$$

where $[x]$ is an elementary set of R and $Pr(X|[x]) = \frac{|X \cap [x]|}{|[x]|}$ is the conditional probability of X given $[x]$.

Obviously, for the set X, the probabilistic approximation of X computed for the threshold equal to the smallest positive conditional probability $Pr(X \mid [x])$ is equal to the standard upper approximation of X. Additionally, the probabilistic approximation of X computed for the threshold equal to 1 is equal to the standard lower approximation of X.

3 Arbitrary Binary Relations

In this section we will discuss first lower and upper approximations and then probabilistic approximations based on an arbitrary binary relation R.

3.1 Lower and Upper Approximations

First we will quote some definitions from [14, 16]. Let U be a finite nonempty set, called the *universe*, let R be a binary relation on U, and let x be a member of U. The relation R is a generalization of the indiscernibility relation. In general, R does not need to be reflexive, symmetric, or transitive. Basic granules defined by a relation R are called R-*successor* and R-*predecessor* sets.

An R-*successor* set of x, denoted by $R_s(x)$, is defined by

$$R_s(x) = \{y \mid xRy\}.$$

An R-*predecessor* set of x, denoted by $R_p(x)$, is defined by

$$R_p(x) = \{y \mid yRx\}.$$

Let X be a subset of U. A set X is R-*successor definable* if and only if $X = \emptyset$ or X is a union of some R-successor sets.

A set X is R-*predecessor definable* if and only if $X = \emptyset$ or X is a union of some R-predecessor sets.

Singleton, Subset and Concept Approximations. An R-*singleton successor lower approximation* of X, denoted by $\underline{appr}_s^{singleton}(X)$, is defined by

$$\{x \in U \mid R_s(x) \subseteq X\}.$$

The singleton successor lower approximations were studied in many papers, see, e.g., [8, 9, 20–23, 30–33, 35, 37–39, 41].

An R-*singleton predecessor lower approximation* of X, denoted by $\underline{appr}_p^{singleton}(X)$, is defined as follows

$$\{x \in U \mid R_p(x) \subseteq X\}.$$

The singleton predecessor lower approximations were studied in [30].

An R-*singleton successor upper approximation* of X, denoted by $\overline{appr}_s^{singleton}(X)$, is defined as follows

$$\{x \in U \mid R_s(x) \cap X \neq \emptyset\}.$$

The singleton successor upper approximations, like singleton successor lower approximations, were also studied in many papers, e.g., [8, 9, 20, 21, 30–33, 35, 37–39, 41].

An R-*singleton predecessor upper approximation* of X, denoted by $\overline{appr}_p^{singleton}(X)$, is defined as follows

$$\{x \in U \mid R_p(x) \cap X \neq \emptyset\}.$$

The singleton predecessor upper approximations were introduced in [30].

An R-*subset successor lower approximation* of X, denoted by $\underline{appr}_s^{subset}(X)$, is defined by

$$\cup \{R_s(x) \mid x \in U \text{ and } R_s(x) \subseteq X\}.$$

The subset successor lower approximations were introduced in [8, 9].

An R-*subset predecessor lower approximation* of X, denoted by $\underline{appr}_p^{subset}(X)$, is defined by

$$\cup \{R_p(x) \mid x \in U \text{ and } R_p(x) \subseteq X\}.$$

The subset predecessor lower approximations were studied in [30].

An R-*subset successor upper approximation* of X, denoted by $\overline{appr}_s^{subset}(X)$, is defined by

$$\cup \{R_s(x) \mid x \in U \text{ and } R_s(x) \cap X \neq \emptyset\}.$$

The subset successor upper approximations were introduced in [8, 9].

An R-*subset predecessor upper approximation* of X, denoted by $\overline{appr}_p^{subset}(X)$, is defined by

$$\cup \{R_p(x) \mid x \in U \text{ and } R_p(x) \cap X \neq \emptyset\}.$$

The subset predecessor upper approximations were studied in [30].

An R-*concept successor lower approximation* of X, denoted by $\underline{appr}_s^{concept}(X)$, is defined by

$$\cup \{R_s(x) \mid x \in X \text{ and } R_s(x) \subseteq X\}.$$

The concept successor lower approximations were introduced in [8, 9].

An R-*concept predecessor lower approximation* of X, denoted by $\underline{appr}_p^{concept}(X)$, is defined by

$$\cup \{R_p(x) \mid x \in X \text{ and } R_p(x) \subseteq X\}.$$

The concept predecessor lower approximations were introduced, for the first time, in [13].

An R-*concept successor upper approximation* of X, denoted by $\overline{appr}_s^{concept}(X)$, is defined by

$$\cup \{R_s(x) \mid x \in X \text{ and } R_s(x) \cap X \neq \emptyset\}$$

The concept successor upper approximations were studied in [8, 9, 23].

An R-*concept predecessor upper approximation* of X, denoted by $\overline{appr}_p^{concept}(X)$, is defined by

$$\cup \{R_p(x) \mid x \in X \text{ and } R_p(x) \cap X \neq \emptyset\}$$

The concept predecessor upper approximations were studied in [30].

