A Mathematical Theory of Fuzzy Numbers
Granular Computing Approach

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Abstract. The essence of granular computing (GrC) is to replace the concept of points in classical mathematics by that of granules. Usual fuzzy number systems are obtained by using type I fuzzy sets as granules. These fuzzy number systems have a common weakness - lack of existence theorem. Let R be the real number system, the trapezoidal membership functions at $r \in R$ is a base of fuzzified topological neighborhood system FNS(r). By taking FNS(r) as the granule, a new (but not type I) fuzzy number system $\mathcal{F}$ is formed. Surprisingly, we have found that such a new $\mathcal{F}$ is abstractly isomorphic to the classical real number system.

Keywords: Fuzzy numbers, Fuzzified topological neighborhood systems, Granular computing, Qualitative fuzzy set, Topology.

1 Introduction

What is a real number, a vector, a point in Euclidean plane or etc.? Due to the nature of mathematics, these questions are answered in a "whole sale" style. Namely, mathematicians have to define first the real number system, the vector spaces, Euclidean planes, and etc. (often axiomatically), then answer to the question by saying that an element of them is a real number, a vector, a point and etc. respectively.

What is a fuzzy number? There are plenty of answers. What is a fuzzy number system? There are no very clear cut answers. The "standard" constructions of fuzzy numbers (of type I fuzzy sets) are, more or less, given in the following ways: Let R be the set of real numbers.

1. Type I hypothesis: For each real number $r \in R$, there is associated a unique membership function $f_r : R \longrightarrow I$ that represents a very special fuzzy set, namely, the fuzzy number associated to r. Let $F_R = \{f_r \mid r \in R\}$ be the collection of such membership functions.
2. Constructions of mathematical structure on F_R: Binary operations, such as "addition" and "multiplication", are then introduced into F_R (and may be some other structures). The collection F_R, together with such a mathematical structure, forms the "standard" fuzzy number system.

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However, the constructions of mathematical structure on F_R are incomplete: It needs to show that there exists (can be constructed) a set F_R of membership functions such that (1) for each real number there is a unique membership function in F_R, and (2) the "sum" and "product" of "addition" and "multiplication" of two members in F_R are some members of F_R again, in the jargon of mathematics, there is a construction of F_R such that F_R is closed under algebraic operations. (1) is satisfied by hypothesis, but (2) has not be shown in literature.

For example, by Type I hypothesis, membership functions f_6, f_2, f_3, f_4 and $f_{1.5}$ have been selected. But, there are no proofs for the following equalities:

$$f_6 = f_2 \odot f_3 = f_4 \odot f_{1.5} = f_5 \odot f_{1.2} = \ldots$$

We shall call these equalities consistency conditions. In other words, without proving the consistency condition, F_R, as a mathematical system, may not exists; see the second paragraph of Section 4.

The primary purpose of this paper is to reformulate these "standard" fuzzy numbers into a mathematical system that meets the consistency conditions. This system is, however, not in Type I theory, but in a very general Type II, called qualitative fuzzy set theory [8].

At first, the final theorem is a surprise: the new fuzzy number is a copy of the classical real number system.

However, if we do a little deeper analysis, we can see that this answer should be the expected one. A trapezoidal fuzzy number (its membership function contains an non-empty open crisp interval) is a "real world" approximation of a real number. So the collection of such approximations should converge to the the real number system; we show this in Section 4.

We also illustrate the idea in a more "commonly" used approximations. The collection of n-digits decimals $\forall\, n \in Z^+$ (positive integers) has been used as approximations for centuries. The system of such n-digits $\forall\, n \in Z^+$ does converge to real number system; this is proved by the concept of topology; see Section 3.

What is granular computing (GrC)? In 1996, granular computing (GrC) [12] was coined to label Zadeh's idea: GrC is a new mathematics, in which the concept of points in classical mathematics, is replaced by that of granules. Note that Zadeh's idea actually appeared in [10]. In fact, a more formal example in model theory of such an idea did exist a few years ahead [4]; the non-standard real number system (hyperreal) could be viewed as a "new" system, in which each real number r is replaced by a granule of "r + infinitesimals".

