Basic Equations of Continuum Mechanics

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Abstract The modeling of the behavior of pressure-sensitive materials is embedded in the general continuum mechanics. The basic equations of continuum mechanics can be split into the material-independent and the material-dependent equations. The starting point is the introduction of the kinematics based on pure mathematical considerations. In addition, the velocities and the accelerations of the relevant kinematical variables are presented. The next section is devoted to the introduction of the action on the continuum and the inner reaction. Starting with such properties like forces and stresses finally the static equilibrium is stated. The last part of the material-independent equations is the introduction of the balances. Limiting our discussions by thermo-mechanical actions only, the balance of mass, momentum, moment of momentum, energy and entropy are deduced. The specific properties and features of the pressure-sensitive materials are presented in the next sections. Within this chapter the general ideas of material modeling (deductive approach) are given. Finally, some examples of special constitutive equations for incompressible and compressible materials are presented. These examples are mostly related to rubber-like materials.

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1 Kinematics and Deformations

1.1 Lagrangian and Eulerian Description

A body $\mathscr{B}$ is an assemblage of material points, which is bounded by boundary points that means the surface of $\mathscr{B}$. Material bodies are introduced in continuum mechanics with the help of the method of sections. By this method the body $\mathscr{B}$ can be separated from the surrounding. The introduction of the surface and the body is arbitrary, which is helpful for the formulation of the balance equations (see Sect. 3).

The movement of material bodies can be presented by the motion of their material points which should be identified. If the material points are related to points in the Euclidean space $\mathbb{E}^3$ and if one point 0 is fixed in this space, then the position of the material points is determined by the position vector $\boldsymbol{x}(t)$ at arbitrary time t. To distinguish the material points of the body $\mathscr{B}$ each of them gets a label: at the time $t = t_0$ the material point is characterized by the position vector $\boldsymbol{x}(t_0) \equiv \boldsymbol{X}$. t_0 is the natural initial state, which changes should be computed. In many cases it holds $t_0 = 0$.

In the Cartesian coordinate system with the origin 0 and the basis vectors $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ the movement of the material point $\boldsymbol{X}$ can be presented as follows

$$\begin{aligned} &\boldsymbol{x} = x_i \boldsymbol{e}_i, \qquad \boldsymbol{X} = X_i \boldsymbol{e}_i, \qquad \boldsymbol{x}(\boldsymbol{X}, t_0) = \boldsymbol{x}_0 \equiv \boldsymbol{X}, \\ &\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) - \text{trajectory of } \boldsymbol{X}, \\ &\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t) - \text{point } \boldsymbol{X}, \text{ which is in the moment } t \text{ located at } \boldsymbol{x} \end{aligned} \tag{1}$$

In Fig. 1 the trajectory of $\boldsymbol{X}$ is shown. If the Jacobi determinant is not equal to 0

$$\det\left(\frac{\partial x_i}{\partial X_j}\right) \equiv \left|\frac{\partial x_i}{\partial X_j}\right| \neq 0, \tag{2}$$

the following unique and invertible relation exists

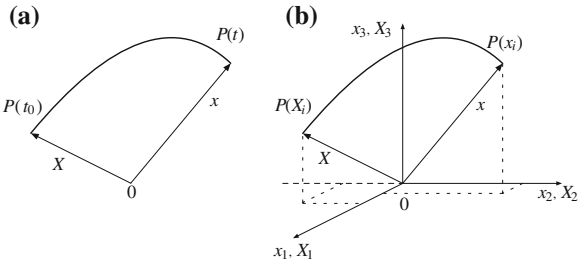

Fig. 1 Trajectory of a *material point*: **a** position vectors, **b** Cartesian coordinates

$$\boldsymbol{x}(\boldsymbol{X}, t) \Longleftrightarrow \boldsymbol{X}(\boldsymbol{x}, t) \tag{3}$$

Now we can introduce the configuration (sometimes called placement).

Definition 1.1 (Configuration) The configuration of a body is defined if we have for any time t an invertible unique mapping for the material points $\boldsymbol{X}$ with $\boldsymbol{x}$. For the initial $t = t_0$ one has the reference configuration, for the actual time t one has the actual configuration.

The placement of the body is defined by its configuration. The motion is presented as follows.

Definition 1.2 (Motion) The motion is given by the changes of the configurations $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$ (t is a parameter).

For further discussions let us introduce two approaches for the description of continuum mechanics problems.

Definition 1.3 (Lagrangian or material description) The changes of properties prescribed to the material point will be given with respect to $\boldsymbol{X}$. In this case the properties are functions of $\boldsymbol{X}$ and t.

