Partial Approximation of Multisets and Its Applications in Membrane Computing

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Abstract. Partial nature of real–life problems requires working out partial approximation schemes. Partial approximation of sets is based on classical set theory. Its generalization for multisets gives a plausible opportunity to introduce an abstract concept of "to be close enough to a membrane" in membrane computing. The paper presents important features of general (maybe partial) multiset approximation spaces, their lattice theory properties, and shows how partial multiset approximation spaces can be applied to membrane computing.

Keywords: Rough set theory, multiset theory, partial approximation of multisets, lattice theory, membrane computing.

1 Introduction

Studies of set approximations were originally invented by Pawlak in the early 1980's [1, 2]. There are many different generalizations of classical Pawlakian rough set theory, among others, for multisets. A possible approach may rely on equivalence multiset relations [3], or general multirelations [4].

Partial nature of real–life problems, however, requires working out partial approximation schemes. The framework called the partial approximation of sets [5, 6] is based on classical set theory similarly to rough set theory. It was generalized for multisets [7, 8] in connection with membrane computing introduced by Păun in 2000 [9–11]. Membrane computing was motivated by biological and chemical processes in which an object has to be close enough to a membrane in order to be able to pass through it. Looking at regions as multisets, partial approximation of multisets gives a plausible opportunity to introduce the abstract, not necessarily space–like, concept of "to be close enough to a membrane". The paper presents the most important features of partial multiset approximation spaces, their lattice theory properties and applications to membrane computing.

The paper is organized as follows. Having reviewed the fundamental notions of multiset theory, Section 3 presents the concept of general multiset approximation space. Section 4 shows its generalized Pawlakian variant which is applied to membrane computing in Section 5.

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2 Fundamental Notions of Multiset Theory

Let U be a finite nonempty set. A *multiset* M, or *mset* M for short, over U is a mapping $M : U \to \mathbb{N} \cup \{\infty\}$, where $\mathbb{N}$ is the set of natural numbers. If $M(a) \neq 0$, it is said that a belongs to M, otherwise a does not belong to M. The set $M^* = \{a \in U \mid M(a) \neq 0\}$ is called the *support* of M.

The mset M is the *empty mset*, denoted by $\emptyset$ if $M^* = \emptyset$. An mset M is *finite* if $M(a) < \infty$ for all $a \in M^*$.

Let $\mathcal{MS}(U)$ denote the set of all msets over U.

Basic set–theoretical relations can be generalized for msets as follows.

Definition 1. *Let M, M_1, M_2 be msets over U.*

1. Multiplicity *relation for an mset M over U is: $a \in M$ $(a \in U)$ if $M(a) \geq 1$.*
2. *Let $n \in \mathbb{N}^+$ be a positive integer. n–*times multiplicity *relation for an mset M over U is the following: $a \in^n M$ $(a \in U)$ if $M(a) = n$.*
3. *$M_1 = M_2$ if $M_1(a) = M_2(a)$ for all $a \in U$ (mset* equality *relation).*
4. *$M_1 \sqsubseteq M_2$ if $M_1(a) \leq M_2(a)$ for all $a \in U$ (mset* inclusion *relation).*

The next definitions give the generalizations for msets of the basic set–theoretical operations.

Definition 2. *Let $M, M_1, M_2 \in \mathcal{MS}(U)$ be msets over U and $\mathcal{M} \subseteq \mathcal{MS}(U)$ be a set of msets over U.*

1. *$(M_1 \sqcap M_2)(a) = \mathsf{min}\{M_1(a), M_2(a)\}$ for all $a \in U$ (*intersection*).*
2. *$(\bigsqcap \mathcal{M})(a) = \mathsf{min}\{M(a) \mid M \in \mathcal{M}\}$ for all $a \in U$.*
3. *$(M_1 \sqcup M_2)(a) = \mathsf{max}\{M_1(a), M_2(a)\}$ for all $a \in U$ (*set–type union*).*
4. *$(\bigsqcup \mathcal{M})(a) = \mathsf{sup}\{M(a) \mid M \in \mathcal{M}\}$ for all $a \in U$. By definition, $\bigsqcup \emptyset = \emptyset$.*
5. *$(M_1 \oplus M_2)(a) = M_1(a) + M_2(a)$ for all $a \in U$ (*mset addition*).*
6. *For any $n \in \mathbb{N}$, n-*times addition *of M, denoted by $\oplus_n M$, is given by the following inductive definition:*
   (a) $\oplus_0 M = \emptyset$;
   (b) $\oplus_1 M = M$;
   (c) $\oplus_{n+1} M = \oplus_n M \oplus M$.
7. *$(M_1 \ominus M_2)(a) = \mathsf{max}\{M_1(a) - M_2(a), 0\}$ for all $a \in U$ (*mset subtraction*).*

By the n-times addition, the *n-times inclusion* relation ($\sqsubseteq^n$) can be defined.

