Aggregated Beliefs and Informational Cascades

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In the 1992 paper [1] Bikchandani et al. show how it may be rational for Bayesian agents in a sequential decision making scenario to ignore their private information and conform to the choices made by previous agents. If this occurs, an agent ignoring her private information is said to be *in a cascade*.

To illustrate, consider the following example: a set of agents must decide which of two restaurants to choose, one lying on the left side of the street, one on the right, with one being the better – L or R. Initially, agents have no information about which; each agent i has prior probabilities $Pr_i(L) = Pr_i(R)$. Every agent has two choices: either to go to the restaurant on the left, l_i, or to go the one on the right, r_i. All agents prefer to go to the better restaurant, and are punished for making the wrong choice, specified by pay-offs $u_i(l_i,L) = u_i(r_i,R) = v_1 > 0$ and $u_i(l_i,R) = u_i(r_i,L) = v_2 < 0$ with $v_1 + v_2 = 0$.

Before choosing, every agent receives a *private signal* indicating that either the restaurant on the left (L_i) or the one on the right (R_i) is the better one. The signals are assumed to be equally informative and positively correlated with the true state, in the sense that $Pr(L_i|L) = Pr(R_i|R) = q > .5$ and $Pr(L_i|R) = Pr(R_i|L) = 1-q$. Given this setup, rational agents will follow their private signal, the majority choosing the better restaurant.

If agents are assumed to *choose sequentially* and *observe the choice of those choosing before them*, a cascade may result, possibly leading the majority to pick the worse option. The argument for this [1] rests on higher-ordering reasoning *not represented in the Bayesian framework*, and goes as follows. Given either L_1 or R_1, agent 1 will choose as her signal indicates, hereby revealing her signal to all subsequent agents. Agent 2 therefore as two pieces of information: his own signal together with that deduced from the choice of 1. If 2 receives the same signal as 1, he will make the same choice; given two opposing signal, assume he will invoke a self-biased tie-breaking rule, and go by his own signal. In both cases, 2's choice will also reveal his private signal to all subsequent agents. Assume that 1 and 2 received signals L_1, L_2. Then *no matter which signal* 3 *receives, she will choose* l_3: agent 3 will have three pieces of information, either L_1, L_2, L_3 or L_1, L_2, R_3. In either case, when conditionalizing on these, the posterior probability of L being the true state will be higher than that of R. So 3 will choose l_3, and thereby be in a cascade. Further, agent 4 will also be in a cascade: as 3 chooses l_3 no matter what, *her choice does not reveal her private signal*, why also 4 has three pieces of information, either L_1, L_2, L_4 or L_1, L_2, R_4. 4 is thus in the same epistemic situation as 3, and will choose l_4. As 4 is in a cascade, his choice will not reveal his private signal, *and the situation thus repeats for all subsequent agents*.

Notice that *cascades may not be truth conducive*: there is a $Pr(L_1|R) \cdot Pr(L_2|R)$ risk that *all* agents will choose the wrong restaurant – e.g., if signals are correct with probability $\frac{2}{3}$, all agents choose wrong with probability $\frac{1}{9}$.

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Aim and Methodology. We construct a formal model that completely represents the reasoning made by agents in the sequential setup, for any input string of private signals. The type of model constructed is a dynamic epistemic logic variant of a state machine, in lack of terms called a *system*. A system operates by having for each state (Kripke model) some set of *transition rules* which as a function of the current state pick the next update to be invoked, hereby specifying the ensuing state. It is initiated from some *initial state* and terminates when an *end condition* is met.

The informational cascades system ($\mathcal{IC}$) constructed captures the following four elements of each agent's turn: *i)* earlier agents' actions are observed from which *ii)* their private signals (beliefs) are deduced and combined with *iii)* the private signal (belief) of the current agent after which *iv)* the chosen action is executed, observed by all.

$\mathcal{IC}$ diverges from the model of [1] in a number of aspects: it is *not probabilistic*, but *qualitative*; related, information aggregation is not done by Bayesian conditionalization, but by the *aggregation of perceived beliefs*; a *finite* set of agents is used; and *no pay-off structure nor rationality is assumed*, agency instead captured by transition rules.