By taking a granule at p as the largest neighborhood system LNS(p) among all topologically equivalent NS(p) (see Example 1, Item 4), in GrC 2011 [9], LNS was axiomatized. That means Zadeh's idea is formally realized:

- This set of axioms defines the mathematics of GrC.

Using Zadeh's style of expressing, the granule, that has been axiomatized, is a granular variable that takes neighborhood $N \in$ LNS(p) as values. The concept of neighborhood systems (NS) was introduced in [7,6], and LNS(p) can also be regarded as a NS(p) that meets the super set condition (see Definition 1).

This paper is organized as follows: In Section 2, we review the concept of topological neighborhood system TNS(p), $p \in R$, then in Section 3, the algebraic operations are introduced among TNS(p) $\in 2^{2^R}$. In Section 4, we fuzzify the idea, and in Section 5, we state our conclusions.

2 Topological Neighborhood System (TNS)

First, we need to recall the concept of topology. Next few paragraphs are taken from [5] and the axioms are from (Chapter 1, Exercise B).

Definition 1. *The pair* $(U, TNS(U))$ *is called a topological space (or TNS-space), if* $TNS(U) = \{TNS(p) : p \in U\}$ *is defined as follows: For each* $p \in U$*, let* $TNS(p)$ *be the family of all subsets, called neighborhoods, that satisfies the following axioms:*

1. *If* $N \in TNS(p)$*, then* $p \in N$*;*
2. *If* N *and* M *are members of* $TNS(P)$*, then* $N \cap M \in TNS(p)$*;*
3. *superset condition: If* $N \in TNS(p)$ *and* $N \subset M$*, then* $M \in TNS(p)$*;*
4. *If* $N \in TNS(p)$*, then there is a member* M *of* $TNS(p)$ *such that* $M \subset N$ *and* $M \in TNS(y)$ *for each* y *in* M *(that is,* M *is a neighborhood of each of its points).*

Definition 2. *A base* $\mathcal{B}(p)$ *of* $TNS(p)$ *of a point* p *is a family of neighborhoods such that every neighborhood* $N \in TNS(p)$ *contains a member of the family* $\mathcal{B}(p)$*.*

Example 1. (Bases of TNS of R)

1. $\forall\ p \in R$*, let us consider the collection* $\mathcal{B}(p) = \{N_1(p) = (p - 1/10^n, p + 1/10^n), n \in Z^+\}$*. This collection* $\mathcal{B}(p)$ *is a base of the* $TNS(p)$*.* N_1 *is the uncertainty region of the n-digits decimal number of* p*.*
2. *Let* $TNS(p)$ *be the maximal collection of subsets, in which each subset contains an* $N_1(p)$ *for some* $n \in Z^+\}$*, where* Z^+ *is positive integers. Then* $TNS(U) = \{TNS(p)\}$ *is the TNS of real number system; it is routine to verify that the four axioms in Definition 1 are satisfied.*
3. *Another base of TNS of real numbers can also be defined by using base other than 10 the collection* $\{N_2(p) | N_2(p) = (p - \epsilon, p + \epsilon)$*, where* $\epsilon = 1/b^n$*,where* b *is any positive integer. This base leads to the same maximal collection* $TNS(U)$*.*
4. *In the theory of neighborhood system (NS), which is the ultimate generalization of topology, the notation for maximal collection is* $LNS(U)$*. So, if we interpret the topology as a special kind of NS, then* $TNS(U) = LNS(U)$*. This may explain a bit more on LNS that is mentioned in the Section of Introduction.*

3 The Real Number System $\mathcal{R}_T$ Defined by Topology

Let the universe R of discourse in this section be the real number system. The main idea is to introduce the algebraic structure into $\mathcal{R}_T \subset 2^{2^R}$, for example, $\text{TNS}(p) \oplus \text{TNS}(q) \in 2^{2^R}$ and $\text{TNS}(p) \otimes \text{TNS}(q) \in 2^{2^R}$, between two TNS.