Definition 3. *Let $M_1 \neq \emptyset, M_2$ be two msets over U.*
For any $n \in \mathbb{N}$, $M_1 \sqsubseteq^n M_2$ if $\oplus_n M_1 \sqsubseteq M_2$ but $\oplus_{n+1} M_1 \not\sqsubseteq M_2$.

Corollary 1. *Let $M_1 \neq \emptyset, M_2$ be two msets over U and $n \in \mathbb{N}$.*

1. *$M_1 \sqsubseteq^n M_2$ if and only if $nM_1(a) \leq M_2(a)$ for all $a \in U$ and there is an $a' \in U$ such that $(n+1)M_1(a') > M_2(a')$.*
2. *$M_1 \sqsubseteq^0 M_2$ if and only if $M_1 \not\sqsubseteq M_2$.*
3. *For all $n \in \mathbb{N}^+$, $M_1 \sqsubseteq^n M_2$ if and only if $\oplus_n M_1 \sqsubseteq^1 M_2$.*

3 Some Lattice Theory Properties of Set of Multisets

The next proposition is an immediate consequence of Definition 1 and 2 (for the lattice theory notions, see, e.g., [12–14]).

Proposition 1. $\langle \mathcal{MS}(U), \sqcap, \sqcup \rangle$ *is a complete lattice, that is*

1. *(a) operations $\sqcup$ and $\sqcap$ are idempotent, commutative and associative;*
   (b) operations $\sqcup$ and $\sqcap$ fulfill the absorption laws for all $M_1, M_2 \in \mathcal{MS}(U)$:
   $$M_1 \sqcap (M_1 \sqcup M_2) = M_1 \text{ and } M_1 \sqcup (M_1 \sqcap M_2) = M_1;$$
2. *$\bigsqcup \mathcal{M}$ and $\bigsqcap \mathcal{M}$ exist for every $\mathcal{M} \subseteq \mathcal{MS}(U)$.*

In addition, $\langle \mathcal{MS}(U), \sqsubseteq \rangle$ is a partially ordered set in which $M_1 \sqsubseteq M_2$ if and only if $M_1 \sqcup M_2 = M_2$, or equivalently, $M_1 \sqcap M_2 = M_1$ for all $M_1, M_2 \in \mathcal{MS}(U)$.

A set $\mathcal{M}$ of finite msets over U is called a *macroset* $\mathcal{M}$ over U [15].
We define the following two fundamental macrosets:

1. $\mathcal{MS}^n(U)$ $(n \in \mathbb{N})$ is the set of all msets M over U such that $M(a) \leq n$ for all $a \in U$, and
2. $\mathcal{MS}^{<\infty}(U) = \bigcup_{n=0}^{\infty} \mathcal{MS}^n(U)$.

Note that $\mathcal{MS}^0(U) = \emptyset$ and $\mathcal{MS}^n(U) \subsetneqq \mathcal{MS}^{n+1}(U)$ $(n = 0, 1, 2, \ldots)$. Moreover, $\mathcal{MS}^n(U)$ $(n \in \mathbb{N})$ is finite and $\mathcal{MS}^{<\infty}(U)$ is countably infinite.

$M_1 \sqcup M_2, M_1 \sqcap M_2 \in \mathcal{MS}^n(U)$ $(M_1, M_2 \in \mathcal{MS}^n(U))$ and the finiteness of $\mathcal{MS}^n(U)$ immediately imply that $\langle \mathcal{MS}^n(U), \sqcup, \sqcap \rangle$ $(n \in \mathbb{N}^+)$ is a complete sublattice of the lattice $\langle \mathcal{MS}(U), \sqcup, \sqcap \rangle$. Its top element is the mset M such that $M^* = U$, $M(a) = n$ $(a \in U)$, and its bottom element is the empty mset $\emptyset$.

