The Logic of *a Priori* and *a Posteriori* Rationality in Strategic Games

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Abstract. We propose a logic for describing the interaction between knowledge, preference, and the freedom to act, and their interactions with the norms of *a Priori* and *a Posteriori* rationality, which we have argued for in previous work [3]. We then apply it to strategic games to characterise weak dominance and Nash equilibrium.

In [3] we proposed a model for rational decision making in which the facts about knowledge, preference and freedom to act are clearly separated from the norms of reasoning. Even the transitivity of the preference relation is considered normative, in our approach. The factual basis for decision making is modelled using what we call 'decision frames' and their multi-agent extensions, 'social decision frames'. We proposed two norms for decision-making, called '*a Priori* rationality' and '*a Posteriori* rationality', which apply to reasoning before making the decision, and after. Before making a decision, one is concerned with making the best, or at least an optimal, decision in ignorance of the effect of contextual factors, especially, in the social setting, the actions of other agents. After making a decision, one is more interested in which the decision was optimal given the conditions that actually applied. We went on to show that these two general norms specialise to the familiar norms of game theory: avoiding (weakly) dominated strategies (*a Priori*) and wanting to have made a best response (*a Posteriori*). The level of abstraction allowed us to provide a uniform account of both pure-strategy and mixed-strategy games.

Here, we propose a language for describing and reasoning with these norms, and show how it formalises and so justifies some of the processes of reasoning that we use to make decisions.

In Section 1 we introduce decision frames and the corresponding concepts of *a Priori* and *a Posteriori* rationality. In Section 2 we propose a language for describing these, and show how it can be embedded in a more powerful language in which an axiomatisation can be given. We go on in Section 3 to show how this is applied to strategic game theory.

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1 The Facts and Norms of Decision-Making

The perspective of our analysis is that of your evaluating a decision that you have just made, to determine whether or not it was a good one. To model this, we propose in [3] the following structures:

Definition 1. *A* decision frame $F = \langle W, \sim, \approx, \leqslant \rangle$ *consists of a non-empty set* W *of* possible decision situations*, with binary relations* $\sim$*,* $\approx$ *and* $\leqslant$ *on* W*, where*

$\sim$ *is an equivalence relation*	$u \sim v$ *represents that* v *would have been possible in* u *had you acted differently, given the contingencies of* u *that are beyond your control (freedom).*
$\approx$ *is an equivalence relation*	$u \approx v$ *represents that in situation* u *you would not know that you weren't in situation* v *(epistemic indistinguishability).*
$\leqslant$ *is a relation*	$u \leqslant v$ *represents that you regard situation* v *as at least as good as situation* u*.*

Importantly, the structures only represent the facts related to your decision, not the norms. In particular, the relation $\leqslant$ is not required to have any special properties (such as transitivity). Nonetheless, certain norms are definable on the basis of these facts. In [3] we argue that the fundamental norm of decision-making is that you should avoid situations in which you know that a strictly better alternative was possible had you chosen differently. This is ambiguous between two readings of the counterfactual. On the first, *a Priori* reading, you consider only what was known to you at the time of making the decision. On the second, *a Posteriori* reading, you also consider what is known to you after making the decision, specifically those contingent factors such as the actual actions of other agents and the actual circumstances relevant to your decision that you could not have known in advance.

The first of these norms, is formalised using a generalisation of the $\leqslant$ relation, called '*a Priori* free preference' that factors in the contribution of knowledge and freedom.