Sets $\underline{appr}_s^{subset}(X)$, $\underline{appr}_s^{concept}(X)$, $\overline{appr}_s^{subset}(X)$, $\overline{appr}_s^{concept}(X)$ and $\overline{appr}_p^{singleton}(X)$ are R-successor definable, while sets $\underline{appr}_p^{subset}(X)$, $\underline{appr}_p^{concept}(X)$, $\overline{appr}_p^{subset}(X)$, $\overline{appr}_p^{concept}(X)$ and $\overline{appr}_s^{singleton}(X)$ are R-predecessor definable for any approximation space (U, R), see. e.g., [8, 10, 24].

Modified Singleton Approximations. Definability and duality of lower and upper approximations of a subset X of the universe U are basic properties of rough approximations defined for the standard lower and upper approximations [25, 26].

To avoid problems with inclusion for singleton approximations, the following modification of the corresponding definitions were introduced in [14]:

An R-*modified singleton successor lower approximation* of X, denoted by $\underline{appr}_s^{modsingleton}(X)$, is defined by

$$\{x \in U \mid R_s(x) \subseteq X \text{ and } R_s(x) \neq \emptyset\}.$$

An R-*modified singleton predecessor lower approximation* of X, denoted by $\underline{appr}_p^{modsingleton}(X)$, is defined by

$$\{x \in U \mid R_p(x) \subseteq X \text{ and } R_p(x) \neq \emptyset\}.$$

An R-*modified singleton successor upper approximation* of X, denoted by $\overline{appr}_s^{modsingleton}(X)$, is defined by

$$\{x \in U \mid R_s(x) \cap X \neq \emptyset \text{ or } R_s(x) = \emptyset\}.$$

An R-*modified singleton predecessor upper approximation* of X, denoted by $\overline{appr}_p^{modsingleton}(X)$, is defined by

$$\{x \in U \mid R_p(x) \cap X \neq \emptyset \text{ or } R_p(x) = \emptyset\}.$$

Largest Lower and Smallest Upper Approximations. For any relation R, the R-subset successor (predecessor) lower approximation of X is the largest R-successor (predecessor) definable set contained in X. It follows directly from the definition.

On the other hand, the smallest R-successor definable set containing X does not need to be unique. It was observed, for the first time, in [13].

Any R-smallest successor upper approximation, denoted by $\overline{appr}_s^{smallest}(X)$, is defined as a R-successor definable set with the smallest cardinality containing X. An R-smallest successor upper approximation does not need to be unique.

An R-smallest predecessor upper approximation, denoted by $\overline{appr}_p^{smallest}(X)$, is defined as an R-predecessor definable set with the smallest cardinality containing X. Likewise, an R-smallest predecessor upper approximation does not need to be unique.

Dual Approximations. As it was shown in [38], singleton approximations are *dual* for any relation R. In [16] it was proved that modified singleton approximations are also dual. On the other hand it was shown in [38] that if R is not an equivalence relation then subset approximations are not dual. Moreover, concept approximations are not dual as well, unless R is reflexive and transitive [14].

Two additional approximations were defined in [38]. The first approximation, denoted by $\underline{appr}_s^{dualsubset}(X)$, was defined by

$$\neg(\overline{appr}_s^{subset}(\neg X))$$

while the second one, denoted by $\overline{appr}_s^{dualsubset}(X)$ was defined by

$$\neg(\underline{appr}_s^{subset}(\neg X)),$$

where $\neg X$ denotes the complement of X.

These approximations are called an *R-dual subset successor lower* and *R-dual subset successor upper approximations*, respectively. Obviously, we may define as well an *R-dual subset predecessor lower approximation*

$$\neg(\overline{appr}_p^{subset}(\neg X))$$

and an *R-dual subset predecessor upper approximation*

$$\neg(\underline{appr}_p^{subset}(\neg X)).$$

By analogy we may define dual concept approximations. Namely, an *R-dual concept successor lower approximation* of X, denoted by $\underline{appr}_s^{dualconcept}(X)$ is defined by

$$\neg(\overline{appr}_s^{concept}(\neg X)).$$

An *R-dual concept successor upper approximation* of X, denoted by $\overline{appr}_s^{dualconcept}(X)$ is defined by

$$\neg(\underline{appr}_s^{concept}(\neg X)).$$

The set denoted by $\underline{appr}_p^{dualconcept}(X)$ and defined by the following formula

$$\neg(\overline{appr}_p^{concept}(\neg X))$$

will be called an *R-dual concept predecessor lower approximation*, while the set $\overline{appr}_p^{dualconcept}(X)$ defined by the following formula

$$\neg(\underline{appr}_p^{concept}(\neg X))$$

will be called an *R-dual concept predecessor upper approximation.*

These four R-dual concept approximations were introduced in [14].

Again, by analogy we may define dual approximations for the smallest upper approximations. The set, denoted by $\underline{appr}_s^{dualsmallest}(X)$ and defined by

$$\neg(\overline{appr}_s^{smallest}(\neg X)),$$

will be called an *R-dual smallest successor lower approximation* of X while the set denoted by $\underline{appr}_p^{dualsmallest}(X)$ and defined by

$$\neg(\overline{appr}_p^{smallest}(\neg X)).$$

will be called an *R-dual smallest predecessor lower approximation* of X.

