QMC Galerkin Discretization of Parametric Operator Equations

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Abstract We extend recent results from Kuo et al. (SIAM J Numer Anal 50:3351–3374, 2012) of QMC quadrature and Finite Element discretization for parametric, scalar second order elliptic partial differential equations to general QMC-Galerkin discretizations of parametric operator equations, which depend on possibly countably many parameters. Such problems typically arise in the numerical solution of differential and integral equations with random field inputs. The present setting covers general second order elliptic equations which are possibly indefinite (Helmholtz equation), or which are given in saddle point variational form (such as mixed formulations). They also cover nonsymmetric variational formulations which appear in space-time Galerkin discretizations of parabolic problems or countably parametric nonlinear initial value problems (Hansen and Schwab, Vietnam J. Math 2013, to appear).

1 Introduction

The efficient numerical computation of statistical quantities for solutions of partial differential and of integral equations with random inputs is a key task in uncertainty quantification in engineering and in the sciences. The quantity of interest being expressed as a mathematical expectation, the efficient computation of these quantities involves two basic steps: (i) approximate (numerical) solution of the operator equation, and (ii) numerical integration. In the present note, we outline a general strategy towards these two aims which is based on (i) stable Galerkin discretization and (ii) Quasi Monte-Carlo (QMC) integration by a randomly shifted, first order lattice rule following [6, 17, 22].

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J. Dick et al. (eds.), *Monte Carlo and Quasi-Monte Carlo Methods 2012*, Springer Proceedings in Mathematics & Statistics 65,
DOI 10.1007/978-3-642-41095-6_32,

QMC (and other) quadrature methods require the *introduction of coordinates of integration* prior to numerical quadrature. In the context of random field inputs with nondegenerate covariance operators, a *countable number of coordinates* is required to describe the random input data, e.g. by a Karhunen-Loève expansion. Therefore, in the present note, we consider in particular that the operator equation contains not only a finite number of random input parameters, but rather depends on *random field inputs*, i.e. it contains random functions of space and, in evolution problems, of time which describe uncertainty in the problem under consideration. Combined QMC – Finite Element error analysis for scalar diffusion problems with random coefficients was obtained recently in [9, 17]. In the present note, we indicate how the main conclusions in [17] extend to larger classes of problems.

2 Parametric Operator Equations

2.1 Abstract Saddle Point Problems

Throughout, we denote by $\mathscr{X}$ and $\mathscr{Y}$ two reflexive Banach spaces over $\mathbb{R}$ (all results will hold with the obvious modifications also for spaces over $\mathbb{C}$) with (topological) duals $\mathscr{X}'$ and $\mathscr{Y}'$, respectively. By $\mathscr{L}(\mathscr{X},\mathscr{Y}')$, we denote the set of bounded linear operators $A : \mathscr{X} \to \mathscr{Y}'$. The Riesz representation theorem associates each $A \in \mathscr{L}(\mathscr{X},\mathscr{Y}')$ in a one-to-one correspondence with a bilinear form $\mathbf{b}(\cdot,\cdot) : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$ by means of

$$\mathbf{b}(v,w) = \langle w, Av\rangle_{\mathscr{Y}\times\mathscr{Y}'} \quad \text{for all } v \in \mathscr{X}, w \in \mathscr{Y} . \tag{1}$$

Here and in what follows, we indicate spaces in duality pairings $\langle\cdot,\cdot\rangle$ by subscripts.

We shall be interested in the solution of linear operator equations $Au = f$ and make use of the following solvability result which is a straightforward consequence of the closed graph theorem, see, e.g., [1] or [8, Chap. 4].

Proposition 1. *A bounded, linear operator $A \in \mathscr{L}(\mathscr{X},\mathscr{Y}')$ is boundedly invertible* if and only if *its bilinear form satisfies* inf-sup conditions*: ex.* $\alpha > 0$ *s.t.*

$$\inf_{0\neq v\in\mathscr{X}} \sup_{0\neq w\in\mathscr{Y}} \frac{\mathbf{b}(v,w)}{\|v\|_{\mathscr{X}}\|w\|_{\mathscr{Y}}} \geq \alpha \ , \quad \inf_{0\neq w\in\mathscr{Y}} \sup_{0\neq v\in\mathscr{X}} \frac{\mathbf{b}(v,w)}{\|v\|_{\mathscr{X}}\|w\|_{\mathscr{Y}}} \geq \alpha \ . \tag{2}$$

If (2) *holds then for every $f \in \mathscr{Y}'$ the operator equation*

$$\textit{find } u \in \mathscr{X} : \quad \mathbf{b}(u,v) = \langle f, v\rangle_{\mathscr{Y}'\times\mathscr{Y}} \quad \forall v \in \mathscr{Y} \tag{3}$$

admits a unique solution $u \in \mathscr{X}$ and there holds $\|u\|_{\mathscr{X}} = \|A^{-1}f\|_{\mathscr{X}} \leq \alpha^{-1}\|f\|_{\mathscr{Y}'}$.

2.2 Parametric Operator Families

We shall be interested in QMC quadratures applied to solutions of *parametric families of operators* A. From partial differential equations with random field input (see, e.g. [27]), we consider, in particular, operator families which depend on infinitely many parameters (obtained, for example, by Karhunen-Loève expansion of random input functions). To this end, we denote by $\boldsymbol{y} := (y_j)_{j\geq 1} \in \mathscr{U}$ the possibly (for random field inputs with nondegenerate covariance kernels) countable set of parameters. We assume the parameters to take values in a bounded parameter domain $\mathscr{U} \subseteq \mathbb{R}^{\mathbb{N}}$. Then, in particular, each realization of $\boldsymbol{y}$ is a sequence of real numbers. Two main cases arise in practice: first, the *"uniform case"*: the parameter domain $\mathscr{U} = [-1/2, 1/2]^{\mathbb{N}}$ and, second, the *"truncated lognormal case"*: the parameter domain $\mathscr{U} \subset \mathbb{R}^{\mathbb{N}}$. In both cases, we account for randomness in inputs by equipping these parameter domains with countable product probability measures (thereby stipulating *mathematical independence* of the random variables y_j). Specifically,

$$\varrho(dy) = \bigotimes_{j\geq 1} \varrho_j(y_j)dy_j \ , \quad \boldsymbol{y} \in \mathscr{U} \tag{4}$$

where, for $j \in \mathbb{N}$, $\varrho_j(y_j) \geq 0$ denotes a probability density on $(-1/2, 1/2)$; for example, $\varrho_j(y_j) = 1$ denotes the uniform density, and in the truncated lognormal case, $\varrho_j = \gamma_1$, the Gaussian measure truncated to the bounded parameter domain $(-1/2, 1/2) \subset \mathbb{R}$, normalized so that $\gamma_1([-1/2, 1/2]) = 1$.