Definition 2. *The relation* $\leqslant_F$ *of* a Priori free preference *is defined by*

$$u \leqslant_F v \text{ iff } u' \leqslant v' \text{ for all } u' \approx u \text{ and all } v' \approx v \text{ such that } u' \sim v'.$$

Decision situation u *is* a Priori rational *iff there is no* $v \sim u$ *such that* $v >_F u$*.*

Various specific cases of the *a Priori* free preference relation are worthy of mention. Firstly, assuming that your freedom to choose is unlimited ($\sim$ is the universal relation) and your knowledge unbounded ($\approx$ is the identity relation),

$\leqslant_F = \leqslant$, so the free preference relation is a generalisation of the ordinary preference relation. Keeping knowledge unbounded, but allowing for limitations on your freedom to choose, we get that $\leqslant_F = \sim \cap \leqslant$. In other words, you only compares your current situation with one that you may have been in had you chosen differently. This is a special case of a more familiar *ceteris paribus* restriction on preference comparisons, which requires the compared situations to be equivalent *ceteris paribus*. In the present setting, two situations are equivalent *ceteris paribus* iff they are free alternatives.[1] When your freedom to choose is unlimited ($\sim$ is universal) but there are some limitations to your knowledge, $u \leqslant_F v$ iff $u' \leqslant v'$ for all $u' \approx u$ and all $v' \approx v$. In other words, since you do not know that you are in situation u, only that you are in one of the situations in $[u]$ (the $\approx$-equivalence class of u) and, likewise, were you in situation v, you would know only that you were in one of the situations in $[v]$, to judge that v is (as far as you know) at least as good as u, there should be no $u' \in [u]$ and $v' \in [v]$ for which $u' \nleqslant v'$. Further discussion of the justification of these various preference relations, including our assuming neither reflexivity nor transitivity of $\leqslant$ are contained in [3] (p.186). *A Priori* rationality is closely related to the norm of avoiding weakly dominated strategies in game theory, to be discussed in Section 3.

The second norm of *a Posteriori* rationality is formalised using the relation of '*a Posteriori* free preference' which is the restriction of *a Priori* free preference to alternatives that were in fact possible had you chosen differently, even if you didn't know this at the time.

Definition 3. *The relation $\leqslant_{F'}$ of* a Posteriori free preference *is defined by*

$$u \leqslant_{F'} v \text{ iff } u' \leqslant v' \text{ for all } u' \approx u \text{ and all } v' \approx v \text{ such that } u \sim u' \sim v' \sim v.$$

Decision situation u is a Posteriori rational *iff there is no $v \sim u$ such that $v >_{F'} u$.*

A posteriori rationality is closely related to the norm of achieving a best response in game theory, to be discussed in Section 3.

2 A Logic of Rational Decisions

To describe decision frames and the corresponding norms we will use the following hybrid modal language.

[1] Dealing with the *ceteris paribus* aspect of preference comparisons is a matter of degree. Assume that $\leqslant$ already involves all those *ceteris paribus* considerations that are not concerned with freedom of choice. So, for example, if v is a situation that is the same as your current situation u in all such relevant respects except that you have one million dollars (more?) in your bank account, You may judge $u \leqslant v$ (and probably that $u < v$) but, unhappily, you are unlikely to be free to choose between u and v and so $u \nleqslant_F v$. We could have started with a basic preference order $\preceq$, modelled *ceteris paribus* equivalence (not including considerations of achievability) by an equivalence relation CP and then defined $\leqslant = CP \cap \preceq$. Adding freedom as a *ceteris paribus* condition, we would still have $\leqslant_F = \sim \cap CP \cap \preceq = \sim \cap \leqslant$.

Definition 4. *Given disjoint countably infinite sets* Prop *of* propositional variables *and* Nom *of* nominals*, the language L consists of the following* formulas

$$\varphi ::= p \mid i \mid R \mid R' \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid G\varphi \mid K\varphi \mid C\varphi \mid @_i\varphi$$

for $p \in$ Prop*,* $i \in$ Nom*.*

We interpret $G\varphi$ to mean that φ holds in all situations that would have been at least as good for you as the present situation. They may, of course, be no longer possible, as a result of your decision. $K\varphi$ means, as usual, that you know that φ in the present situation, or, more precisely, that φ holds in all situations that you could be in, given your knowledge. We do not, of course, assume that you know precisely which situation you are in. $C\varphi$ means that φ holds in all situations in which you could have been, had you acted differently. The sense of 'could have' here takes into account all those factors that are beyond your control, including the actual actions of other agents and other contingent factors. Finally, R and R' mean that present situation is *a Priori* or *a Posteriori* rational, respectively.