$\langle \mathcal{MS}^{<\infty}(U), \sqcup, \sqcap \rangle$ is also a sublattice of the lattice $\langle \mathcal{MS}(U), \sqcup, \sqcap \rangle$. However, it is not a complete lattice since it lacks a top element. Nevertheless, $\langle \mathcal{MS}^{<\infty}(U), \sqsubseteq \rangle$ is a meet-semilattice such that $\bigsqcap \mathcal{M}$ exists in $\mathcal{MS}^{<\infty}(U)$ for every nonempty $\mathcal{M} \subseteq \mathcal{MS}^{<\infty}(U)$. Consequently, $\bigsqcup \mathcal{M}$ exists in $\mathcal{MS}^{<\infty}(U)$ for every subset $\mathcal{M} \subseteq \mathcal{MS}^{<\infty}(U)$ which has an upper bound in $\mathcal{MS}^{<\infty}(U)$, and

$$\bigsqcup \mathcal{M} = \bigsqcap \{M' \in \mathcal{MS}^{<\infty}(U) \mid \forall M \in \mathcal{M}(M \sqsubseteq M')\}.$$

4 General Multiset Approximation Spaces

A general mset approximation space has four components:

– a *domain* of the approximation space whose members are approximated;
– some distinguished members of the domain as the *basis* of approximations;
– *definable msets* deriving from base msets in some way as possible approximations of the members of the domain;
– an *approximation pair* determining the lower and upper approximations of the msets of the domain using definable msets.

Definable msets represent our available knowledge about the domain. They can be thought of as *tools*, in particular, base msets as *primary tools*, definable msets as *derived tools*. The way of getting derived tools from primary tools shows how primary tools are used. An approximation pair prescribes the *utilization* of primary and derived tools in a whole approximation process.

Definition 4. *The ordered 5–tuple* $\mathsf{MAS}(U) = \langle \mathcal{MS}^{<\infty}(U), \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u}\rangle$ *is a* (general) mset approximation space *over* U *with the domain* $\mathcal{MS}^{<\infty}(U)$ *if*

1. $\mathfrak{B} \subseteq \mathcal{MS}^{<\infty}(U)$ *and if* $B \in \mathfrak{B}$*, then* $B \neq \emptyset$ *(in notation* $\mathfrak{B} = \{B_\gamma \mid \gamma \in \Gamma\}$*);* $\mathfrak{B}$ *is called the* base system*, its members are called the* base msets*;*
2. $\mathfrak{D}_{\mathfrak{B}} \subseteq \mathcal{MS}^{<\infty}(U)$ *is an extension of* $\mathfrak{B}$ *satisfying the following minimal requirement: if* $B \in \mathfrak{B}$*, then* $\oplus_n B \in \mathfrak{D}_{\mathfrak{B}}$ *for all* $n \in \mathbb{N}$*; members of* $\mathfrak{D}_{\mathfrak{B}}$ *are called* definable msets*;*
3. *the functions* $\mathsf{l}, \mathsf{u} : \mathcal{MS}^{<\infty}(U) \to \mathcal{MS}^{<\infty}(U)$ *(called* lower *and* upper approximation functions*) form a* weak approximation pair $\langle \mathsf{l}, \mathsf{u}\rangle$ *if*
   - *(C0)* $\mathsf{l}(\mathcal{MS}^{<\infty}(U)), \mathsf{u}(\mathcal{MS}^{<\infty}(U)) \subseteq \mathfrak{D}_{\mathfrak{B}}$ *(*definability *of* l, u*);*
   - *(C1) the functions* l *and* u *are monotone, i.e., for all* $M_1, M_2 \in \mathcal{MS}^{<\infty}(U)$ *if* $M_1 \sqsubseteq M_2$*, then* $\mathsf{l}(M_1) \sqsubseteq \mathsf{l}(M_2)$*,* $\mathsf{u}(M_1) \sqsubseteq \mathsf{u}(M_2)$ *(*monotonicity *of* l, u*);*
   - *(C2)* $\mathsf{u}(\emptyset) = \emptyset$ *(*normality *of* u*);*
   - *(C3) if* $M \in \mathcal{MS}^{<\infty}(U)$*, then* $\mathsf{l}(M) \sqsubseteq \mathsf{u}(M)$ *(*weak approximation property*).*

Corollary 2. $\mathsf{l}(\emptyset) = \emptyset$ *(*normality *of* l*).*

$\mathsf{MAS}(U)$ is *total* if for any $M \in \mathcal{MS}^{<\infty}(U)$ there is a definable mset $D \in \mathfrak{D}_{\mathfrak{B}}$ such that $M \sqsubseteq D$, and it is *partial* otherwise. If $\mathfrak{D}_{\mathfrak{B}}$ is the smallest set of msets satisfying condition 2 in Definition 4, $\mathsf{MAS}(U)$ is total if and only if there is a $B \in \mathfrak{B}$ such that $B(a) \geq 1$ for all $a \in U$.