Definition 1.4 (Eulerian or spatial description) The changes of properties prescribed to the material point properties will be given with respect to $\boldsymbol{x}$. In this case the properties are functions of $\boldsymbol{x}$ and t.

1.2 Time Derivatives and Nabla Operator

The properties prescribed to an arbitrary material point can be given in the material or spatial description. For the scalar function φ presenting a property one gets with respect to Eq. (3)

$$\begin{aligned} &\varphi = \varphi(\boldsymbol{X}, t) = \varphi(X_1, X_2, X_3, t) && \text{material description,} \\ &\varphi = \varphi[\boldsymbol{X}(\boldsymbol{x}, t), t] = \varphi(\boldsymbol{x}, t) = \varphi(x_1, x_2, x_3, t) && \text{spatial description.} \end{aligned}$$

Similar formulations can be introduced for tensorial functions of arbitrary order. In dependence of the assumed presentation of the function φ two time derivatives can be defined: the spatial one and the material one. The spatial derivative is given as

$$\frac{\partial \varphi(\boldsymbol{x}, t)}{\partial t} = \left.\frac{\partial \varphi(\boldsymbol{x}, t)}{\partial t}\right|_{\boldsymbol{x} \text{ fixed}}$$

The material derivative is

$$\frac{\partial \varphi(\boldsymbol{X}, t)}{\partial t} = \left.\frac{\partial \varphi(\boldsymbol{X}, t)}{\partial t}\right|_{\boldsymbol{X} \text{ fixed}}$$

Below the material derivative is denoted by

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

For the material velocity and acceleration the following definitions hold true.

Definition 1.5 (Material derivatives) The material derivative of the position vector $\boldsymbol{x}(\boldsymbol{X},t)$ results in the velocity vector $\boldsymbol{v}(\boldsymbol{X},t)$, the derivative of $\boldsymbol{v}(\boldsymbol{X},t)$ in the acceleration vector $\boldsymbol{b}(\boldsymbol{X},t)$

$$\boldsymbol{v}(\boldsymbol{X},t)=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(\boldsymbol{X},t)=\dot{\boldsymbol{x}}(\boldsymbol{X},t),\qquad \boldsymbol{b}(\boldsymbol{X},t)=\dot{\boldsymbol{v}}(\boldsymbol{X},t)=\ddot{\boldsymbol{x}}(\boldsymbol{X},t) \tag{4}$$

The spatial descriptions of $\boldsymbol{v}$ and $\boldsymbol{b}$ one gets, if $\boldsymbol{X}$ will be substituted by $\boldsymbol{x}$

$$\boldsymbol{v}=\boldsymbol{v}[\boldsymbol{X}(\boldsymbol{x},t),t]=\boldsymbol{v}(\boldsymbol{x},t),\qquad \boldsymbol{b}=\boldsymbol{b}[\boldsymbol{X}(\boldsymbol{x},t),t]=\boldsymbol{b}(\boldsymbol{x},t) \tag{5}$$

Using the nabla operator the material derivative can be introduced

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}=\left.\frac{\partial\varphi}{\partial t}\right|_{\boldsymbol{x}\text{ fixed}}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\varphi|_{\boldsymbol{x}\text{ fixed}}=\left.\frac{\partial\varphi}{\partial t}\right|_{\boldsymbol{x}\text{ fixed}}+\boldsymbol{v}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi \tag{6}$$

1.3 Strains and Deformation Gradient

Let us discuss the transform of line, surface and volume elements from the reference to the actual configuration. The introduction of the deformation gradient (gradient of the position vector) $\boldsymbol{F}$ can be helpful in this case.

Definition 1.6 (Deformation gradient) If the deformation of a body can be with the help of the equation of motion

$$\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{X},t)$$

transferred from the reference to the actual configuration, the following equation presents the material deformation gradient

$$\boldsymbol{F}=[\nabla_{\boldsymbol{X}}\boldsymbol{x}(\boldsymbol{X},t)]^{\mathrm{T}} \tag{7}$$

$\boldsymbol{F}$ transform a material line element $\mathrm{d}\boldsymbol{X}$ of the reference configuration into a material line element $\mathrm{d}\boldsymbol{x}$ of the actual configuration, i.e.

$$\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{X}=\mathrm{d}\boldsymbol{x}$$

With the help of the deformation gradient one can express the relations for the transform of a surface or volume element from the reference to the actual configuration. The surface element $\mathrm{d}A_0$ in the reference configuration has the size $\mathrm{d}X_1\mathrm{d}X_2$. Considering the orientation of the surface element one can write

$$\mathrm{d}\boldsymbol{A}_0 = \mathrm{d}\boldsymbol{X}_1 \times \mathrm{d}\boldsymbol{X}_2$$