The semantics of L is the standard semantics for hybrid logic, taking G, K and C to be the normal modal operators for the relations $\leqslant$, $\approx$ and $\sim$. R and R' are zero-ary operators that hold in the *a Priori* and *a Posteriori*rational situations, respectively. That R and R' cannot be given an explicit definition in terms of the other operators is easy to shown by a bisimulation argument. This makes the derivation of logical principles relating them somewhat difficult and to solve this problem we will embed the language L in the following, much more powerful language.

Definition 5. *The language of* cpdl[2] *over a sets* Prop *of* propositional variables, Nom *of* nominals *and* AtProg *of* atomic programs *consists of the sets* Form *of* formulas *and* Prog *of* programs *given by*

$$\begin{aligned} \varphi \in \text{Form} &::= p \mid i \mid [\pi]\varphi \mid \neg\varphi \mid (\varphi \wedge \varphi) \\ \pi \in \text{Prog} &::= a \mid \varphi? \mid \overline{\pi} \mid \pi^{\circ} \mid \pi^{*} \mid (\pi;\pi) \mid (\varphi \cup \varphi) \end{aligned}$$

for $i \in$ Nom*,* $p \in$ Prop *and* $\alpha \in$ AtProg*.*
Abbreviations: $\top = p \vee \neg p$*,* $U = \alpha \cup \overline{\alpha}$ *(universal),* $I = \top?$ *(identity),* $\langle\pi\rangle = \neg[\pi]\neg$*,* $(\pi \cap \rho) = \overline{(\overline{\pi} \cup \overline{\rho})}$*,* $(\pi \subset \rho) = \overline{\pi \cap \overline{\rho}}$ *plus the usual Booleans:* $\bot$, $\vee$, $\rightarrow$, $\leftrightarrow$*. Also, where no confusion can arise, especially in the case of atomic programs, we further abbreviate* $(\pi \cap \rho)$ *as* $\pi\rho$*. A formula* φ *is* pure *iff it contains no propositional variables.*

Definition 6. *A structure* $F = \langle W, R\rangle$ *is a* cpdl frame *if* $R(\alpha) \subseteq W^2$ *for each* $\alpha \in$ AtProg*. A structure* $M = \langle W, R, V\rangle$ *is a* cpdl model *if* $\langle W, R\rangle$ *is a* cpdl *frame,* $V(i) \in W$ *for each* $i \in$ Nom*, and* $V(p) \subseteq W$ *for each* $p \in$ Prop*.*

[2] What we are calling cpdl (Combinatory pdl)is also known as 'full-cpdl'. The language of cpdl without $^{-}$ and $^{\circ}$ is also known as 'hybrid pdl' [1].

Definition 7. *Given a* CPDL *model* $M = \langle W, R, V\rangle$*, and a state* $u \in W$*, we define* $[\![\varphi]\!]^M \subseteq W$ *for each* $\varphi \in$ FORM *and* $[\![\pi]\!]^M \subseteq W^2$ *for each* $\pi \in$ PROG *as follows:*