There may be more than one msets with the same lower and upper approximations. If $M \in \mathcal{MS}^{<\infty}(U)$, the set

$$\mathcal{RM}(M) = \{M' \in \mathcal{MS}^{<\infty}(U) \mid \mathsf{l}(M) = \mathsf{l}(M') \text{ and } \mathsf{u}(M) = \mathsf{u}(M')\}$$

is called the *rough mset connected to* M.

Of course, l and u are neither additive nor multiplicative in general.

Proposition 2. *Let* $\mathsf{MAS}(U) = \langle \mathcal{MS}^{<\infty}(U), \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u}\rangle$ *be a general mset approximation space over* U*. Then, for any* $M_1, M_2 \in \mathcal{MS}^{<\infty}(U)$*,*

1. $\mathsf{l}(M_1) \sqcup \mathsf{l}(M_2) \sqsubseteq \mathsf{l}(M_1 \sqcup M_2)$*,* $\mathsf{l}(M_1 \sqcap M_2) \sqsubseteq \mathsf{l}(M_1) \sqcap \mathsf{l}(M_2)$*,*
2. $\mathsf{u}(M_1) \sqcup \mathsf{u}(M_2) \sqsubseteq \mathsf{u}(M_1 \sqcup M_2)$*,* $\mathsf{u}(M_1 \sqcap M_2) \sqsubseteq \mathsf{u}(M_1) \sqcap \mathsf{u}(M_2)$*,*

i.e., lower and upper approximations are superadditive and submultiplicative.

Proof. $M_1, M_2 \sqsubseteq M_1 \sqcup M_2$ and $M_1 \sqcap M_2 \sqsubseteq M_1, M_2$, and so, by the monotonicity of l, $\mathsf{l}(M_1), \mathsf{l}(M_2) \sqsubseteq \mathsf{l}(M_1 \sqcup M_2)$ and $\mathsf{l}(M_1 \sqcap M_2) \sqsubseteq \mathsf{l}(M_1), \mathsf{l}(M_2)$, and the statement (1) immediately follows. Statement (2) can be proved similarly. □

It is reasonable to assume that the base msets and their n-times additions are exactly approximated from "lower side". In certain cases, it is also required of definable msets.

Definition 5. *A weak approximation pair* $\langle \mathsf{l}, \mathsf{u} \rangle$ *is*

(C4) granular *if* $B \in \mathfrak{B}$ *implies* $\mathsf{l}(\oplus_n B) = \oplus_n B$ *(*$n \in \mathbb{N}$*) (in other words,* l *is granular),*
(C5) standard *if* $D \in \mathfrak{D}_{\mathfrak{B}}$ *implies* $\mathsf{l}(D) = D$ *(in other words,* l *is standard).*

Of course, if l is standard, the granularity of l also holds. The next proposition gives a necessary and sufficient condition that l is standard.

Proposition 3. *Let* $\mathsf{MAS}(U) = \langle \mathcal{MS}^{<\infty}(U), \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u} \rangle$ *be a general mset approximation space over* U*.*

l *is standard if and only if* $\mathsf{l}(\mathcal{MS}^{<\infty}(U)) = \mathfrak{D}_{\mathfrak{B}}$ *and* l *is idempotent, i.e.,* $\forall M \in \mathcal{MS}^{<\infty}(U)\, (\mathsf{l}(\mathsf{l}(M)) = \mathsf{l}(M))$*.*

Proof. ($\Rightarrow$) By *(C0)*, $\mathsf{l}(\mathcal{MS}^{<\infty}(U)) \subseteq \mathfrak{D}_{\mathfrak{B}}$. On the other hand, for any $D \in \mathfrak{D}_{\mathfrak{B}}$, $\mathsf{l}(D) = D \in \mathsf{l}(\mathcal{MS}^{<\infty}(U))$, since l is standard, i.e., $\mathfrak{D}_{\mathfrak{B}} \subseteq \mathsf{l}(\mathcal{MS}^{<\infty}(U))$. Thus, $\mathsf{l}(\mathcal{MS}^{<\infty}(U)) = \mathfrak{D}_{\mathfrak{B}}$.

Further, let $M \in \mathcal{MS}^{<\infty}(U)$. $\mathsf{l}(M) \in \mathfrak{D}_{\mathfrak{B}}$ according to the condition *(C0)*, and so $\mathsf{l}(\mathsf{l}(M)) = \mathsf{l}(M)$, since l is standard.