Often, mathematical expectations w.r. to the probability measure ϱ of (functionals of) the solutions $u(\boldsymbol{y})$ of operator equations depending on the parameter vector $\boldsymbol{y}$ are of interest. One object of this note is to address error analysis of QMC evaluation of such, possibly infinite dimensional, integrals. A key role in QMC convergence analysis is played by *parametric regularity* of integrand functions, in terms of weighted (reproducing kernel) Hilbert spaces which were identified in recent years as pivotal for QMC error analysis (see, e.g., [20, 21, 30, 30, 33]) and QMC rule construction (see, e.g., [4, 5, 26]). By $\mathbb{N}_0^{\mathbb{N}}$ we denote the set of all sequences of nonnegative integers, and by $\mathfrak{F} = \{\nu \in \mathbb{N}_0^{\mathbb{N}} : |\nu| < \infty\}$ the set of "finitely supported" such sequences, i.e., sequences of nonnegative integers which have only a finite number of nonzero entries. For $\nu \in \mathfrak{F}$, we denote by $\mathfrak{n} \subset \mathbb{N}$ the set of coordinates j such that $\nu_j \neq 0$, *with* j *repeated* $\nu_j \geq 1$ *many times*. Analogously, $\mathfrak{m} \subset \mathbb{N}$ denotes the supporting coordinate set for $\mu \in \mathfrak{F}$.

We consider *parametric* families of continuous, linear operators which we denote as $A(\boldsymbol{y}) \in \mathscr{L}(\mathscr{X}, \mathscr{Y}')$. We now make precise the dependence of $A(\boldsymbol{y})$ on the parameter sequence $\boldsymbol{y}$ which is required for our regularity and approximation results.

Assumption 1. *The parametric operator family* $\{A(\boldsymbol{y}) \in \mathscr{L}(\mathscr{X}, \mathscr{Y}') : \boldsymbol{y} \in \mathscr{U}\}$ *is a regular* p*-analytic operator family for some* $0 < p \leq 1$*, i.e.,*

1. *$A(\boldsymbol{y}) \in \mathscr{L}(\mathscr{X},\mathscr{Y}')$ is boundedly invertible for every $\boldsymbol{y} \in \mathscr{U}$ with uniformly bounded inverses $A(\boldsymbol{y})^{-1} \in \mathscr{L}(\mathscr{Y}',\mathscr{X})$, i.e., there exists $C_0 > 0$ such that*

$$\sup_{\boldsymbol{y}\in\mathscr{U}} \|A(\boldsymbol{y})^{-1}\|_{\mathscr{L}(\mathscr{Y}',\mathscr{X})} \leq C_0 \tag{5}$$

and

2. *For any fixed $\boldsymbol{y} \in \mathscr{U}$, the operators $A(\boldsymbol{y})$ are analytic with respect to each y_j such that there exists a nonnegative sequence $b = (b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ such that*

$$\forall \nu \in \mathfrak{F}\backslash\{0\}: \quad \sup_{\boldsymbol{y}\in\mathscr{U}} \left\|(A(0))^{-1}(\partial^\nu_{\boldsymbol{y}} A(\boldsymbol{y}))\right\|_{\mathscr{L}(\mathscr{X},\mathscr{X})} \leq C_0 b^\nu \,. \tag{6}$$

Here $\partial^\nu_{\boldsymbol{y}} A(\boldsymbol{y}) := \partial^{\nu_1}_{y_1}\partial^{\nu_2}_{y_2}\cdots A(\boldsymbol{y})$; the notation b^ν signifies the (finite due to $\nu \in \mathfrak{F}$) product $b_1^{\nu_1} b_2^{\nu_2} \ldots$ where we use the convention $0^0 := 1$.

We verify the abstract assumptions in the particular setting of *affine parameter dependence*; this case arises, for example, in diffusion problems where the diffusion coefficients are given in terms of a Karhunen-Loève expansion (see, e.g. [28] for such Karhunen-Loève expansions and their numerical analysis, in the context of elliptic PDEs with random coefficients). Then, there exists a family $\{A_j\}_{j\geq 0} \subset \mathscr{L}(\mathscr{X},\mathscr{Y}')$ such that $A(\boldsymbol{y})$ can be written in the form

$$\forall \boldsymbol{y} \in \mathscr{U}: \quad A(\boldsymbol{y}) = A_0 + \sum_{j\geq 1} y_j A_j \,. \tag{7}$$

We shall refer to $A_0 = A(0)$ as "nominal" operator, and to the operators A_j, $j \geq 1$ as "fluctuation" operators. In order for the sum in (7) to converge, we impose the following assumptions on the sequence $\{A_j\}_{j\geq 0} \subset \mathscr{L}(\mathscr{X},\mathscr{Y}')$. In doing so, we associate with the operator A_j the bilinear forms $\mathbf{b}_j(\cdot,\cdot) : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$ via

$$\forall v \in \mathscr{X},\ w \in \mathscr{Y}: \quad \mathbf{b}_j(v,w) = {}_{\mathscr{Y}}\langle w, A_j v\rangle_{\mathscr{Y}'}\,, \quad j = 0,1,2\ldots.$$

Assumption 2. *The family $\{A_j\}_{j\geq 0}$ in (7) satisfies the following conditions:*

1. *The "nominal" or "mean field" operator $A_0 \in \mathscr{L}(\mathscr{X},\mathscr{Y}')$ is boundedly invertible, i.e. (cf. Proposition 1) there exists $\alpha_0 > 0$ such that*

$$\inf_{0\neq v\in\mathscr{X}} \sup_{0\neq w\in\mathscr{Y}} \frac{\mathbf{b}_0(v,w)}{\|v\|_{\mathscr{X}}\|w\|_{\mathscr{Y}}} \geq \alpha_0\,, \quad \inf_{0\neq w\in\mathscr{Y}} \sup_{0\neq v\in\mathscr{X}} \frac{\mathbf{b}_0(v,w)}{\|v\|_{\mathscr{X}}\|w\|_{\mathscr{Y}}} \geq \alpha_0\,. \tag{A1}$$

2. *The "fluctuation" operators $\{A_j\}_{j\geq 1}$ are small with respect to A_0 in the following sense: there exists a constant $0 < \kappa < 2$ such that for α_0 as in (A1) holds*

$$\sum_{j\geq 1} b_j \leq \kappa < 2\,, \quad \text{where} \quad b_j := \|A_0^{-1} A_j\|_{\mathscr{L}(\mathscr{X},\mathscr{X})}\,, \quad j = 1,2,\ldots\,. \tag{A2}$$

Condition (**A2**) (and, hence, Assumption 2) is sufficient for the bounded invertibility of $A(\boldsymbol{y})$, uniformly w.r. to the parameter vector $\boldsymbol{y} \in \mathscr{U}$.