$$
\begin{array}{ll}
[\![i]\!]^M & = \{V(i)\} \\
[\![p]\!]^M & = V(p) \\
[\![[\pi]\varphi]\!]^M & = \{u \in W \mid v \in [\![\varphi]\!]^M \text{ for each } v \in W \text{ such that } \langle u, v\rangle \in [\![\pi]\!]^M\} \\
[\![\neg\varphi]\!]^M & = W \backslash [\![\varphi]\!]^M \\
[\![(\varphi \wedge \psi)]\!]^M & = [\![\varphi]\!]^M \cap [\![\psi]\!]^M \\
[\![\alpha]\!]^M & = R(\alpha) \\
[\![\varphi?]\!]^M & = \{\langle u, u\rangle \mid u \in [\![\varphi]\!]^M\} \\
[\![\overline{\pi}]\!]^M & = \{\langle u, v\rangle \mid \langle u, v\rangle \notin [\![\pi]\!]^M\} \\
[\![\pi^\circ]\!]^M & = \{\langle u, v\rangle \mid \langle v, u\rangle \in [\![\pi]\!]^M\} \\
[\![\pi^*]\!]^M & = \textit{the smallest transitive, reflexive relation containing } [\![\pi]\!]^M \\
[\![\pi; \rho]\!]^M & = \{\langle u, v\rangle \mid \langle v, w\rangle \in [\![\pi]\!]^M \text{ and } \langle w, v\rangle \in [\![\rho]\!]^M \text{ for some } w \in W\} \\
[\![(\pi \cup \rho)]\!]^M & = [\![\pi]\!]^M \cup [\![\rho]\!]^M
\end{array}
$$

When M *is clear from the context, we write* $[\![\varphi]\!]^M$ *as* $[\![\varphi]\!]$*. Note that* $[\![(\pi \cap \rho)]\!]^M = [\![\pi]\!]^M \cap [\![\rho]\!]^M$ *and* $[\![(\pi \subset \rho)]\!]^M = W$ *if* $[\![\pi]\!]^M \subseteq [\![\rho]\!]^M$*, otherwise* $\varnothing$*.*

As usual, a formula is valid on a model if $M, u \models \varphi$ *for all* u*, valid on a frame* F *if it is valid on all models* $\langle F, V\rangle$ *and simply valid if it is valid on all frames.*

Theorem 1. *[1] There is an axiomatisation* K *of* CPDL *which is sound and such that for every extension* $\mathsf{K}\Gamma$ *of* K *with pure formulas* Γ *as axioms, if a formula is consistent in* $\mathsf{K}\Gamma$ *then it has a countable model on a frame in which all the formulas in* Γ *are valid. The system* $\mathsf{K}\Gamma$ *is therefore also complete for that class of frames.*

Comment on Axiomatisation and Complexity. Although the validity problem for CPDL is known to be highly undecidable [8], it has a number of well-known decidable fragments, include PDL itself and its extension to allow $^\circ$ and either $\cap$ or $\overline{a}$, i.e., $\overline{}$ restricted to atomic programs, but not both [2] [5]. Hybrid PDL namely PDL with nominals is also decidable [7] but even non-hybrid PDL with unrestricted $\overline{}$ is not. Frame consequence is undecidable even for PDL [8] with premises restricted to pure formulas, and so the decidability of validity for these fragments of CPDL cannot be automatically extended to specific classes of frames defined by pure formulas, despite the existence of a complete axiomatisation.

In order to describe a social decision frame as a frame $F = \langle W, R\rangle$ we take ATPROG $= \{\mathsf{g}, \mathsf{k}, \mathsf{c}\}$ with $R(\mathsf{g}) = \leqslant$, $R(\mathsf{k}) = \approx$ and $R(\mathsf{c}) = \sim$. Then the relations of *a Priori* and *a Posteriori* free preference can be defined as

$$
\begin{array}{ll}
\mathsf{F} = \overline{\mathsf{k}; \mathsf{c}\overline{\mathsf{g}}; \mathsf{k}} & (\textit{a Priori} \text{ free preference}) \\
\mathsf{F}' = \overline{\mathsf{ck}; \mathsf{c}\overline{\mathsf{g}}; \mathsf{ck}} & (\textit{a Posteriori} \text{ free preference})
\end{array}
$$