The transform into the actual configuration of $\mathrm{d}\boldsymbol{A}_0$ results in the element $\mathrm{d}\boldsymbol{A}$

$$\mathrm{d}\boldsymbol{A} = \mathrm{d}\boldsymbol{x}_1 \times \mathrm{d}\boldsymbol{x}_2 = (\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x}_1) \times (\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{x}_2)$$

With respect to

$$(\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X}_1) \times (\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X}_2) = (\det \boldsymbol{F})(\boldsymbol{F}^{\mathrm{T}})^{-1} \cdot (\mathrm{d}\boldsymbol{X}_1 \times \mathrm{d}\boldsymbol{X}_2)$$

finally one obtains

$$\mathrm{d}\boldsymbol{A} = (\det \boldsymbol{F})(\boldsymbol{F}^{-1})^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_0 \tag{8}$$

The volume element in the reference configuration is defined as

$$\mathrm{d}V_0 = |(\mathrm{d}\boldsymbol{X}_1 \times \mathrm{d}\boldsymbol{X}_2) \cdot \mathrm{d}\boldsymbol{X}_3|,$$

and in the actual configuration as

$$\mathrm{d}V = |[(\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X}_1) \times (\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X}_2)] \cdot (\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X}_3)|$$

After some manipulations one obtains

$$\mathrm{d}V = |\det \boldsymbol{F}| \mathrm{d}V_0 \tag{9}$$

It holds true $\det \boldsymbol{F} \neq 0$ for all t if the continuity w.r.t. t can be assumed. In addition, for $t = t_0$ we obtain $\det \boldsymbol{F} = 1$.

1.4 Velocities, Velocity Gradients

The velocity $\boldsymbol{v}(\boldsymbol{X}, t)$ of a material point $\boldsymbol{X}$ is defined by

$$\boldsymbol{v}(\boldsymbol{X}, t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(\boldsymbol{X}, t) \equiv \dot{\boldsymbol{x}}(\boldsymbol{X}, t) = \frac{\partial}{\partial t}\boldsymbol{x}(\boldsymbol{X}, t)$$

Since

$$\boldsymbol{x}(\boldsymbol{X}, t) \Longleftrightarrow \boldsymbol{X}(\boldsymbol{x}, t)$$

in the spatial description of $\boldsymbol{v}$ can be presented as

$$\boldsymbol{v}[\boldsymbol{X}(\boldsymbol{x},t),t]=\boldsymbol{v}(\boldsymbol{x},t)$$

For the acceleration we get

$$\boldsymbol{b}=\dot{\boldsymbol{v}}=\frac{\partial^2}{\partial t^2}\boldsymbol{x}(\boldsymbol{X},t)=\ddot{\boldsymbol{x}}(\boldsymbol{X},t)$$

For the velocity in the Eulerian description we have

$$\boldsymbol{v}(\boldsymbol{x},t)=\boldsymbol{v}(\boldsymbol{x},t)=\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}$$

and for the acceleration

$$\boldsymbol{b}=\boldsymbol{b}(\boldsymbol{x},t)=\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$$

Let us introduce the gradient of the velocities $\boldsymbol{L}$.

Definition 1.7 (Velocity gradient tensor) The spatial velocity gradient tensor $\boldsymbol{L}$ of a given velocity field $\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{x},t)$ is defined by

$$\boldsymbol{L}(\boldsymbol{x},t)=[\nabla_{\boldsymbol{x}}\boldsymbol{v}(\boldsymbol{x},t)]^{\mathrm{T}} \tag{10}$$

With the help of $\boldsymbol{L}$ the time derivatives of the material line, surface and volume elements can be computed in the actual configuration

$$(\mathrm{d}\boldsymbol{x})^{\cdot}=\boldsymbol{L}\cdot\mathrm{d}\boldsymbol{x}, \tag{11}$$

$$(\mathrm{d}\boldsymbol{A})^{\cdot}=[(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v})\boldsymbol{I}-\boldsymbol{L}^{\mathrm{T}}]\cdot\mathrm{d}\boldsymbol{A}, \tag{12}$$

$$(\mathrm{d}V)^{\cdot}=(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v})\mathrm{d}V \tag{13}$$

$\boldsymbol{I}$ is the second-order unit tensor.