Definition 3. *The real number system R is a complete order field.*

The real number system can be defined in many ways; we take the axiomatic approach ([1] p.98). Here "order field" refers to the usual "college algebra" (four kinds of operations and order relations that satisfy various kinds of laws), and term "complete" means: Any bounded below set has the greatest lower bound. For example, the rational number system Q is not a complete order field because the set $A = \{x \in Q | x > \sqrt{2}\} \subset Q$ dose not have the greatest lower bound, while R is a complete order field. For R, the greatest lower bound of its subset $B = \{x \in R | x > \sqrt{2}\} \subset R$ is $\sqrt{2}$.

Before, we give the new definition of fuzzy numbers, we will show in this section that the real number system can be defined by the collection of topological neighborhood system (TNS).

Definition 4. (GrC based real number system $\mathcal{R}_T$) *GrC based real number system is:*

$$\mathcal{R}_T = \{\bar{p} \mid \bar{p} = TNS(p), p \in R\},$$

with appropriate algebraic structure that will be introduced below.

For mathematical students, this is a fairly routine to verify that $\mathcal{R}_T$ defined above, indeed, forms the complete ordered field. Since this is in computer science paper, we shall sketch few key points.

Definition 5. (Subset operations in an algebraic system) *Let $X, Y \subseteq E$ be two subsets of an algebraic system $(E, \cdot)$. The operator $\circ$ between X and Y is defined as*

$$X \circ Y = \{x \cdot y | \ \forall\ x \in X,\ \forall\ y \in Y\},$$

or in terms of the convolution of characteristic functions

$$\chi_{X \circ Y}(z) = \max_{x \cdot y = z} \min(\chi_X(x), \chi_Y(y)),\ (x, y, z) \in R^3.$$

where $\cdot$ is a binary operator in E.

Proposition 1. *If $\cdot$ is commutative or associative, then $\circ$ is commutative or associative respectively.*

In practice (for example, in the coset multiplication in group theory), we do not use new notation $\circ$, but (by abuse of notation) use the notation $\cdot$ of the given binary operations. By applying Definition 5 twice (to neighborhoods ($\subseteq R$), then to TNS ($\subseteq 2^R$), we have

Definition 6. (External algebraic operations) $\forall\ \bar{p},\ \bar{q} \in \mathcal{R}_T$,

$$\bar{p} \oplus' \bar{q} \equiv \{N(p) + N(q) \mid N(p) \in \mathit{TNS}(p), N(q) \in \mathit{TNS}(q)\}$$

$$\bar{p} \odot' \bar{q} \equiv \{N(p) \cdot N(q) \mid N(p) \in \mathit{TNS}(p), N(q) \in \mathit{TNS}(q)\},$$

where (in the following formulas, we use "$\cdot$", instead of "$\circ$"),

$$N(p) + N(q) = \{r_1 + r_2 | r_1 \in N(p), r_2 \in N(q)\},$$

$$(\chi_{N(p)+N(q)}(z) = \max_{x+y=z} \min(\chi_{N(p)}(x), \chi_{N(q)}(y)),\ (x, y, z) \in R^3).$$

$$N(p) \cdot N(q) = \{r_1 \cdot r_2 | r_1 \in N(p), r_2 \in N(q)\},$$

$$(\chi_{N(p)\cdot N(q)}(z) = \max_{x\cdot y=z} \min(\chi_{N(p)}(x), \chi_{N(q)}(y)),\ (x, y, z) \in R^3),$$

These 2 external operations induce the following 2 inclusions.

Proposition 2. $\forall\ \bar{p},\ \bar{q} \in \mathcal{R}_T$,

$$\bar{p} \oplus' \bar{q} \subseteq \overline{p+q}; \quad \bar{p} \odot' \bar{q} \subseteq \overline{p \cdot q}.$$

We shall explain the first inclusion: $\bar{p} \oplus' \bar{q}$ consists of all possible $\{N(p) + N(q)\}$. From Example 1, there are bases (of $1/10^t$-neighborhoods) for TNS(p) and TNS(q). Namely, there are $(p - 1/10^n, p + 1/10^n) \subseteq N(p)$, for some integer n, and $(q - 1/10^m, q + 1/10^m) \subseteq N(q)$, for some integer m. Obviously, we can find a $1/10^s$-neighborhood of $p + q$ so that $(p + q - 1/10^s, p + q + 1/10^s) \subseteq (p - 1/10^n, p + 1/10^n) + (q - 1/10^m, q + 1/10^m)$. This inclusion implies that $N(p) + N(q) \in$ TNS$(p+q)$, by the Axiom of supper set condition in Definition 1. Similar proof works for the other inclusion too.