This enables us to embed L in CPDL as follows:

$$\begin{array}{ll} G\colon=[\mathsf{g}] & R:=[\mathsf{c}\mathsf{F}\overline{\mathsf{F}^\circ}]\bot \\ K\colon=[\mathsf{k}] & R'\colon=[\mathsf{c}\mathsf{F}'\overline{\mathsf{F}'^\circ}]\bot \\ C:=[\mathsf{c}] & \end{array}$$

Theorem 2. *A frame is a decision frame iff the following pure formulas* D *are valid on* F*:*

$$\begin{array}{lll} \neg @_i K\neg i & \neg @_i C\neg i & \textit{(reflexivity of } \approx \textit{ and } \sim\textit{)} \\ \neg(\neg KK\neg i \wedge K\neg i) & \neg(\neg CC\neg i \wedge C\neg i) & \textit{(transitivity of } \approx \textit{ and } \sim\textit{)} \\ @_i K\neg j \rightarrow @jK\neg i & @_i C\neg j \rightarrow @jC\neg i & \textit{(symmetry of } \approx \textit{ and } \sim\textit{)} \end{array}$$

Proof. It is enough to check that R and R' are satisfied by precisely the *a Priori* and *a Posteriori* rational situations.

Corollary 1. *The system* $\mathsf{K}D$ *is a complete axiomatisation of the formulas valid in decision frames.*

Proof. It follows from Theorem 1 and 2.

In the full paper we will consider larger fragments of CPDL extending L but which are self-contained in terms of axiomatisation, i.e. to identify exactly which auxiliary operators are needed.

Our language and its CPDL-extension can easily be extended to the multi-agent setting. For a given finite set A of agents, we define the language $L(A)$ to have operators G_a, K_a, C_a, R_a and R'_a and interpret the resulting formulas in 'social decision frames'.

Definition 8. *A* social decision frame $F = \langle W, \approx, \sim, \leqslant\rangle$ *for* A *consists of a decision frame* $F_a = \langle W, \approx_a, \sim_a, \leqslant_a\rangle$ *for each* $a \in A$.

Theorems 2 and Corollary 1 can then be extended to social decision frames, using the corresponding embedding into CPDL with AtProg $= \{\mathsf{g}_a, \mathsf{k}_a, \mathsf{c}_a\}_{a\in A}$ and the corresponding $\mathsf{K}D_A$.

3 Games

Our primary examples of social decision frames are taken from the concept of a strategic game in Game Theory.

Definition 9. *Given a set* A *of* agents*, sets* D_a *of* strategies *(for each* $a \in A$*), and* utility *functions*

$$U_a\colon \prod_{a\in A} D_a \rightarrow \mathbb{R}$$

the strategic game frame $G(A, D, U)$ *is the social decision frame* $\langle W, \sim, \approx, \leqslant\rangle$ *given by*

$W = \prod_{a \in A} D_a$
$w \sim_a v$ *iff* $w_b = v_b$ *for all* $b \neq a$ *in* A
$w \approx_a v$ *iff* $w_a = v_a$
$w \leqslant_a v$ iff $U_a(w) \leqslant U_a(v)$

For example, consider the game between players a and b, whose possible strategies are $\{A, B, C\}$ and $\{X, Y, Z\}$ respectively, with utilities given by the table on the left of Figure 1, with (x, y) representing a utility of x to a and y to b for the corresponding outcome. This determines the strategic game frame shown on the right, with a's relations shown with solid lines, and b's with dotted lines. We assume reflexivity and transitivity without displaying the additional links explicitly. The $\approx$ relations are also not shown, since they can be calculated from the capacity relations in a strategic game frame. So, for example, $AY \approx_a AZ$, $AZ \sim_a BZ$, $AY <_a AZ$, $AY \leqslant_b AZ \geqslant AY$.