These two approximations were introduced in [16].

Approximations with Mixed Idempotency. Smallest upper approximations, introduced in Section 3.1, and subset lower approximations are the only approximations discussed so far that satisfy the Mixed Idempotency Property, so

$$\underline{appr}_s(X) = \overline{appr}_s(\underline{appr}_s(X))(\underline{appr}_p(X) = \overline{appr}_p(\underline{appr}_p(X))), \tag{1}$$

and

$$\overline{appr}_s(X) = \underline{appr}_s(\overline{appr}_s(X))(\overline{appr}_p(X) = \underline{appr}_p(\overline{appr}_p(X))). \tag{2}$$

For the following approximations, defined sets satisfy the above two conditions. The upper approximation, denoted by $\overline{appr}_s^{subset\text{-}concept}(X)$ and defined by

$$\underline{appr}_s^{subset}(X) \cup \bigcup\{R_s(x) \mid x \in X - \underline{appr}_s^{subset}(X) \text{ and } R_s(x) \cap X \neq \emptyset\}$$

will be called an *R-subset-concept successor upper approximation* of X.

The upper approximation, denoted by $\overline{appr}_p^{subset\text{-}concept}(X)$ and defined by

$$\underline{appr}_p^{subset}(X) \cup \bigcup\{R_p(x) \mid x \in X - \underline{appr}_p^{subset}(X) \text{ and } R_p(x) \cap X \neq \emptyset\}$$

will be called an *R-subset-concept predecessor upper approximation* of X. The upper approximation, denoted by $\overline{appr}_s^{subset\text{-}subset}(X)$ and defined by

$$\underline{appr}_s^{subset}(X) \cup \bigcup\{R_s(x) \mid x \in U - \underline{appr}_s^{subset}(X) \text{ and } R_s(x) \cap X \neq \emptyset\}$$

will be called an *R-subset-subset successor upper approximation* of X.

The upper approximation, denoted by $\overline{appr}_p^{subset\text{-}subset}(X)$ and defined by

$$\underline{appr}_p^{subset}(X) \cup \bigcup\{R_p(x) \mid x \in U - \underline{appr}_p^{subset}(X) \text{ and } R_p(x) \cap X \neq \emptyset\}$$

will be called an *R-subset-subset predecessor upper approximation* of X.

These four upper approximations, together with $\underline{appr}_s^{subset}$ (or $\underline{appr}_p^{subset}$, respectively), satisfy Mixed Idempotency Property.

Note that for these four upper approximations corresponding dual lower approximations may be defined as well. These definitions are skipped since they are straightforward.

3.2 Probabilistic Approximations

By analogy with standard approximations defined for arbitrary binary relations, we will introduce three kinds of probabilistic approximations for such relations: singleton, subset and concept. For simplicity, we restrict our attention only to R-successor sets as the basic granules. Obviously, analogous three definitions based on R-predecessor sets may be easily introduced as well.

A *singleton probabilistic approximation* of X with the threshold α, $0 < \alpha \leq 1$, denoted by $appr_{\alpha}^{singleton}(X)$, is defined by

$$\{x \mid x \in U,\ Pr(X|R_s(x)) \geq \alpha\},$$

where $Pr(X|R_s(x)) = \frac{|X \cap R_s(x)|}{|R_s(x)|}$ is the conditional probability of X given $R_s(x)$.

A *subset probabilistic approximation* of the set X with the threshold α, $0 < \alpha \leq 1$, denoted by $appr_{\alpha}^{subset}(X)$, is defined by

$$\cup\{R_s(x) \mid x \in U,\ Pr(X|R_s(x)) \geq \alpha\}.$$

A *concept probabilistic approximation* of the set X with the threshold α, $0 < \alpha \leq 1$, denoted by $appr_{\alpha}^{concept}(X)$, is defined by

$$\cup\{R_s(x) \mid x \in X,\ Pr(X|R_s(x)) \geq \alpha\}.$$

Obviously, for the concept X, the probabilistic approximation of a given type (singleton, subset or concept) of X computed for the threshold equal to the smallest positive conditional probability $Pr(X \mid R_s(x))$ is equal to the standard upper approximation of X of the same type. Additionally, the probabilistic approximation of a given type of X computed for the threshold equal to 1 is equal to the standard lower approximation of X of the same type.

Results of many experiments on probabilistic approximations were published in [1–7, 12, 17, 18].

4 Conclusions

We discussed 36 basic definitions of lower and upper approximations based on a relation R that is not an equivalence relation. For such a relation R, there are six probabilistic approximations, generalizations of 12 corresponding lower and upper approximations. How to convert remaining 24 lower and upper approximations to 12 respective probabilistic approximations is an open problem.

Note that other definitions of approximations, called *local*, were discussed in [6, 13, 15]. First, local lower and upper approximations were introduced in [13, 15], then these approximations were generalized to probabilistic in a few different ways in [6].

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