The two inclusions induce two following internal operations in $\mathcal{R}_T$

Definition 7. $\forall\ \bar{p},\ \bar{q} \in \mathcal{R}_T$,

$$\bar{p} \oplus \bar{q} \equiv \overline{p+q} : \quad \bar{p} \odot \bar{q} \equiv \overline{p \cdot q}$$

Definition 8. *Algebraic system with two operators* $(E, \circ_1, \circ_2)$ *is called a bi-operator algebra.*

For example $(\mathcal{R}_T, \oplus, \odot)$ just introduced and $(R, +, \cdot)$ are bi-operator algebras.

Let us side track a little bit. The existence of the 2 internal operations imply the consistent conditions. For example $\bar{6} = \overline{2 \cdot 3} \equiv \bar{2} \odot \bar{3} = \ldots \quad \bar{6} = \overline{1+5} \equiv \bar{1} \oplus \bar{5} = \ldots.$

Next, we introduce the order relation into $(\mathcal{R}_T, \oplus, \odot)$ by

$$\bar{p} < \bar{q} \Leftrightarrow p < q.$$

Now, we have $(\mathcal{R}_T, \oplus, \odot, >)$. With these, we shall prove the following main theorem.

Theorem 1. $(\mathcal{R}_T, \oplus, \odot, >)$ *is a complete order field.*

Let $p \in R$, then the map: $\bar{p} \longrightarrow p$ is a one-to-one onto map, because R is a Hausdorff space. In Definition 7, we have defined $\bar{p} \oplus \bar{q} \equiv \overline{p+q}$, and $\bar{p} \otimes \bar{q} \equiv \overline{p \cdot q}$, so the two compositions below,

$$\bar{p} \oplus \bar{q} \longrightarrow \overline{p+q} \longrightarrow p+q; \quad \bar{p} \otimes \bar{q} \longrightarrow \overline{p \cdot q} \longrightarrow p \cdot q,$$

imply that the map from $(\mathcal{R}_T, \oplus, \odot)$ to $(R, +, \cdot)$ is an isomorphism of bi-operator algebras. Since this isomorphism (and its inverse) preserves all the identities and inequalities among elements, the isomorphism actually is a complete order field isomorphism. QED.

4 Fuzzy Number System $\mathcal{F}$

This section is the main subject of this paper. A new mathematic system called "fuzzy number system" will be formally defined.

Let $N(0)$ be a neighborhood of 0 in R. We claim that $N(0) \neq N(0) + N(0)$: From group theory, the equality holds only if $NS(0)$ is a abelian subgroup of R. There is no bounded abeliam subgroup in R, so we proved the claim. This shows that for characteristic functions, and hence for membership functions, F_R cannot be closed under the "addition" that is defined by convolution; similar conclusion can be drawn for "multiplications". This counter example implies that F_R, together with the algebraic operations defined by convolutions, such as [3], are not closed under algebraic operations. To show that F_R with some algebraic operations is a well-defined mathematical system, an explicit proof of closed-ness is needed; that seems lacking in the literature.

4.1 The Universe of Membership Functions

First, we have to specify the membership functions. In Type I fuzzy control, the outputs are control functions. So in most cases, they are continuous functions. Therefore the membership functions used in control are likely the continuous functions. Unfortunately, this choice will exclude out the classical sets from fuzzy set theory (characteristic functions are not continuous functions). So we choose the functions of continuous almost everywhere (a.e.) as our universe of discourse, where "continuous a.e." means a function whose continuous points are almost everywhere, in other words, whose set of discontinuous points has measure zero [2].

4.2 Fuzzification of Topology

Definition 9. (fuzzification of neighborhood system) *The fuzzification FNS(p), called fuzzy neighborhood system, of topological neighborhood system* $TNS(p)$ *consists of all membership functions* f_i *defined on* R *that contain (as "inclusion" of fuzzy sets), at least, one subset* N *that is an element* $TNS(p)$.