	b			
		X	Y	Z
a	A	3, 1	1, 0	1, 0
	B	3, 2	0, 1	2, 0
	C	4, 2	0, 3	1, 3

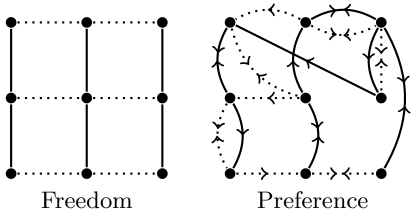

Fig. 1. A two-player game and its representation as a strategic game frame

Theorem 3. *The extension* $\mathsf{K}G$ *of* $\mathsf{K}DA$ *with the following axioms is complete for the class of strategic game frames:*

$\mathbf{G_1}$	$\vdash [U \cap \overline{\mathsf{c}_A}]\bot$	*(connected)*
$\mathbf{G_2}$	$\vdash [\mathsf{c}_{\bar{a}} \cap \overline{\mathsf{k}}_a]\bot$	*(isolated)*
$\mathbf{G_3}$	$\vdash [(\mathsf{c}_a;\mathsf{c}_b) \cap \overline{(\mathsf{c}_b;\mathsf{c}_a)}]\bot$	*(unordered)*
$\mathbf{G_4}$	$\vdash [\mathsf{c}_a \cap \mathsf{k}_a \cap \overline{\top?}]\bot$	*(deterministic)*
$\mathbf{G_5}$	$\vdash [g_a^* \cap \overline{\mathsf{g}_a}]\bot \wedge [U \cap \overline{(\mathsf{g}_a \cup \mathsf{g}_a{}^\circ)}]\bot$	*(linear)*

Proof. The formulas correspond to the following frame conditions:

$\mathbf{G_1}$ is valid iff $(u)_A = W$ for every $u \in W$	(connected)
$\mathbf{G_2}$ is valid iff every $a \in A$ is isolated	(isolated)
$\mathbf{G_3}$ is valid iff $\sim_a;\sim_b = \sim_b;\sim_a$ for every $a, b \in A$	(unordered)
$\mathbf{G_4}$ is valid iff $(u)[u] = \{u\}$ for every $u \in W$	(deterministic)
$\mathbf{G_5}$ is valid iff $\leqslant$ is reflexive, transitive, and total	(linear)

Each formulas are all pure, so by Theorem 1 $\mathsf{K}G$ is complete for the class of frames satisfying these conditions. By Theorem 1 of [3] (p. 194) a frame is isomorphic to a strategic game frame iff each of these conditions holds and in addition, it has a 'small' value-size, which means that the number of sets of indifferent situations (i.e. equivalence classes under the equivalence relation $u \leqslant v$ and $v \leqslant u$) is of cardinality $\leqslant 2^{\aleph_0}$. Thus $\mathsf{K}G$ is a sound for strategic frames, and complete if every formula satisfiable in such a frame is also satisfiable on a frame with small value-size. But this is guaranteed by the existence of countable models in Theorem 1.

Again, the full version of the paper will contain an exploration of which of these axioms can be stated in fragments extending L with auxiliary operators.

Standard game theoretic concepts such as 'best response', 'Nash equilibrium', 'dominated strategy,' etc. all lift to the slightly more abstract setting of strategic game frames, as shown in [3].

Definition 10. *Given a strategic game frame $G(A, D, U)$, agent a's strategy $d \in D_a$ is (weakly)* dominated *by another strategy $d' \in D_a$ iff*

1. *d' is sure to be at least as good as d: $w[^a_{d'}] \geqslant_a w[^a_d]$ for all $w \in W(A, D)$,*[3] *and*
2. *d' may be better than d: $w[^a_{d'}] >_a w[^a_d]$ for some $w \in W(A, D)$.*

For example, b's strategy Z is dominated by Y because (1) $AY \geqslant_b AZ$, $BY \geqslant_b BZ$, $CY \geqslant_b CZ$, and (2) $BY >_v BZ$. In fact $AY \geqslant_{bF} AZ$ but $AY \nleqslant_{bF} AZ$, so $AY >_{bF} AZ$ and since $AY \sim_b AZ$, the decision situation AZ is not *a Priori* rational for b. This connection between *a Priori* rationality and domination is quite general.