Theorem 1. *Under Assumption 2, for every realization* $\boldsymbol{y} \in \mathscr{U} = [-1/2, 1/2]^{\mathbb{N}}$ *of the parameter vector, the parametric operator* $A(\boldsymbol{y})$ *is boundedly invertible. Specifically, for the bilinear form* $\mathbf{b}(\boldsymbol{y}; \cdot, \cdot) : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$ *associated with* $A(\boldsymbol{y}) \in \mathscr{L}(\mathscr{X}, \mathscr{Y}')$ *via*

$$\mathbf{b}(\boldsymbol{y}; w, v) :=_{\mathscr{Y}} \langle v, A(\boldsymbol{y})w \rangle_{\mathscr{Y}'} \tag{8}$$

there hold uniform (w.r. to $\boldsymbol{y} \in \mathscr{U}$*) inf-sup conditions* (2) *with* $\alpha = (1-\kappa/2)\alpha_0 > 0$,

$$\forall \boldsymbol{y} \in \mathscr{U} : \quad \inf_{0 \neq v \in \mathscr{X}} \sup_{0 \neq w \in \mathscr{Y}} \frac{\mathbf{b}(\boldsymbol{y}; v, w)}{\|v\|_{\mathscr{X}} \|w\|_{\mathscr{Y}}} \geq \alpha \ , \quad \inf_{0 \neq w \in \mathscr{Y}} \sup_{0 \neq v \in \mathscr{X}} \frac{\mathbf{b}(\boldsymbol{y}; v, w)}{\|v\|_{\mathscr{X}} \|w\|_{\mathscr{Y}}} \geq \alpha \ . \tag{9}$$

In particular, for every $f \in \mathscr{Y}'$ *and for every* $\boldsymbol{y} \in \mathscr{U}$*, the parametric operator equation*

$$\text{find } u(\boldsymbol{y}) \in \mathscr{X} : \quad \mathbf{b}(\boldsymbol{y}; u(\boldsymbol{y}), v) = \langle f, v \rangle_{\mathscr{Y}' \times \mathscr{Y}} \quad \forall v \in \mathscr{Y} \tag{10}$$

admits a unique solution $u(\boldsymbol{y})$ *which satisfies the a-priori estimate*

$$\sup_{y \in \mathscr{U}} \|u(\boldsymbol{y})\|_{\mathscr{X}} \leq C \|f\|_{\mathscr{Y}'} \ . \tag{11}$$

Proof. We use Proposition 1, which gives necessary and sufficient conditions for bounded invertibility; also, $1/\alpha$ is a bound for the inverse. By Assumption 2, the nominal part A_0 of $A(\boldsymbol{y})$ in (7) is boundedly invertible, and we write for every $\boldsymbol{y} \in \mathscr{U}$: $A(\boldsymbol{y}) = A_0 \left(I + \sum_{j \geq 1} y_j A_0^{-1} A_j \right)$. We see that $A(\boldsymbol{y})$ is boundedly invertible iff the Neumann Series in the second factor is. Since $|y_j| \leq 1/2$, a sufficient condition for this is (**A2**) which implies, with Proposition 1, the assertion with $\alpha = \alpha_0(1 - \kappa/2)$. □

From the preceding considerations, the following is readily verified.

Corollary 1. *The affine parametric operator family* (7) *satisfies Assumption 1 with*

$$C_0 = \frac{1}{(1-\kappa/2)\alpha_0} \quad \text{and} \quad b_j := \|A_0^{-1} A_j\|_{\mathscr{L}(\mathscr{X}, \mathscr{X})} \ , \quad \text{for all } j \geq 1 \ .$$

Examples for families of parametric operator equation include certain linear and parabolic evolution equations [12], linear second order wave equations [13], nonlinear elliptic equations [11], elliptic problems in random media with multiple scales [14], and elliptic and parabolic control problems [15].

2.3 Analytic Parameter Dependence of Solutions

The dependence of the solution $u(\boldsymbol{y})$ of the parametric, variational problem (10) on the parameter vector $\boldsymbol{y}$ is analytic, with precise bounds on the growth of the partial derivatives. The following bounds of the parametric solution's dependence on the parameter vector $\boldsymbol{y}$ will, as in [17], allow us to prove dimension independent rates of convergence of QMC quadratures.

Theorem 2. *Under Assumption 1, for every $f \in \mathscr{Y}'$ and for every $\boldsymbol{y} \in \mathscr{U}$, the unique solution $u(\boldsymbol{y}) \in \mathscr{X}$ of the parametric operator equation*

$$A(\boldsymbol{y})\, u(\boldsymbol{y}) = f \qquad \text{in } \mathscr{Y}' \tag{12}$$

depends analytically on the parameters, and the partial derivatives of the parametric solution family $u(\boldsymbol{y})$ satisfy the bounds

$$\sup_{\boldsymbol{y}\in\mathscr{U}} \|(\partial_{\boldsymbol{y}}^{\nu} u)(\boldsymbol{y})\|_{\mathscr{X}} \le C_0 |\nu|!\, \tilde{b}^{\nu} \|f\|_{\mathscr{Y}'} \quad \text{for all } \nu \in \mathfrak{F}, \tag{13}$$

*where $0! := 1$ and where the sequence $\tilde{b} = (\tilde{b}_j)_{j\ge1} \in \ell^p(\mathbb{N})$ is defined by $\tilde{b}_j = b_j$ with b_j as in (**A2**) in the affine case (7), and with $\tilde{b}_j = b_j / \ln 2$ for all $j \in \mathbb{N}$ in the truncated lognormal case (6).*

For a proof in the case of stationary diffusion problems we refer, for example, to [3], for control problems to [15]. The regularity estimates (13) (and, therefore, also sparsity and, as shown ahead, QMC convergence) results are available for linear parabolic and hyperbolic PDE problems [12, 13], and for solutions of nonlinear, parametric initial value problems on possibly infinite dimensional state spaces [10].

2.4 Spatial Regularity of Solutions

Convergence rates of Galerkin discretizations will require regularity of the parametric solution $u(\boldsymbol{y})$. To state it, we assume given *scales of smoothness spaces* $\{\mathscr{X}_t\}_{t\ge0}$ and $\{\mathscr{Y}_t'\}_{t\ge0}$, with

$$\mathscr{X} = \mathscr{X}_0 \supset \mathscr{X}_1 \supset \mathscr{X}_2 \supset \dots, \qquad \mathscr{Y}' = \mathscr{Y}_0' \supset \mathscr{Y}_1' \supset \mathscr{Y}_2' \supset \dots. \tag{14}$$

The scales $\{\mathscr{X}_t\}_{t\ge0}$ and $\{\mathscr{Y}_t'\}_{t\ge0}$ (and analogously $\{\mathscr{X}_t'\}_{t\ge0}$, $\{\mathscr{Y}_t\}_{t\ge0}$) are defined for noninteger values of $t \ge 0$ by interpolation.