An advantage of $\mathcal{IC}$ over the model from [1] is that $\mathcal{IC}$ fully specifies the intended scenario formally: all steps are defined for any string of private signals and all higher-order reasoning is represented. $\mathcal{IC}$ is thus a complete model for informational cascades.

In the present, only the system $\mathcal{IC}$ and results are presented, together with novel machinery required to define a system. A detailed walk-through of a cascading run with arguments for modeling choices, the presupposed definitions and references may be found in [2], the extended version of the present abstract.

Transition Rules and Systems

Before commencing with definitions, let us fix notation for presupposed machinery. Assume a finite set of agents $\mathcal{A} = \{1, 2, ..., n\}$, atoms $P \in \Phi$, a set Prop_Φ given by $\varphi ::= P \mid \varphi \mid \neg\varphi \mid \varphi \wedge \psi \mid B_i\varphi \mid K_i\varphi$ and definitions of *pointed epistemic plausibility models* (EPMs) $(\mathbf{S}, s_0) = (S, \leq_i, \|\cdot\|, s_0)_{i\in\mathcal{A}}$, *propositions* $\|\varphi\|_\mathbf{S} \subseteq S$ over EPMs for $\varphi \in \mathsf{Prop}_\Phi$, *pointed action plausibility models with postconditions* (APMs) $(\mathbf{E}, \sigma_0) = (\Sigma, \preceq_i, pre, post, \sigma_0)_{i\in\mathcal{A}}$, *doxastic programs* $\Gamma \subseteq \Sigma$ over APMs, *action priority update product* $\mathbf{S} \otimes \mathbf{E}$ (an anti-lexicographic belief revision operation on EPMs and APMs), and *dynamic modalities* $[\Gamma]$ with associated propositions $\|[\Gamma]\varphi\|_\mathbf{S}$.

Transition Rules. A *transition rule* $\mathcal{T}$ is an expression $\varphi \rightsquigarrow [X]\psi$ where $\varphi, \psi \in \mathsf{Prop}_\Phi$. Transition rules are *prescriptive* and read "if φ, then the next update must be such that after it, ψ".

Solutions. A set of transition rules dictates the choice for the next APM by finding the transition rule(s)'s *solution*. A *solution* to $\mathcal{T} = \varphi \rightsquigarrow [X]\psi$ over pointed EPM $(\mathbf{S}, s)$ is a doxastic program Γ such that $\mathbf{S}, s \models \varphi \rightarrow [\Gamma]\psi$. Γ is a solution to the set $\mathbb{T} = \{\mathcal{T}_1, ..., \mathcal{T}_n\}$ with $\mathcal{T}_k = \varphi_k \rightsquigarrow [X]\psi_k$ over $(\mathbf{S}, s)$ if $\mathbf{S}, s \models \bigwedge_1^n(\varphi_k \rightarrow [\Gamma_k]\psi_k)$, i.e. if Γ is a solution to all $\mathcal{T}_i$ over $(\mathbf{S}, s)$ *simultaneously*.[1] Finally, a set of doxastic programs $\mathbb{G}$ is a solution to $\mathbb{T}$ over $\mathbf{S}$ iff for every t of $\mathbf{S}$, there is a $\Gamma \in \mathbb{G}$ such that Γ is a solution to $\mathbb{T}$ over $(\mathbf{S}, t)$.

[1] Note the analogy with numerical equations; for both $2+x=5$ and $\{2+x=5, 4+x=7\}$, $x=3$ is the (unique) solution.

Next APM Choice. If $\mathbb{G}$ is a solution to $\mathbb{T}$ over $\mathbf{S}$, then given any state from $\mathbf{S}$, the transition rules in $\mathbb{T}$ will specify one (or more) programs from $\mathbb{G}$ as the *next APM choice*, denoted $next(\mathbf{S})_{\mathbb{T},\mathbb{G}}$, subscripts omitted. A *deterministic* choice will be made if $\mathbb{G}$ is selected suitably, i.e. if it contains a *unique* Γ for each s. In the ensuing, solution spaces will be chosen thusly.