Theorem 4. *In a model M based on a strategic game frame $G(D, U, A)$, a strategy w_a is dominated iff $M, w \models \neg R_a$.*

Proof. By [3], Theorem 2, p.199, w_a is dominated iff w is *a Priori* rational for a.

Definition 11. *A decision situation w in a strategic game frame $G(A, D, U)$ is a* best response *for agent a iff there is no strategy $d \in D_a$ such that $w <_a w[^a_d]$ It is a* Nash equilibrium *iff it is a best response for all agents.*

For example, AY is a best response for a because neither $AY <_a BY$ nor $AY <_a CY$. But it is not a best response for b because $AY <_b AX$. This game has no Nash equilibrium. Since $(AY)_a$ is $\{AY, BY, CY\}$ and $(AY)_b$ is $\{AX, AY, AZ\}$, we can also check that $AY <_{F'b} AX$ but there is no w such that $AY <_{aF'} w$, so AY is *a Posteriori* rational for a but not for b. Again, this connection is general.

[3] $w[^a_d]$ is the strategy profile obtained by replacing a's strategy in w by d, i.e.

$$w[^a_d](b) = \begin{cases} d & \text{if } b = a \\ w_b & \text{otherwise} \end{cases}$$

Theorem 5. *In a model M based on a strategic game frame $G(D, U, A)$, a situation w is Nash equilibrium iff $M, w \models \bigwedge_{a \in A} R'_a$.*

Proof. By [3], Theorem 3, p.200. a Nash equilibrium is a situation that is *a Posteriori* rational for all agents.

We can extend this analysis from 'pure strategy' games to 'mixed strategy' games, in which the players randomise their choice of strategy.

Definition 12. *Given a strategic game frame $G(A, D, U)$ with finite*[4] *D, the* mixed-strategy extension *of G is the strategic game frame $G^*(A, D^*, U^*)$ in which D^*_a is the set of probability functions $\delta \colon D_a \to [0, 1]$ and for each $\delta \in D^*$,*

$$U^*_a(\delta) = \sum_{s \in \prod_{b \in A} D_b} u_a(s) \prod_{b \in A} \delta_b(s_b)$$

A frame is a mixed-strategy game frame *iff it is isomorphic to G^* for some strategic game frame G.*

There are, of course, formulas that are valid in every mixed-strategy game frame that are not valid in every strategic game frame and so cannot be derived from **G**. A central example is the following.

Theorem 6. $\mathsf{K}G \not\vdash \langle U \rangle \bigwedge_{a \in A} R'_a$ *but this formula is valid on all mixed-strategy strategic game frames.*

Proof. Let M be any model based on a strategic game frame G^*. By [6], every mixed-strategy game has a Nash equilibrium w, and so by Theorem 5, $M, w \models \bigwedge_{a \in A} R'_a$. So the formula is valid on all mixed-strategy strategic game frames. Yet the the frame in Figure 1 does not validate this formula since it lacks a Nash equilibrium, so by Theorem 3, it is not derivable in $\mathsf{K}G$.

4 Concluding Remarks

We have presented a logic that formalises the approach to rational decision-making adopted in [3]. Many salient features of games can be modelled using strategic game frames, which conveniently generalise over pure and mixed strategy games. Our logical investigations, however, are far from complete. In particular, we would like to investigate other fragments of CPDL that are sufficient for use in game theory. A particularly interesting open problem is the axiomatisation of the class of mixed-strategy games. Theorem 6 gives one example of a formula valid over these frames that is not valid in, for example, pure strategy game frames. But approaches to the computation of Nash equilibria (e.g.[4]) suggest many structural features that could be analysed logically.

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4 The restriction to finite D is essential, because U^* is calculated as a finite sum. This restriction prevent us from forming G^{**} because D^* is always infinite.

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