Instances of smoothness scales (14) in the context of the diffusion problem considered in [3, 17] are, in a *convex domain* D, the choices $\mathscr{X} = H_0^1(D)$, $\mathscr{X}_1 = (H^2 \cap H_0^1)(D)$, $\mathscr{Y}' = H^{-1}(D)$, $\mathscr{Y}_1' = L^2(D)$. In a nonconvex polygon (or polyhedron), analogous smoothness scales are available, but involve Sobolev spaces with weights (see, e.g., [25]). In the ensuing convergence analysis of QMC – Galerkin discretizations of (12), we assume $f \in \mathscr{Y}_t'$ for some $t > 0$ implies that

$$\sup_{y \in \mathscr{U}} \|u(y)\|_{\mathscr{X}_t} = \sup_{y \in \mathscr{U}} \|A(y)^{-1} f\|_{\mathscr{X}_t} \leq C_t \|f\|_{\mathscr{Y}_t'} \, . \tag{15}$$

Such regularity is available for a wide range of parametric differential equations (see [10, 15, 27] and the references there). For the analysis of Multi-Level QMC Galerkin discretizations, however, stronger bounds which combined (15) and (13) are necessary (see [19]).

2.5 Discretization

As the inverse $A(y)^{-1}$ is not available explicitly, we will have to compute, for given QMC quadrature points $\boldsymbol{y} \in \mathscr{U}$, an approximate inverse. We consider the case when it is obtained by *Galerkin discretization*: we assume given two one-parameter families $\{\mathscr{X}^h\}_{h>0} \subset \mathscr{X}$ and $\{\mathscr{Y}^h\}_{h>0} \subset \mathscr{Y}$ of subspaces of equal, finite dimension N_h, which are dense in $\mathscr{X}$ resp. in $\mathscr{Y}$, i.e.

$$\forall u \in \mathscr{X} : \quad \limsup_{h \to 0} \inf_{0 \neq u^h \in \mathscr{X}^h} \|u - u^h\|_{\mathscr{X}} = 0 \tag{16}$$

and likewise for $\{\mathscr{Y}^h\}_{h>0} \subset \mathscr{Y}$. We also assume the *approximation property*:

$$\forall 0 < t \leq \bar{t} : \; \exists C_t > 0 : \; \forall u \in \mathscr{X}_t \; \forall 0 < h \leq h_0 : \quad \inf_{w^h \in \mathscr{X}^h} \|u - w^h\|_{\mathscr{X}} \leq C_t h^t \|u\|_{\mathscr{X}_t} \, . \tag{17}$$

The maximum amount of smoothness in the scale $\mathscr{X}_t$, denoted by $\bar{t}$, depends of the problem class under consideration and on the Sobolev scale: e.g. for elliptic problems in polygonal domains, it is well known that choosing for $\mathscr{X}_t$ the usual Sobolev spaces will allow (15) with t only in a rather small interval $0 < t \leq \bar{t}$, whereas choosing $\mathscr{X}_t$ as *weighted Sobolev spaces* will allows large values of $\bar{t}$ (see [25]).

Proposition 2. *Assume that the subspace sequences* $\{\mathscr{X}^h\}_{h>0} \subset \mathscr{X}$ *and* $\{\mathscr{Y}^h\}_{h>0} \subset \mathscr{Y}$ *are stable, i.e. that there exists* $\bar{\alpha} > 0$ *and* $h_0 > 0$ *such that for every* $0 < h \leq h_0$*, there hold the* uniform (w.r.t. to $\boldsymbol{y} \in \mathscr{U}$) discrete inf-sup conditions

$$\forall \boldsymbol{y} \in \mathscr{U} : \quad \inf_{0 \neq v^h \in \mathscr{X}^h} \sup_{0 \neq w^h \in \mathscr{Y}^h} \frac{\mathbf{b}(\boldsymbol{y}; v^h, w^h)}{\|v^h\|_{\mathscr{X}} \|w^h\|_{\mathscr{Y}}} \geq \bar{\alpha} > 0 \tag{18}$$

and

$$\forall \boldsymbol{y} \in \mathscr{U} : \quad \inf_{0 \neq w^h \in \mathscr{Y}^h} \sup_{0 \neq v^h \in \mathscr{X}^h} \frac{\mathbf{b}(\boldsymbol{y}; v^h, w^h)}{\|v^h\|_{\mathscr{X}} \|w^h\|_{\mathscr{Y}}} \geq \bar{\alpha} > 0 \, . \tag{19}$$

Then, for every $0 < h \le h_0$, *and for every* $\boldsymbol{y} \in \mathscr{U}$, *the Galerkin approximation* $u^h \in \mathscr{X}^h$, *given by*

$$\text{find } u^h(\boldsymbol{y}) \in \mathscr{X}^h : \quad \mathbf{b}(\boldsymbol{y}; u^h(\boldsymbol{y}), v^h) = \langle f, v^h \rangle_{\mathscr{Y}' \times \mathscr{Y}} \quad \forall v^h \in \mathscr{Y}^h \tag{20}$$

admits a unique solution $u^h(\boldsymbol{y})$ *which satisfies the a-priori estimate*

$$\sup_{\boldsymbol{y} \in \mathscr{U}} \|u^h(\boldsymbol{y})\|_{\mathscr{X}} \le \bar{\alpha}^{-1} \|f\|_{\mathscr{Y}'} \,. \tag{21}$$

Moreover, there exists a constant $C > 0$ *such that for all* $\boldsymbol{y} \in \mathscr{U}$ *holds* quasioptimality

$$\|u(\boldsymbol{y}) - u^h(\boldsymbol{y})\|_{\mathscr{X}} \le C \bar{\alpha}^{-1} \inf_{0 \ne w^h \in \mathscr{X}^h} \|u(\boldsymbol{y}) - w^h\|_{\mathscr{X}} \,. \tag{22}$$

We remark that under Assumption 2, the validity of the discrete inf-sup conditions (18), (19) for the "nominal" bilinear forms $\mathbf{b}_0(\boldsymbol{y}; \cdot, \cdot)$ with constant $\bar{\alpha}_0 > 0$ independent of h implies (18), (19) for the form $\mathbf{b}(\boldsymbol{y}; \cdot, \cdot)$ with constant $\bar{\alpha} = (1 - \kappa/2)\bar{\alpha}_0 > 0$.

3 QMC Integration

For a given bounded, linear functional $G(\cdot) : \mathscr{X} \to \mathbb{R}$, we are interested in computing expected values of

$$F(\boldsymbol{y}) := G(u(\cdot, \boldsymbol{y})) \,, \qquad \boldsymbol{y} \in \mathscr{U} \,, \tag{23}$$

(respectively of its parametric Galerkin approximation $u^h(\boldsymbol{y}) \in \mathscr{X}_h \subset \mathscr{X}$ defined in (20)). The expected value of F is an infinite-dimensional, iterated integral of the functional $G(\cdot)$ of the parametric solution:

$$\int_{\mathscr{U}} F(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} = \int_{\mathscr{U}} G(u(\cdot, \boldsymbol{y})) \,\mathrm{d}\boldsymbol{y} = G\left(\int_{\mathscr{U}} u(\cdot, \boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \right) . \tag{24}$$

The issue is thus the numerical evaluation of Bochner integrals of $\mathscr{X}$-valued functions over the infinite dimensional domain of integration $\mathscr{U}$. We also observe that for the parametric operator equation (12), to evaluate F at a single QMC point $\boldsymbol{y} \in \mathscr{U}$ requires the approximate (Galerkin) solution of one instance of the operator equation for $u(\cdot, \boldsymbol{y}) \in \mathscr{X}$. This introduces an additional *Galerkin discretization error*, and can be accounted for as in [17] in the present, more general, setting with analogous proofs.