System. A *system* is a tuple $\mathcal{S} = \langle \mathbf{S_0}, \mathbb{T}, \mathbb{G}, end\rangle$ where $\mathbf{S_0}$ is an EPM, called the *initial state*, $\mathbb{T}$ is a set of sets $\mathbb{T}(\mathbf{S})$ each a set of transition rules (those of EPM $\mathbf{S}$),[2] $\mathbb{G}$ is a set of sets $\mathbb{G}(\mathbb{T}(\mathbf{S}))$, each a set of doxastic programs (the solution space for $\mathbb{T}(\mathbf{S})$), and $end \in \mathsf{Prop}_\Phi$ is called the *end condition*.

A system provides for each EPM from some chosen set, a set of transition rules with associated solution space, and is run using next APM choice. The first next APM choice is made when the actual state of $\mathbf{S_0}$ is specified. A system runs until either the end condition is met, or until it constructs an EPM for which no transition rules are specified or no solution is available. Care must be taken to avoid the latter possibilities.

An Informational Cascades System Based on Aggregated Beliefs

Atoms. Let Φ consist of two "types" of atomic doxastic propositions; $\{L\}$ with $\neg L =: R$, representing respectively that the restaurant on the left or the one on the right is better, and $\{\alpha_i L, \alpha_i R\}_{i\in\mathcal{A}}$ with $\alpha_i L \cap \alpha_i R = \emptyset$, representing i's restaurant choice.[3] $\alpha_i R$ is *not* short for $\neg\alpha_i L$ as i may not yet have made *any* choice.

Aggregated Beliefs. To accumulate information, a notion of *perceived aggregated beliefs* is used. Introduce an operator $A_{i|G}$, representing the beliefs of agent i when aggregating information from her beliefs about the beliefs of agents from group G. $A_{i|G}$ is defined using simple majority 'voting' with a self-bias tie-breaking rule:

$$\mathbf{S}, s \models A_{i|G}\varphi \text{ iff } \alpha + |\{j \in G : \mathbf{S}, s \models B_i B_j \varphi\}| > \beta + |\{j \in G : \mathbf{S}, s \models B_i B_j \neg\varphi\}|$$

with tie-breaking parameters α, β given by $\alpha = ½$ if $s \in (B_i\varphi)_\mathbf{S}$, else $\alpha = 0$, and $\beta = ½$ if $s \in (B_i\neg\varphi)_\mathbf{S}$, else $\beta = 0$. This definition leaves agent i's aggregated beliefs undetermined iff both i is agnostic whether φ and there is no strict majority on the matter.

Overview of $\mathcal{IC}$. As mentioned, each agent's turn consists of four steps. In $\mathcal{IC}$ defined below, these consist of: *i*) EPM $\mathbf{S_i}$, the initial state of i's turn, *ii*) APM $\mathbf{I_{i-1}}$, invoking the *interpretation* of agent $i-1$'s executed action, supplying i with information about $i-1$'s beliefs, *iii*) APM $\mathbf{P_i}$, the *private signal* of i, forming her private beliefs about L/R, and *iv*) either l_i or r_i, the action i finally executes. The initial state of $i+1$ is then given by $\mathbf{S_{i+1}} := ((\mathbf{S_i} \otimes \mathbf{I_{i-1}}) \otimes \mathbf{P_i}) \otimes next((\mathbf{S_i} \otimes \mathbf{I_{i-1}}) \otimes \mathbf{P_i})$, with $next((\mathbf{S_i} \otimes \mathbf{I_{i-1}}) \otimes \mathbf{P_i}) \in \{l_i, r_i\}$.

Three sets of transition rules are used to run the system: the first is a singleton, always invoking interpretation of the previous agent's action. The second is also a singleton, invoking a private signal specified by a vector defined together with the system. Third, a set of two rules which specify the choice of the agent as a function of her aggregated beliefs, hereby specifying the used agent type.

[2] It is assumed that *model names matter*: though $\mathbf{S} = \mathbf{S} \otimes \Gamma$, we allow that $\mathbb{T}(\mathbf{S}) \neq \mathbb{T}(\mathbf{S} \otimes \Gamma)$.