1.5 Strains and Strain Measures

The deformation gradient $\boldsymbol{F}$ is related to the whole motion that means it contains the rigid body motions. The strains can be obtained if we can find expression which are free from the rigid body motions. In this situation the theorem on the polar decomposition can be helpful.

Theorem 1.1 (Polar decomposition) *Any non-singular second-order tensor $\boldsymbol{T}$ with $\det\boldsymbol{T}\neq 0$ can be represented in a unique manner by a decomposition into the positive definite symmetric tensors $\boldsymbol{U}$ or $\boldsymbol{V}$ and orthogonal tensor $\boldsymbol{R}$*

$$\boldsymbol{T} = \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{R}$$

$\boldsymbol{R} \cdot \boldsymbol{U}$ is named the right $\boldsymbol{V} \cdot \boldsymbol{R}$ the left polar decomposition.

The deformation gradient tensor $\boldsymbol{F}$ is always non-singular ($\det \boldsymbol{F} \neq 0$). So we get

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{R} \tag{14}$$

The polar decomposition of $\boldsymbol{F}$ yields the following deformation tensors

$$\begin{array}{ll} \boldsymbol{U} = \left(\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F}\right)^{1/2} & \text{right stretch tensor,} \\ \boldsymbol{V} = \left(\boldsymbol{F} \cdot \boldsymbol{F}^{\mathrm{T}}\right)^{1/2} & \text{left stretch tensor,} \\ \boldsymbol{C} = \boldsymbol{U}^2 = \left(\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F}\right) & \text{right Cauchy-Green tensor,} \\ \boldsymbol{B} = \boldsymbol{V}^2 = \left(\boldsymbol{F} \cdot \boldsymbol{F}^{\mathrm{T}}\right) & \text{left Cauchy-Green tensor} \end{array} \tag{15}$$

For application purposes, for example the formulation of constitutive equations, it is better to introduce a strain measure resulting in the reference configuration for pure rigid body motions the value zero instead of one. One possibility is the Green-Lagrange strain tensor

$$\boldsymbol{G}(\boldsymbol{X}, t) = \frac{1}{2}[\boldsymbol{C}(\boldsymbol{X}, t) - \boldsymbol{I}] = \frac{1}{2}(\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{I}\right) \tag{16}$$

In many continuum mechanics problems one needs the volume strain as a characteristic property.

Definition 1.8 (Volume strain) If we divide the difference of the material volume elements $\mathrm{d}V$ and $\mathrm{d}V_0$ in the actual and the reference configurations by $\mathrm{d}V_0$ we get the volume strain ε_{V}

$$\varepsilon_{\mathrm{V}} = \frac{\mathrm{d}V - \mathrm{d}V_0}{\mathrm{d}V_0} \tag{17}$$

With $\mathrm{d}V = (\det \boldsymbol{F})\mathrm{d}V_0$ yields

$$\frac{\mathrm{d}V - \mathrm{d}V_0}{\mathrm{d}V_0} = \frac{(\det \boldsymbol{F} - 1)\mathrm{d}V_0}{\mathrm{d}V_0} = \det \boldsymbol{F} - 1$$

and

$$\varepsilon_{\mathrm{V}} = \det \boldsymbol{F} - 1 = \sqrt{\det\left(\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F}\right)} - 1 = \sqrt{\det \boldsymbol{C}} - 1 = \sqrt{\det\left(2\boldsymbol{G} + \boldsymbol{I}\right)} - 1 \tag{18}$$

Definition 1.9 (Volume conservation) Volume conservation (isochoric motion) is related to the constraint

$$\varepsilon_{\mathrm{V}} \equiv 0$$

With Eq. (18) it follows $\det \boldsymbol{F} = 1$.

The starting point for the analysis of the strain rates is the spatial rate tensor $\boldsymbol{L}(\boldsymbol{x},t)$. The following equation is valid

$$\boldsymbol{L}(\boldsymbol{x},t)=[\nabla_{\boldsymbol{x}}\boldsymbol{v}(\boldsymbol{x},t)]^{\mathrm{T}}=\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}$$

$\boldsymbol{L}$ is a second-order tensor, which can be split additively into symmetric and antisymmetric tensors

$$\begin{aligned}\boldsymbol{L}&=\frac{1}{2}\left(\boldsymbol{L}+\boldsymbol{L}^{\mathrm{T}}\right)+\frac{1}{2}\left(\boldsymbol{L}-\boldsymbol{L}^{\mathrm{T}}\right)\\&=\frac{1}{2}\left[(\nabla_{\boldsymbol{x}}\boldsymbol{v})^{\mathrm{T}}+\nabla_{\boldsymbol{x}}\boldsymbol{v}\right]+\frac{1}{2}\left[(\nabla_{\boldsymbol{x}}\boldsymbol{v})^{\mathrm{T}}-\nabla_{\boldsymbol{x}}\boldsymbol{v}\right]=\boldsymbol{D}+\boldsymbol{W}\end{aligned}$$

Definition 1.10 (Strain rate tensor) The symmetric part of $\boldsymbol{L}$

$$\boldsymbol{D}=\frac{1}{2}\left(\boldsymbol{L}+\boldsymbol{L}^{\mathrm{T}}\right)$$

is the strain rate tensor.