($\Leftarrow$) Let $D \in \mathfrak{D}_{\mathfrak{B}}$. Since $\mathfrak{D}_{\mathfrak{B}} = \mathsf{l}(\mathcal{MS}^{<\infty}(U))$, there exists at least one $M \in \mathsf{l}(\mathcal{MS}^{<\infty}(U))$ such that $D = \mathsf{l}(M)$. l is idempotent, and so

$$\mathsf{l}(D) = \mathsf{l}(\mathsf{l}(M)) = \mathsf{l}(M) = D,$$

that is, l is standard. □

An important question is how lower and upper approximations relate to the approximated mset.

Definition 6. *A weak approximation pair* $\langle \mathsf{l}, \mathsf{u} \rangle$ *is*

(C6) lower semi–strong *if* $\mathsf{l}(M) \sqsubseteq M$ *(*$M \in \mathcal{MS}^{<\infty}(U)$*) (*$\mathsf{l}$ *is* contractive*);*
(C7) upper semi–strong *if* $M \sqsubseteq \mathsf{u}(M)$ *(*$M \in \mathcal{MS}^{<\infty}(U)$*) (*$\mathsf{u}$ *is* extensive*);*
(C8) strong *if it is lower and upper semi–strong simultaneously, i.e., each subset* $M \in \mathcal{MS}^{<\infty}(U)$ *is bounded by* $\mathsf{l}(M)$ *and* $\mathsf{u}(M)$*:* $\mathsf{l}(M) \sqsubseteq S \sqsubseteq \mathsf{u}(M)$*.*

Definition 7. *The general mset approximation space* $\mathsf{MAS}(U)$ *is a* weak/granular/standard/lower semi–strong/upper semi–strong/strong *mset approximation space if the approximation pair* $\langle \mathsf{l}, \mathsf{u} \rangle$ *is weak/granular/standard/lower semi–strong/upper semi–strong/strong, respectively.*

5 Generalized Pawlakian Multiset Approximation Spaces

It is a natural assumption that $\mathfrak{D}_{\mathfrak{B}}$ is obtained (derived) from $\mathfrak{B}$ by some sorts of set and/or mset type transformations (for the most important cases, see [8]). In this case, an mset approximation space is surely partial if there exists at least one object in U which does not belong to any base mset.

In order to build a generalized Pawlakian mset approximation space, first, we define $\mathfrak{D}_{\mathfrak{B}}$ as follows.

Definition 8. $\mathsf{MAS}(U)$ *is a* strictly set–union type *mset approximation space if* $\mathfrak{D}_{\mathfrak{B}}$ *is given by the following inductive definition:*

1. $\emptyset \in \mathfrak{D}_{\mathfrak{B}}$*;*
2. $\mathfrak{B} \subseteq \mathfrak{D}_{\mathfrak{B}}$*;*
3. *if* $\mathfrak{B}^{\oplus} = \{\oplus_n B \mid B \in \mathfrak{B},\ n = 1, 2, \dots\}$ *and* $\mathfrak{B}' \subseteq \mathfrak{B}^{\oplus}$*, then* $\bigsqcup \mathfrak{B}' \in \mathfrak{D}_{\mathfrak{B}}$*.*

In a general mset approximation space $\mathsf{MAS}(U)$, $\bigsqcup\{D' \in \mathfrak{D}_{\mathfrak{B}} \mid D' \sqsubseteq D\} \sqsubseteq D$. On the other hand, D is definable, and so $D \in \{D' \in \mathfrak{D}_{\mathfrak{B}} \mid D' \sqsubseteq D\}$, i.e., $D \sqsubseteq \bigsqcup\{D' \in \mathfrak{D}_{\mathfrak{B}} \mid D' \sqsubseteq D\}$ also holds. Thus,

$$D = \bigsqcup\{D' \in \mathfrak{D}_{\mathfrak{B}} \mid D' \sqsubseteq D\}.$$

This formula indicates set–union nature of definable sets which can be sharpened in strictly set–union type mset approximation spaces as follows.

Proposition 4. *Let* $\mathsf{MAS}(U) = \langle \mathcal{MS}^{<\infty}(U), \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u}\rangle$ *be a strictly set–union type mset approximation space over* U*.*

1. *For any definable set* $D \in \mathfrak{D}_{\mathfrak{B}}$*,*
$$D = \bigsqcup\{\oplus_n B \mid n \in \mathbb{N}^+, B \in \mathfrak{B}, B \sqsubseteq^n D\}.$$
2. *If* $\mathsf{MAS}(U)$ *is also granular and lower semi–strong, for any* $M \in \mathcal{MS}^{<\infty}(U)$*,*
$$\mathsf{l}(M) = \bigsqcup\{\oplus_n B \mid n \in \mathbb{N}^+, B \in \mathfrak{B}, B \sqsubseteq^n M\}.$$

Proof.