[3] $\alpha_i L$ and $\alpha_i R$ are *post-factual action descriptions*, not the actions themselves, as these are captured using APMs, see point 3. in the definition of the system $\mathcal{IC}$ below.

The System $\mathcal{IC}$. Define the system $\mathcal{IC} = \langle \mathbf{S_1}, \mathbb{T}, \mathbb{G}, end \rangle$ as follows: let the initial state be $\mathbf{S_1}$ (Fig. 1) and set $end := \alpha_m L \vee \alpha_m R$ with $m = max(\mathcal{A})$. That is, the system initiates with all agents uninformed about whether L or R, and terminates when the last agent has chosen at which restaurant to dine.

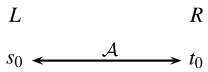

Fig. 1. The EPM $\mathbf{S_1}$ representing the initial uncertainty about the better restaurant. All agents know one restaurant is better, but does neither know nor believe which one. Labels L and R indicate truth of the atom, e.g. $s_0 \in \|L\|_{\mathbf{S_1}}$. For all $P \in \{\alpha_i L, \alpha_i R\}_{i \in \mathcal{A}}$, $\|P\|_{\mathbf{S_1}} = \emptyset$ as no agent has chosen.

Set $\mathbf{S_{n+1}} := (((\mathbf{S_n} \otimes \mathbf{I_{n-1}})) \otimes \mathbf{P_n}) \otimes next(((\mathbf{S_n} \otimes \mathbf{I_{n-1}})) \otimes \mathbf{P_n})$, and give $\mathbb{T}$ and $\mathbb{G}$ by

1. $\mathbb{T}(\mathbf{S_n}) = \{\mathcal{I}_{n-1} = \top \rightsquigarrow [X]\top\}$ with $\mathbb{G}(\{\mathcal{I}_{n-1}\}) = \{\mathbf{I_{n-1}}\}$, where $\mathbf{I_{n-1}}$ is the one state interpretation APM with preconditions

$$pre(i_{n-1}) := \alpha_{n-1} L \to A_{n-1|\mathcal{A}} L \wedge \alpha_{n-1} R \to A_{n-1|\mathcal{A}} R$$

with special case $\mathbf{I_0}$ having $pre(i_0) = post(i_0) = \top$.

2. $\mathbb{T}(\mathbf{S_n} \otimes \mathbf{I_{n-1}}) = \{\mathcal{P}_n = \top \rightsquigarrow [X]\top\}$ with $\mathbb{G}(\{\mathcal{P}_n\}) = \{(\mathbf{P_n}, x_n)\}$, where $\mathbf{P_n}$ is the private signal APM (Fig. 2) indexed for n, with x_n the actual state as given by a *private signal vector* $\mathbf{P} = (x_1, x_2, ..., x_m)$ with $x_k \in \{\sigma_L, \sigma_R\}$, determining whether n receives a signal that L (σ_L) or that R (σ_R).

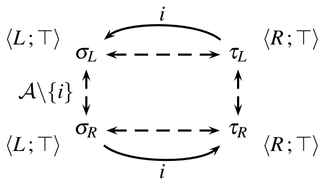

Fig. 2. APM $\mathbf{P_i}$: i receives private signal while others remain uninformed about *which*. State labels $\langle \varphi; \psi \rangle$ specify pre- and postconditions. Transitive and reflexive arrows are not drawn.

3. $\mathbb{T}(((\mathbf{S_n} \otimes \mathbf{I_{n-1}})) \otimes (\mathbf{P_n}, x_n)) = \{\mathcal{A}_L, \mathcal{A}_R\}$ (Fig. 3) with $\mathbb{G}(\{\mathcal{A}_L, \mathcal{A}_R\}) = \{l_n, r_n\}$, the singleton doxastic programs over the APM in Fig. 3, all indexed for n.

Aggregator rules:

$\mathcal{A}_L = A_{i|G} L \rightsquigarrow [X]\alpha_i L$

$\mathcal{A}_R = A_{i|G} R \rightsquigarrow [X]\alpha_i R$

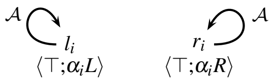

Fig. 3. Aggregator transition rules specifying an agent type who bases decisions on aggregated beliefs, and the APM $\mathbf{A_i}$ over which the two possible actions for agent i is given; i may choose to go to either the restaurant on the left (l_i) or the one on the right (r_i).