Observe that in this case all membership functions have some "flat top"; we call them "trapezoidal" membership functions.

Definition 10. (GrC Based Fuzzy numbers) *Fuzzy number system is:*

$$\mathcal{F} = \{\tilde{p} \mid \tilde{p} = FNS(p), p \in R\},$$

with appropriate algebraic structure that will be introduced below.

Here are the external operations:

Definition 11. $\forall \tilde{p},\ \tilde{q} \in \mathcal{F}$,

$$\tilde{p} \oplus' \tilde{q} \equiv \{f_p \boxplus f_q | f_p \in \tilde{p}, f_q \in \tilde{q}\},$$

$$f_p \boxplus' f_q(z) \equiv \max_{x+y=z} \min(f_p(x), f_q(y)),\ (x, y, z) \in R^3.$$

$$\tilde{p} \otimes' \tilde{q} \equiv \{f_p \boxtimes f_q | f_p \in \tilde{p}, f_q \in \tilde{q}\}.$$

$$f_p \boxtimes' f_q \equiv \max_{x \cdot y=z} \min(f_p(x), f_q(y)),\ (x, y, z) \in R^3.$$

The two external operations induce the following fuzzy-inclusions

Proposition 3. $\forall \tilde{p},\ \tilde{q} \in \mathcal{F}$,

$$\tilde{p} \oplus' \tilde{q} \subseteq \widetilde{p+q}; \qquad \tilde{p} \otimes' \tilde{q} \subseteq \widetilde{p \cdot q}.$$

Note that $1/10^t$-neighborhoods are also a base for $\tilde{p}$ (as well as $\bar{p}$), so the same reasoning for $\bar{p}$ (Proposition 2) does work for $\tilde{p}$ mathematically.

The 2 inclusions induce the following 2 internal operations

Definition 12. $\forall \tilde{p},\ \tilde{q} \in \mathcal{F}$,

$$\widetilde{p+q} \equiv \tilde{p} \oplus \tilde{q}; \qquad \widetilde{p \cdot q} \equiv \tilde{p} \otimes \tilde{q}.$$

These operations are simple generalizations of the convolutions defined on characteristic functions (the generalized formulas were used in [3]). As in $\mathcal{R}_T$, we will introduce the order relation into $(\mathcal{F}, \oplus, \odot)$ by

$$\tilde{p} < \tilde{q} \Leftrightarrow p < q.$$

So we have $(\mathcal{F}, \oplus, \odot, >)$.

Again, let us have some side tracks, the 2 internal operations imply the consistent conditions: $\tilde{r} = \tilde{p} \otimes \tilde{q} = \tilde{r} \oplus \tilde{s} \quad \forall\ p,\ q,\ r,\ s\ \in R$ for all possible decompositions of the real number r with respect to multiplication and additions respectively. For example, $\tilde{6} = \widetilde{2 \cdot 3} \equiv \tilde{2} \odot \tilde{3} = \ldots \quad \tilde{6} = \widetilde{1+5} \equiv \tilde{1} \oplus \tilde{5} = \ldots$.

With these, we shall prove the following main theorem

Theorem 2. $(\mathcal{F}, \oplus, \odot, >)$ *is a complete order field.*

The same proof for Theorem 1 will work for $\mathcal{F}$ by considering a similar map $\tilde{p} \longrightarrow p$.

5 Conclusion – The Meanings of Computing in $\mathcal{F}$ and $\mathcal{R}_T$

In applications, we often use n-digits, say n=2, decimals, instead of the real numbers R, to do the computation. Then, the infinitely many numbers in the interval [0, 1] have reduced to 100 2-digits representations, $0.00, 0.01, \ldots 0.99$; each number represents a crisp interval (a neighborhood). So 2-digits computation is an interval or neighborhood computing. The theory in Section 3 (TNS or GrC computing) says, if n increases, $1/10^n$-interval computing will converge to real number computing.

If we fuzzified the 2-digits numbers, then it means we are computing in trapezoidal fuzzy numbers, the theory in Section 4 (LNS or GrC computing) guarantees that, if we decrease the length of the flat top, the computations will be eventually converge to the real number computations.

This paper gives n-digits and fuzzy n-digits computing some theoretical foundations.

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