In [3] and the present paper, the summability of the fluctuation operators A_j, $j \ge 1$, plays an important role for proving dimension-independent convergence rates of approximations of the parametric solution maps. Accordingly, we will make the assumption, stronger than Assumption (**A2**) that there exists $0 < p < 1$ such that

$$\sum_{j\geq 1} \|A_j\|^p_{\mathscr{L}(\mathscr{X},\mathscr{Y}')} < \infty \, . \tag{A3}$$

Notice that this condition is, by (**A1**), equivalent to $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$, and implies decay of the fluctuation coefficients A_j, with stronger decay as the value of p becomes smaller. In both [3, 17] and the present paper, the rate of convergence $\mathscr{O}(N^{-1+\delta})$ is attained if (**A3**) is satisfied with $p = 2/3$. Here and throughout what follows, N denotes the number of points used in QMC integration. For values of p between $2/3$ and 1, the rate of convergence in both cases is $\mathscr{O}(N^{-(1/p-1/2)})$.

Recall that the purpose of the present paper is to analyze accuracy and complexity of QMC methods in connection with the Galerkin approximation (20) of (10). To obtain convergence rates, we strengthen Assumption (**A2**) to the requirement

$$\sup_{\boldsymbol{y}\in\mathscr{U}} \|A(\boldsymbol{y})^{-1}\|_{\mathscr{L}(\mathscr{Y}'_t,\mathscr{X}_t)} < \infty \, , \quad 0 \leq t \leq 1 \, . \tag{A4}$$

For application of QMC quadrature rules, the infinite sum in (7) must be truncated to a finite sum of, say, s terms. Below, the parameter s shall be referred to as "QMC-truncation dimension". In order for the dimension truncation to be meaningful, we will assume additionally that the A_j are decreasingly, i.e. the sequence of bounds b_j in (**A2**) is nonincreasing:

$$b_1 \geq b_2 \geq \ldots \geq b_j \geq \cdots \, . \tag{A5}$$

The overall error for the QMC-Galerkin approximation is then a sum of three terms: a *truncation error*, a *QMC error*, and the *Galerkin discretization error*. We bound the three errors and finally combine them to arrive at an overall QMC-Galerkin error bound.

3.1 Finite Dimensional Setting

In this subsection we review QMC integration when the truncation dimension (i.e. the number of integration variables), denoted by s, is assumed to be finite and fixed. The domain of integration is taken to be the s-dimensional unit cube $[-\frac{1}{2}, \frac{1}{2}]^s$ centered at the origin so that QMC integration methods formulated for $[0, 1]^s$ may require a coordinate translation. We thus consider integrals of the form

$$I_s(F) := \int_{[-\frac{1}{2},\frac{1}{2}]^s} F(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \, . \tag{25}$$

In our later applications F will be of the form (23), but for the present it is general and depends only on s variables. An N-point QMC approximation to this integral is an equal-weight rule of the form

$$Q_{s,N}(F) := \frac{1}{N}\sum_{i=1}^{N} F(\boldsymbol{y}^{(i)}),$$

with carefully chosen points $\boldsymbol{y}^{(1)},\ldots,\boldsymbol{y}^{(N)} \in [-\frac{1}{2},\frac{1}{2}]^s$. For classical results on QMC methods, see, e.g. [24, 29].

We shall assume that our integrand F belongs to a *weighted* and *anchored* Sobolev space $\mathscr{W}^a_{s,\boldsymbol{\gamma}}$. This is a Hilbert space over the unit cube $[-\frac{1}{2},\frac{1}{2}]^s$ with norm given by

$$\|F\|^2_{\mathscr{W}^a_{s,\boldsymbol{\gamma}}} := \sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{[-\frac{1}{2},\frac{1}{2}]^{|\mathfrak{u}|}} \left| \frac{\partial^{|\mathfrak{u}|} F}{\partial \boldsymbol{y}_{\mathfrak{u}}}(\boldsymbol{y}_{\mathfrak{u}};0)\right|^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{u}} \,, \tag{26}$$

where $\{1:s\}$ is a shorthand notation for the set of indices $\{1,2,\ldots,s\}$, $\frac{\partial^{|\mathfrak{u}|}F}{\partial \boldsymbol{y}_{\mathfrak{u}}}$ denotes the mixed first derivative with respect to the variables y_j with $j \in \mathfrak{u}$, and $(\boldsymbol{y}_{\mathfrak{u}};0)$ denotes the vector whose jth component is y_j if $j\in\mathfrak{u}$ and 0 if $j\notin\mathfrak{u}$.

A closely related family of weighted spaces are the so-called *unanchored spaces* denoted by $\mathscr{W}^u_{s,\boldsymbol{\gamma}}$. Here, "inactive" arguments of integrands are averaged, rather than fixed at the origin as in (26). Accordingly, the *un*anchored norm $\|\circ\|_{\mathscr{W}^u_{s,\boldsymbol{\gamma}}}$ is given by

$$\|F\|^2_{\mathscr{W}^u_{s,\boldsymbol{\gamma}}} := \sum_{\mathfrak{u}\subseteq\{1:s\}} \frac{1}{\gamma_{\mathfrak{u}}} \int_{[-\frac{1}{2},\frac{1}{2}]^{|\mathfrak{u}|}} \left(\int_{[-\frac{1}{2},\frac{1}{2}]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|} F}{\partial \boldsymbol{y}_{\mathfrak{u}}}(\boldsymbol{y}_{\mathfrak{u}};\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}})\mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}\right)^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{u}} \,. \tag{27}$$

We omit the superscripts a and u in statements which apply for either choice of space; we will also require $u \in \mathscr{W}^a_{s,\boldsymbol{\gamma}}(U;\mathscr{X})$ which is defined as the Bochner space of strongly measurable, $\mathscr{X}$-valued functions for which the (26) (with the $\|\circ\|_{\mathscr{X}}$ norm in place of the absolute value) is finite.

Weighted, anchored spaces $\mathscr{W}^a_{s,\boldsymbol{\gamma}}$ were first introduced by Sloan and Woźniakowski in [32]. By now there are many variants and generalizations, see e.g. [7, 31] and the references there. In (26) the "anchor" is $(0,\ldots,0)$, the center of the unit cube $[\frac{1}{2},\frac{1}{2}]^s$, corresponding to the anchor $(\frac{1}{2},\ldots,\frac{1}{2})$ in the standard unit cube $[0,1]^s$. For parametric operator equations (12) anchoring at the origin is preferable, since *the parametric solution of* (12) *with anchored operators corresponds to the anchored parametric solution*.