1. Since $\mathsf{MAS}(U)$ is strictly set–union type, by Definition 8, there exists $\mathfrak{B}' \subseteq \mathfrak{B}^{\oplus}$ for any $D' \in \mathfrak{D}_{\mathfrak{B}}$ such that $D' = \bigsqcup \mathfrak{B}'$. Hence,
$$\begin{aligned} D &= \bigsqcup\{D' \in \mathfrak{D}_{\mathfrak{B}} \mid D' \sqsubseteq D\} \\ &= \bigsqcup\{\oplus_n B \mid n \in \mathbb{N}^+, B \in \mathfrak{B}, \oplus_n B \sqsubseteq D\} \\ &= \bigsqcup\{\oplus_n B \mid n \in \mathbb{N}^+, B \in \mathfrak{B}, B \sqsubseteq^n D\}. \end{aligned}$$
2. By Corollary 1(3), $B \sqsubseteq^n M$ if and only if $\oplus_n B \sqsubseteq^1 M$ $(n \in \mathbb{N}^+)$. Thus, for any $n \in \mathbb{N}^+$ and $\oplus_n B \sqsubseteq^1 M$ $(B \in \mathfrak{B})$, the granularity and the monotone property of l imply that $\oplus_n B = \mathsf{l}(\oplus_n B) \sqsubseteq \mathsf{l}(M)$, therefore
$$\bigsqcup\{\oplus_n B \mid n \in \mathbb{N}^+, B \in \mathfrak{B}, B \sqsubseteq^n M\} \sqsubseteq \mathsf{l}(M).$$
On the other hand, $\mathsf{l}(M) \in \mathfrak{D}_{\mathfrak{B}}$ and so by Proposition 4(1), and since l is contractive, we obtain
$$\begin{aligned} \mathsf{l}(M) &= \bigsqcup\{\oplus_n B \mid n \in \mathbb{N}^+, B \in \mathfrak{B}, B \sqsubseteq^n \mathsf{l}(M)\} \\ &\sqsubseteq \bigsqcup\{\oplus_n B \mid n \in \mathbb{N}^+, B \in \mathfrak{B}, B \sqsubseteq^n M\}. \end{aligned}$$
Thus, $\mathsf{l}(M) = \bigsqcup\{\oplus_n B \mid n \in \mathbb{N}^+, B \in \mathfrak{B}, B \sqsubseteq^n M\}$. □

Next, we generalize the Pawlakian approximation pair for msets in strictly set–union type mset approximation spaces.

Definition 9. *Let* $\mathsf{MAS}(U) = \langle \mathcal{MS}^{<\infty}(U), \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u}\rangle$ *be a strictly set–union type mset approximation space.*

The functions $\mathsf{l}, \mathsf{u} : \mathcal{MS}^{<\infty}(U) \to \mathcal{MS}^{<\infty}(U)$ *form a* (generalized) Pawlakian mset approximation pair $\langle \mathsf{l}, \mathsf{u}\rangle$ *if for any mset* $M \in \mathcal{MS}^{<\infty}(U)$*,*

1. $\mathsf{l}(M) = \bigsqcup\{\oplus_n B \mid n \in \mathbb{N}^+, B \in \mathfrak{B}$ *and* $B \sqsubseteq^n M\}$*,*
2. $\mathsf{b}(M) = \bigsqcup\{\oplus_n B \mid B \in \mathfrak{B}, B \not\sqsubseteq M,\ B \sqcap M \neq \emptyset$ *and* $B \sqcap M \sqsubseteq^n M\}$*,*
3. $\mathsf{u}(M) = \mathsf{l}(M) \sqcup \mathsf{b}(M)$*,*

where the function b *gives the* boundary *of mset* M*.*

It is easy to check the next proposition by Definition 9.