Given the cumbersome definition of $\mathcal{IC}$, it is worth verifying that the system in fact runs appropriately. By induction it may be shown that for every agent $i \leq m$, the system will produce state $\mathbf{S_{i+1}}$ satisfying $\alpha_i L \vee \alpha_i R$, yielding the following proposition.

Proposition 1. *The system $\mathcal{IC}$ runs until end $:= \alpha_m L \vee \alpha_m R$ is satisfied at $\mathbf{S_{m+1}}$, irrespectively of which initial state or which signal vector $\mathbf{P}$ is used for input.*

In a Cascade. With $\mathcal{IC}$ defined, it is possible to precisely define the notion of *being in a cascade*: agent i is *in a cascade* iff

i) $next((\mathbf{S_i} \otimes \mathbf{I_{i-1}})) \otimes (\mathbf{P_i}, x_i)) = l_i$ for both $x_i \in \{\sigma_L, \sigma_R\}$, or
ii) $next((\mathbf{S_i} \otimes \mathbf{I_{i-1}})) \otimes (\mathbf{P_i}, x_i)) = r_i$ for both $x_i \in \{\sigma_L, \sigma_R\}$.

The definition captures that i acts in accordance with an established majority, *irrespective of her own signal*.[4]

The following lemma captures a crucial property regarding the higher-order reasoning occurring in cascades, namely that *the choice of an agent in a cascade provides no information about their private beliefs* (hence neither about their private signal).

Lemma 1. $\mathbf{S_{n+1}} \otimes \mathbf{I_n} \models B_{n+1}B_nL \vee B_{n+1}B_nR$ *iff n is not in a cascade.*

To state the main result, notation for the agents in cascade who ignored which signals is handy. Let $\mathbf{P}_i$ be the private signals for agents $j < i$, i.e. the initial segment of $\mathbf{P}$ of length $i-1$. Let $C_{Li} = \{j < i : j \text{ is in cascade and } x_j = \sigma_L\}$ and $C_{Ri} = \{j < i : j \text{ is in cascade and } x_j = \sigma_R\}$. We may then state the main result.

Theorem 1. *Agent i is in cascade iff two more agents have received private signal of one type than have received signals of the other type, not counting signals of agents in a cascade. Precisely: i is in cascade of type i) iff*

$$|\{\sigma_L \in \mathbf{P}_i\}| - |C_{Li}| \geq (|\{\sigma_R \in \mathbf{P}_i\}| - |C_{Ri}|) + 2,$$

and agent i is in cascade of type ii) iff

$$(|\{\sigma_L \in \mathbf{P}_i\}| - |C_{Li}|) + 2 \leq |\{\sigma_R \in \mathbf{P}_i\}| - |C_{Ri}|.$$

The theorem provides necessary and sufficient conditions on the private signal string for an agent to be in a cascade. The sufficient conditions are identical to those from [1], see p. 1005-06, here shown for a model which explicitly represents all higher-order reasoning and agent decision making.

Corollary 1. *Cascades in $\mathcal{IC}$ are irreversible: if i is in a cascade of type i) resp. type ii), then for all $k > i$, k will be in a cascade of type i) resp. type ii).*

The corollary captures the quintessential effect of cascades, namely that they propagate through the remaining group.

These results show that the system $\mathcal{IC}$ functions as the informal reasoning supposed in [1]. For proofs, further conclusions, discussion, venues for future research and relevant references, the reader is referred to [2].

References

1. Bikhchandani, S., Hirshleifer, D., Welch, I.: A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades. Journal of Political Economy 100(5), 992–1026 (1992)
2. Rendsvig, R.K.: Aggregated Beliefs and Information Cascades (extended) (2013), `http://vince-inc.com/rendsvig/papers/IC1.pdf`

[4] The definition thus closely mirrors that from the original paper: “An informational cascade occurs if an individual’s action does not depend on his private signal.” [1, p. 1000]