Regarding the choice of weights, from derivative bounds (13), in [17] *product and order dependent ("POD" for short) weights* were derived which are given by

$$\gamma_{\mathfrak{u}} = \Gamma_{|\mathfrak{u}|}\prod_{j\in\mathfrak{u}}\gamma_j > 0\,. \tag{28}$$

Here $|\mathfrak{u}|$ denotes the cardinality (or the "order") of $\mathfrak{u}$. The weights are therefore determined by a specific choice of the sequences $\Gamma_0 = \Gamma_1 = 1, \Gamma_2, \Gamma_3, \dots$ and $\gamma_1, \gamma_2, \gamma_3, \dots$. (a precise choice of $\gamma_\mathfrak{u}$ will be given in (38) ahead).

QMC error analysis is based on the *worst case error* of a QMC rule (or a family of QMC rules). It is defined as supremum of the (bounded, linear) QMC error functional over all functions in the unit ball of $\mathscr{W}_{s,\boldsymbol{\gamma}}$:

$$e^{\mathrm{wor}}(Q_{s,N}; \mathscr{W}_{s,\boldsymbol{\gamma}}) := \sup_{\|F\|_{\mathscr{W}_{s,\boldsymbol{\gamma}}} \le 1} |I_s(F) - Q_{s,N}(F)| \,. \tag{29}$$

Due to linearity of the functionals $I_s(\cdot)$ and $Q_{s,N}(\cdot)$, we have

$$|I_s(F) - Q_{s,N}(F)| \le e^{\mathrm{wor}}(Q_{s,N}; \mathscr{W}_{s,\boldsymbol{\gamma}}) \, \|F\|_{\mathscr{W}_{s,\boldsymbol{\gamma}}} \qquad \text{for all} \quad F \in \mathscr{W}_{s,\boldsymbol{\gamma}} \,. \tag{30}$$

In shifted rank-1 lattice rules, quadrature points in $\mathscr{U}$ are given by

$$\boldsymbol{y}^{(i)} = \mathrm{frac}\left(\frac{i\boldsymbol{z}}{N} + \boldsymbol{\Delta}\right) - \left(\tfrac{1}{2}, \dots, \tfrac{1}{2}\right) \,, \qquad i = 1, \dots, N \,,$$

where $\boldsymbol{z} \in \mathbb{Z}^s$ is the *generating vector*, $\boldsymbol{\Delta} \in [0,1]^s$ is the *shift*, and frac$(\cdot)$ indicates the fractional part of each component in the vector. Subtraction by the vector $(\frac{1}{2}, \dots, \frac{1}{2})$ translates the rule from $[0,1]^s$ to $[-\frac{1}{2}, \frac{1}{2}]^s$. In *randomly shifted lattice rules* the shift $\boldsymbol{\Delta}$ is a vector with independent, uniformly in $[0,1)$ distributed components; we denote the application of the QMC rule to the integrand function F for one draw of the shift $\boldsymbol{\Delta}$ by $Q_{s,N}(\boldsymbol{\Delta}; F)$.

Theorem 3 ([16, Theorem 5]). *Let $s, N \in \mathbb{N}$ be given, and assume that $F \in \mathscr{W}_{s,\boldsymbol{\gamma}}$ for a particular choice of weights $\boldsymbol{\gamma}$, with $\mathscr{W}_{s,\boldsymbol{\gamma}}$ denoting either the anchored space with norm* (26) *or the unanchored space with norm* (27)*.*

In each case, there exists a randomly shifted lattice such that its root-mean-square error (with respect to averages over all shifts) satisfies, for all $\lambda \in (1/2, 1]$,

$$\sqrt{\mathbb{E}\left[|I_s(F) - Q_{s,N}(\cdot; F)|^2\right]} \le \left(\sum_{\emptyset \ne \mathfrak{u} \subseteq \{1:s\}} \gamma_\mathfrak{u}^\lambda \rho(\lambda)^{|\mathfrak{u}|}\right)^{1/(2\lambda)} [\varphi(N)]^{-1/(2\lambda)} \, \|F\|_{\mathscr{W}_{s,\boldsymbol{\gamma}}} \,, \tag{31}$$

where $\mathbb{E}[\cdot]$ denotes the expectation with respect to the random shift which is uniformly distributed over $[0,1]^s$. In (31)*, with $\zeta(x)$ denotes the Riemann zeta function, and $\varphi(N)$ the Euler totient function which satisfies $\varphi(N) \le 9N$ for all $N \le 10^{30}$,*

$$\rho(\lambda) := \begin{cases} \dfrac{2\zeta(2\lambda)}{(2\pi^2)^\lambda} + \dfrac{1}{12^\lambda} & \text{if} \quad \mathscr{W}_{s,\boldsymbol{\gamma}} = \mathscr{W}^a_{s,\boldsymbol{\gamma}} , \\ \dfrac{2\zeta(2\lambda)}{(2\pi^2)^\lambda} & \text{if} \quad \mathscr{W}_{s,\boldsymbol{\gamma}} = \mathscr{W}^u_{s,\boldsymbol{\gamma}} . \end{cases} \tag{32}$$

The result with general weights, restricted to prime N in the anchored case was first obtained in [31, Theorem 3(A)], for general N and unanchored spaces in [16, Theorem 4.1] (with the choice $m = 0$ in the statement of that theorem), and for general N and anchored spaces in [16, Theorem 4.1],

The question of *efficient construction of lattice rules* has received much attention in recent years [30]. Algorithms which obtain the generating vector with favourable (w.r. to N and s) scaling have been obtained for integrands in unanchored spaces in [26], where the first algorithm for fast CBC construction using FFT at a cost of $O(sN \log N)$ was given. Efficient algorithms for construction of so-called embedded families of lattice rules where proposed in [4]. We refer to [16, 18] for a discussion.

3.2 Dimensional Truncation

Given $s \in \mathbb{N}$ and $\boldsymbol{y} \in \mathscr{U}$, we observe that truncating the sum in (7) at s terms amounts to setting $y_j = 0$ for $j > s$. We thus denote by $u^s(\boldsymbol{x}, \boldsymbol{y}) := u(\boldsymbol{x}, (\boldsymbol{y}_{\{1:s\}}; 0))$ the solution of the parametric weak problem (10) corresponding to the parametric operator $A((\boldsymbol{y}_{\{1:s\}}; 0))$ in which the sum (7) is truncated at s terms. Then Theorem 1 remains valid *with constants independent of s* when $u(\cdot, \boldsymbol{y})$ is replaced by its dimensionally truncated approximation $u^s(\cdot, \boldsymbol{y})$.