Proposition 5. *Let* $\mathsf{MAS}(U) = \langle \mathcal{MS}^{<\infty}(U), \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u}\rangle$ *be a strictly set–union type mset approximation space with a Pawlakian mset approximation pair.*

Then $\mathsf{MAS}(U)$ *is a lower semi–strong mset approximation space and* l *is granular. In other words,* $\mathsf{MAS}(U)$ *fulfills the conditions (C0)–(C3), (C4), (C6).*

Definition 10. *A strictly set–union type approximation space with a Pawlakian mset approximation pair is called a* Pawlakian mset approximation space*.*

Proposition 6. *Let* $\mathsf{MAS}(U) = \langle \mathcal{MS}^{<\infty}(U), \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u}\rangle$ *be a Pawlakian mset approximation space. Then*

$$\mathsf{u}(M) = (\mathsf{l}(M) \oplus \mathsf{b}(M)) \ominus (\mathsf{l}(M) \sqcap \mathsf{b}(M)).$$

Proof. For all $a \in U$,

$$\begin{aligned}
\mathsf{u}(M)(a) &= ((\mathsf{l}(M) \oplus \mathsf{b}(M)) \ominus (\mathsf{l}(M) \sqcap \mathsf{b}(M)))(a)\\
&= \mathsf{max}\{(\mathsf{l}(M) \oplus \mathsf{b}(M))(a) - (\mathsf{l}(M) \sqcap \mathsf{b}(M))(a), 0\}\\
&= \mathsf{max}\{\mathsf{l}(M)(a) + \mathsf{b}(M)(a) - \mathsf{min}\{\mathsf{l}(M)(a), \mathsf{b}(M)(a)\}, 0\}\\
&= \begin{cases} \mathsf{max}\{\mathsf{l}(M)(a), 0\}, & \text{if } \mathsf{l}(M)(a) \geq \mathsf{b}(M)(a);\\ \mathsf{max}\{\mathsf{b}(M)(a), 0\}, & \text{if } \mathsf{l}(M)(a) < \mathsf{b}(M)(a);\end{cases}\\
&= \mathsf{max}\{\mathsf{l}(M)(a), \mathsf{b}(M)(a)\}\\
&= (\mathsf{l}(M) \sqcup \mathsf{b}(M))(a).
\end{aligned}$$

□

6 Applications in Membrane Computing

In the membrane application we focus on hierarchical membrane systems with communication rules.

A membrane structure μ of degree m ($m \in \mathbb{N}^+$) is a rooted tree with m nodes. It can be represented by the set $R_\mu \subseteq \{1, \ldots, m\} \times \{1, \ldots, m\}$ where $\langle i, j\rangle \in R_\mu$ means that there is an edge from i (parent) to j (child) of the tree μ which is formulated by $\mathsf{parent}(j) = i$.

Let V be a finite alphabet. The tuple

$$\Pi = \langle V, \mu, w_1, w_2, \ldots, w_m, R_1, R_2, \ldots, R_m \rangle$$

is called a *membrane system* or *P system* if $w_i \in \mathcal{MS}^{<\infty}(V)$ is the *region* of Π, and R_i is a finite set of *rules* of the form symport and antiport ($i = 1, 2, \ldots, m$). For the precise definition, see [8], Definition 6.

If the P system $\Pi = \langle V, \mu, w_1, w_2, \ldots, w_m, R_1, R_2, \ldots, R_m \rangle$ is given, let $\mathsf{MAS}(\Pi) = \langle \mathcal{MS}^{<\infty}(V), \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u} \rangle$ be a strictly set–union type mset approximation space with a generalized Pawlakian approximation pair $\langle \mathsf{l}, \mathsf{u} \rangle$. $\mathsf{MAS}(\Pi)$ is called a *joint membrane approximation space.*

Having given a membrane system Π and its joint membrane approximation space $\mathsf{MAS}(\Pi)$, we can define the boundaries of the regions $w_1, w_2, \ldots, w_m$ as msets with the help of approximative function b specified in Definition 9.

Definition 11. *Let $\Pi = \langle V, \mu, w_1, w_2, \ldots, w_m, R_1, R_2, \ldots, R_m \rangle$ be a P system and $\mathsf{MAS}(\Pi) = \langle \mathcal{MS}^{<\infty}(V), \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u} \rangle$ be its joint membrane approximation space. If $B \in \mathfrak{B}$ and $i = 1, 2, \ldots, m$, let*

$$N(B,i) = \begin{cases} 0, & \text{if } B \sqsubseteq w_i \text{ or } B \sqcap w_i = \emptyset; \\ n, & \text{if } i = 1 \text{ and } B \sqcap w_1 \sqsubseteq^n w_1; \\ \min\{k, n \mid B \sqcap w_i \sqsubseteq^k w_i, \text{ and } B \ominus w_i \sqsubseteq^n w_{\mathsf{parent}(i)}\}, & \text{otherwise.} \end{cases}$$

Then, for $i = 1, \ldots, m$,

$\mathsf{bnd}(w_i) = \bigsqcup \{ \oplus_{N(B,i)} B \mid B \in \mathfrak{B} \}$;
$\mathsf{bnd}^{\mathsf{out}}(w_i) = \mathsf{bnd}(w_i) \ominus w_i$;
$\mathsf{bnd}^{\mathsf{in}}(w_i) = \mathsf{bnd}(w_i) \ominus \mathsf{bnd}^{\mathsf{out}}(w_i)$.

The functions $\mathsf{bnd}(w_i)$, $\mathsf{bnd}^{\mathsf{out}}(w_i)$, $\mathsf{bnd}^{\mathsf{in}}(w_i)$ give *membrane boundaries*, *outside membrane boundaries* and *inside membrane boundaries*, respectively.

The general notion of boundaries given in Definition 9 cannot be used here, because membrane boundaries have to follow the given membrane structure μ. The Pawlakian lower approximations $\mathsf{l}(w_i)$ ($i = 1, \ldots, m$) surely obey the membrane structure, and the Pawlakian upper approximation $\mathsf{u}(w_1)$ and the boundary $\mathsf{b}(w_1)$ are completely within the environment of the membrane structure.

However, the Pawlakian upper approximation $\mathsf{u}(w_i)$, therefore the boundary $\mathsf{b}(w_i)$ ($i = 2, \ldots, m$) do not obey the membrane structure in general. Thus, the Pawlakian boundaries have to be adjusted to the membrane structure by the function bnd. Of course, $\mathsf{b}(w_1) = \mathsf{bnd}(w_1)$, but $\mathsf{b}(w_i) \neq \mathsf{bnd}(w_i)$ ($i = 2, \ldots, m$) in general. Moreover, membrane boundaries $\mathsf{bnd}(w_i)$ ($i = 1, \ldots, m$) are split into two parts, inside and outside membrane boundaries.

As an illustrative example for the membrane boundary, let us take a membrane structure with 1 node, and let the base system $\mathfrak{B}$ consist of three base msets: B_1, B_2, B_3. In the figures below, they are represented by circle, triangle, and square, respectively. For the sake of clarity, only a fragment of the whole mset approximation space is depicted focusing on the membrane boundary solely.

Fig. 1 shows the membrane boundary of the region w_1.

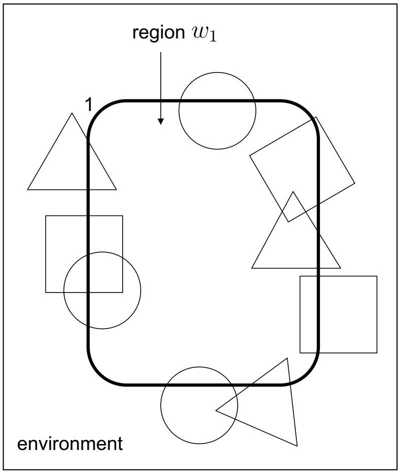

Fig. 1. A membrane boundary

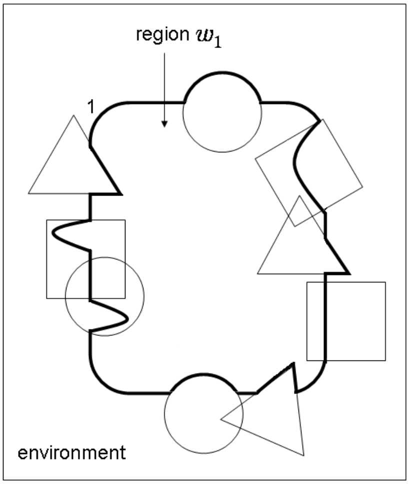

Fig. 2. The membrane boundary, after the membrane computation

Using membrane boundaries, the following constraints for rule executions are prescribed: a rule $r \in R_i$ of a membrane i has to work only in the boundaries of its region. It can be shown that the membrane computation actually works in the membrane boundaries ([8], Theorem 1). Fig. 2 illustrates the membrane boundary just after the membrane computation has halted.

In [8], the authors gave the pseudocode of the whole computation process as well.

7 Conclusion

In the paper, the authors have defined general multiset approximation spaces and have discussed their fundamental approximative properties. Their lattice theory properties have been shown as well. These properties hold not only in Pawlakian but also in general mset approximation spaces.

The importance of defined general multiset approximation spaces can be found, for instance, in their applications in membrane computing. By using the partial multiset approximation technique, the notion of "to be close enough to a membrane", even from inside and outside, has been specified in an abstract way. Thus, by constraining the communication rule executions on these abstract membrane boundaries, the membrane computation can be controlled.

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