Theorem 4. *Under Assumptions* (**A2**), (**A3**), (**A5**), *for every $f \in \mathscr{Y}'$ and for every $\boldsymbol{y} \in \mathscr{U}$ and for every $s \in \mathbb{N}$, the dimensionally truncated, parametric solution $u^s(\cdot, \boldsymbol{y}) = u(\cdot, (\boldsymbol{y}_{\{1:s\}}; 0))$ of the s-term truncated parametric weak problem* (10) *satisfies, with b_j as defined in* (**A2**),

$$\|u(\cdot, \boldsymbol{y}) - u^s(\cdot, \boldsymbol{y})\|_{\mathscr{X}} \le C\alpha^{-1}\|f\|_{\mathscr{Y}'} \sum_{j \ge s+1} b_j \tag{33}$$

for some constant $C > 0$ independent of s, $\boldsymbol{y}$ and f. For every $G(\cdot) \in \mathscr{X}'$

$$|I(G(u)) - I_s(G(u))| \le \tilde{C}\,\alpha^{-1}\|f\|_{\mathscr{Y}'}\|G(\cdot)\|_{\mathscr{X}'} \left(\sum_{j \ge s+1} b_j \right)^2 \tag{34}$$

for some constant $\tilde{C} > 0$ independent of s, f and $G(\cdot)$. In addition, if Assumptions (**A3**) *and* (**A5**) *hold, then*

$$\sum_{j\geq s+1} b_j \leq \min\left(\frac{1}{1/p-1}, 1\right)\left(\sum_{j\geq 1} b_j^p\right)^{1/p} s^{-(1/p-1)} .$$

This result is proved in the affine case (7) in [17, Theorem 5.1], and for operators depending lognormally on $\boldsymbol{y}$ in [2]. It will hold for general probability densities $\varrho(y)$ in (4) whenever the factor measures $\varrho_j(dy_j)$ are centered.

4 Analysis of QMC and Galerkin Disretization

We apply QMC quadrature $Q_{s,N}$ to the dimensionally truncated approximaion $I_s(G(u))$ of the integral (24), where the integrand $F(\boldsymbol{y}) = G(u(\cdot, \boldsymbol{y}))$ is a continuous, linear functional $G(\cdot)$ of the parametric solution $u(\cdot, \boldsymbol{y})$ of the operator equation (10).

As proposed in [7, 23], choices of QMC weights can be based on minimizing the product of worst case error and of (upper bounds for) the weighted norms $\|F\|_{\mathscr{W}_{s,\boldsymbol{\gamma}}}$ in the error bound (30). This idea was combined with the bounds (13) in [17] to identify POD QMC weights (28) as sufficient to ensure a QMC convergence rate of $O(N^{-1+\delta})$ with $O()$ being independent of the truncation dimension s. Another issue raised by the infinite dimensional nature of the problem is to choose the value of s and estimate the truncation error $I(G(u)) - I_s(G(u))$, which was estimated in Theorem 4. The following QMC quadrature error bound is proved in [17, Theorem 5.1] for scalar, parametric diffusion problems; its statement and proof generalize to the parametric operator equations (12) with solution regularity (13).

Theorem 5 (Root-mean-square error bound). *Under Assumptions* (**A2**) *and* (9) *let b_j be defined as in* (**A2**). *For every $f \in \mathscr{Y}'$ and for every $G(\cdot) \in \mathscr{X}'$, let $u(\cdot, \boldsymbol{y})$ denote the solution of the parametric variational problem* (10).

Then for $s, N \in \mathbb{N}$ and weights $\boldsymbol{\gamma} = (\gamma_{\mathrm{u}})$, randomly shifted lattice rules $Q_{s,N}(\cdot;\cdot)$ with N points in s dimensions can be constructed by a component-by-component algorithm such that the root-mean-square error for approximating the finite dimensional integral $I_s(G(u))$ satisfies, for all $\lambda \in (1/2, 1]$, and all $N \leq 10^{30}$

$$\sqrt{\mathbb{E}\left[|I_s(G(u)) - Q_{s,N}(\cdot; G(u))|^2\right]} \leq \frac{C_{\boldsymbol{\gamma}}(\lambda)}{\alpha} N^{-1/(2\lambda)} \|f\|_{\mathscr{Y}'} \|G(\cdot)\|_{\mathscr{X}'} , \quad (35)$$

where $\mathbb{E}[\cdot]$ denotes the expectation with respect to the random shift $\boldsymbol{\Delta}$ (uniformly distributed over $[0, 1]^s$) and $C_{\boldsymbol{\gamma}}(\lambda)$ is independent of s as in [17, Eq. (6.2)].

In [17, Theorem 6.1], a choice of weights which minimizes the upper bound was derived. As the derivation in [17, Theorem 6.1] generalizes verbatim to the presently considered setting we only state the result. Under the assumptions of Theorem 5, for b_j as in (**A2**) suppose that (**A3**) holds, i.e.

$$\sum_{j\geq 1} b_j^p < \infty \qquad \text{for some} \quad 0 < p < 1 \,, \tag{36}$$

For the choice

$$\lambda := \begin{cases} \frac{1}{2-2\delta} & \text{for some} \quad \delta \in (0,1/2) \quad \text{when } p \in (0,2/3] \,, \\ \frac{p}{2-p} & \qquad\qquad\qquad\qquad\qquad \text{when } p \in (2/3,1) \,, \end{cases} \tag{37}$$

the choice of weights

$$\gamma_{\mathfrak{u}} = \gamma_{\mathfrak{u}}^* := \left(|\mathfrak{u}|! \prod_{j\in\mathfrak{u}} \frac{b_j}{\sqrt{\rho(\lambda)}} \right)^{2/(1+\lambda)} \tag{38}$$

with $\rho(\lambda)$ in (32) minimizes the constant $C_{\boldsymbol{\gamma}}(\lambda)$ in the bound (35). To account for the impact of Galerkin discretization of the operator equation, recall Sect. 2.5. For any $\boldsymbol{y} \in \mathscr{U}$, the *parametric FE approximation* $u^h(\cdot,\boldsymbol{y}) \in \mathscr{X}^h$ is defined as in (20). Here, $\boldsymbol{b}(\boldsymbol{y};\cdot,\cdot)$ denotes the parametric bilinear form (8). In particular the FE approximation (20) is defined *pointwise* with respect to the parameter $\boldsymbol{y} \in \mathscr{U}$.

Theorem 6. *Under Assumptions* (**A2**), (9) *and* (15) *for every* $f \in \mathscr{Y}'$ *and for every* $\boldsymbol{y} \in \mathscr{U}$*, the approximations* $u^h(\cdot,\boldsymbol{y})$ *are stable, i.e.* (21) *holds. For every* $f \in \mathscr{Y}_t'$ *with* $0 < t \leq 1$ *exists a constant* $C > 0$ *such that for all* $\boldsymbol{y} \in \mathscr{U}$ *as* $h \to 0$ *holds*

$$\sup_{\boldsymbol{y}\in\mathscr{U}} \|u(\cdot,\boldsymbol{y}) - u^h(\cdot,\boldsymbol{y})\|_{\mathscr{X}} \leq C\, h^t\, \|f\|_{\mathscr{Y}_t'} \,. \tag{39}$$

Proof. Since $f \in \mathscr{Y}_t'$ for some $t > 0$ implies with (15) that $u(\boldsymbol{y}) \in \mathscr{X}_t$ and, with the approximation property (22),

$$\|u(\cdot,\boldsymbol{y}) - u^h(\cdot,\boldsymbol{y})\|_{\mathscr{X}} \leq C\, h^t\, \|u(\cdot,\boldsymbol{y})\|_{\mathscr{X}_t}$$

where the constant C is independent h and of $\boldsymbol{y}$. This proves (39). □

Since we are interested in estimating the error in approximating functionals (24), we will also impose a regularity assumption on the functional $G(\cdot) \in \mathscr{X}'$:

$$\exists\, 0 < t' \leq 1 : \quad G(\cdot) \in \mathscr{X}_{t'}' \tag{40}$$

and the *adjoint regularity*: for t' as in (40), and for every $\boldsymbol{y} \in \mathscr{U}$,

$$w(\boldsymbol{y}) = (A^*(\boldsymbol{y}))^{-1} G \in \mathscr{Y}_{t'} \,, \quad \sup_{\boldsymbol{y}\in\mathscr{U}} \|w(\boldsymbol{y})\|_{\mathscr{Y}_{t'}} \leq C \|G\|_{\mathscr{X}_{t'}'} \,. \tag{41}$$

Moreover, since in the expression (23) only a bounded linear functional $G(\cdot)$ of u rather than the parametric solution u itself enters, the discretization error of $G(u)$ is of main interest in QMC error analysis. An Aubin-Nitsche duality argument shows that $|G(u(\cdot, \boldsymbol{y})) - G(u^h(\cdot, \boldsymbol{y}))|$ converges faster than $\|u(\cdot, \boldsymbol{y}) - u^h(\cdot, \boldsymbol{y})\|_{\mathscr{X}}$: under Assumptions (**A2**), (9), (**A4**), and (15), (41) there exists a constant $C > 0$ such that for every $f \in \mathscr{Y}_t'$ with $0 < t \leq 1$, for every $G(\cdot) \in \mathscr{X}_{t'}'$ with $0 < t' \leq 1$ and for every $\boldsymbol{y} \in \mathscr{U}$, as $h \to 0$, the Galerkin approximations $G(u^h(\cdot, \boldsymbol{y}))$ satisfy

$$\left|G(u(\cdot, \boldsymbol{y})) - G(u^h(\cdot, \boldsymbol{y}))\right| \leq C\, h^{\tau}\, \|f\|_{\mathscr{Y}_t'}\, \|G(\cdot)\|_{\mathscr{X}_{t'}'}\,, \tag{42}$$

where $0 < \tau := t + t'$ and where the constant $C > 0$ is independent of $\boldsymbol{y} \in \mathscr{U}$.

We conclude with bounds for the combined QMC FE approximation of the integral (24). To define the approximation of (24), we approximate the infinite dimensional integral using a randomly shifted lattice rule with N points in s dimensions. The QMC rule with N points for integration over $(-1/2, 1/2)^s$ for one single draw $\boldsymbol{\Delta}$ of the shift will be denoted by $Q_{s,N}(\cdot; \boldsymbol{\Delta})$. For each evaluation of the integrand F, we replace the exact solution $u(\cdot, \boldsymbol{y})$ of the parametric weak problem (10) by the Galerkin approximation $u^h(\cdot, \boldsymbol{y})$ in the subspace $\mathscr{X}^h \subset \mathscr{X}$ of dimension $M^h := \dim \mathscr{X}^h < \infty$.

Thus we may express the overall error as a sum of a *dimension truncation error* (which is implicit when a finite dimensional QMC method is used for an infinite dimensional integral), a *QMC quadrature error*, and a *FE discretization error*:

$$\begin{aligned} &I(G(u)) - Q_{s,N}(G(u^h); \boldsymbol{\Delta}) \\ &\quad = (I - I_s)(G(u)) + (I_s(G(u)) - Q_{s,N}(G(u); \boldsymbol{\Delta})) + Q_{s,N}(G(u - u^h); \boldsymbol{\Delta})\,. \end{aligned}$$

We bound the mean-square error with respect to the random shift by

$$\begin{aligned} \mathbb{E}\left[|I(G(u)) - Q_{s,N}(G(u^h); \cdot)|^2\right] &\leq 3\,|(I - I_s)(G(u))|^2 \\ &+ 3\,\mathbb{E}\left[|I_s(G(u)) - Q_{s,N}(G(u); \cdot)|^2\right] + 3\,\mathbb{E}\left[|Q_{s,N}(G(u - u^h); \cdot)|^2\right]\,. \end{aligned} \tag{43}$$

The dimension truncation error, i.e., the first term in (43), was estimated in Theorem 4. The QMC error, i.e., the second term in (43), is already analyzed in Theorem 5. Finally, for the Galerkin projection error, i.e., for the third term in (43), we apply the property that the QMC quadrature weights $1/N$ are positive and sum to 1, to obtain

$$\mathbb{E}\left[|Q_{s,N}(G(u - u^h); \cdot)|^2\right] \leq \sup_{\boldsymbol{y} \in \mathscr{U}} |G(u(\cdot, \boldsymbol{y}) - u^h(\cdot, \boldsymbol{y}))|^2\,,$$

and apply (42) Then, under the assumptions in Theorems 4, 5 and in (42), we approximate the dimensionally truncated approximation (25) of the integral (24) over $\mathscr{U}$ by the randomly shifted lattice rule from Theorem 5 with N points in s dimensions. For each lattice point we solve the approximate problem (20) with *one common subspace* $\mathscr{X}^h \subset \mathscr{X}$ with $M_h = \dim(\mathscr{X}^h)$ degrees of freedom and with the approximation property (17). Then, there holds the root-mean-square error bound

$$\sqrt{\mathbb{E}\left[|I(G(u)) - Q_{s,N}(\cdot; G(u^h))|^2\right]}$$
$$\leq C \left(\kappa(s,N) \, \|f\|_{\mathscr{Y}'} \, \|G(\cdot)\|_{\mathscr{X}'} + h^\tau \, \|f\|_{\mathscr{Y}'_t} \|G(\cdot)\|_{\mathscr{X}'_{t'}}\right) \,,$$

where $\tau = t + t'$, and, assuming $\varphi(N) \leq CN$, for fixed $\delta > 0$ arbitrary small,

$$\kappa(s,N) = \begin{cases} s^{-2(1/p-1)} & + N^{-(1-\delta)} \quad \text{when } p \in (0, 2/3] \,, \\ s^{-2(1/p-1)} & + N^{-(1/p-1/2)} \quad \text{when } p \in (2/3, 1) \,. \end{cases}$$

Acknowledgements The author is supported by ERC under Grant AdG 247277.

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