Definition 1.11 (Vorticity tensor) The antisymmetric part of $\boldsymbol{L}$

$$\boldsymbol{W}=\frac{1}{2}\left(\boldsymbol{L}-\boldsymbol{L}^{\mathrm{T}}\right)$$

is the vorticity tensor.

1.6 Displacements, Displacement Gradient, Linearizations

Let us express the kinematical variables by the displacement vector and the displacement gradient tensor

- displacement vector in the reference configuration

$$\boldsymbol{u}(\boldsymbol{X},t)=\boldsymbol{x}(\boldsymbol{X},t)-\boldsymbol{X},$$

- displacement vector in the actual configuration

$$\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{x}-\boldsymbol{X}(\boldsymbol{x},t),$$

- displacement gradient tensor

$$\begin{array}{ll}P_0(\boldsymbol{X})\Longrightarrow P(\boldsymbol{x}):&\boldsymbol{x}=\boldsymbol{X}+\boldsymbol{u}(\boldsymbol{X},t),\\Q_0(\boldsymbol{X}+\mathrm{d}\boldsymbol{X})\Longrightarrow Q(\boldsymbol{x}+\mathrm{d}\boldsymbol{x}):&\boldsymbol{x}+\mathrm{d}\boldsymbol{x}=\boldsymbol{X}+\mathrm{d}\boldsymbol{X}+\boldsymbol{u}(\boldsymbol{X}+\mathrm{d}\boldsymbol{X},t)\end{array}\qquad(19)$$

From Eq. (19) one gets

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X} + \mathrm{d}\boldsymbol{X}, t) - \boldsymbol{u}(\boldsymbol{X}, t)$$

and finally

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}\boldsymbol{X} + (\nabla_{\boldsymbol{X}}\boldsymbol{u})^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{X} = (\boldsymbol{I} + \boldsymbol{J}) \cdot \mathrm{d}\boldsymbol{X} \tag{20}$$

Definition 1.12 (Displacement gradient tensor) The material displacement gradient tensor is defined as

$$[\nabla_{\boldsymbol{X}}\boldsymbol{u}(\boldsymbol{X}, t)]^{\mathrm{T}} \equiv \boldsymbol{J}$$

The spatial displacement gradient tensor is defined as

$$[\nabla_{\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x}, t)]^{\mathrm{T}} \equiv \boldsymbol{K}$$

With

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X} &\Longrightarrow (\nabla_{\boldsymbol{X}}\boldsymbol{u})^{\mathrm{T}} = (\nabla_{\boldsymbol{X}}\boldsymbol{x})^{\mathrm{T}} - \boldsymbol{I}, \qquad \boldsymbol{J} = \boldsymbol{F} - \boldsymbol{I}, \\ \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{x} - \boldsymbol{X}(\boldsymbol{x}, t) &\Longrightarrow (\nabla_{\boldsymbol{x}}\boldsymbol{u})^{\mathrm{T}} = \boldsymbol{I} - (\nabla_{\boldsymbol{x}}\boldsymbol{X})^{\mathrm{T}}, \qquad \boldsymbol{K} = \boldsymbol{I} - \boldsymbol{F}^{-1}, \end{aligned}$$

all kinematical tensors can be expressed by $\boldsymbol{u}$ and $\boldsymbol{J}$ or $\boldsymbol{K}$, for example

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{I} + \boldsymbol{J}, \qquad \boldsymbol{F}^{-1} = \boldsymbol{I} - \boldsymbol{K}, \\ \boldsymbol{C} &= (\boldsymbol{I} + \boldsymbol{J})^{\mathrm{T}} \cdot (\boldsymbol{I} + \boldsymbol{J}) = \boldsymbol{I} + \boldsymbol{J} + \boldsymbol{J}^{\mathrm{T}} + \boldsymbol{J}^{\mathrm{T}} \cdot \boldsymbol{J}, \\ \boldsymbol{C}^{-1} &= (\boldsymbol{I} - \boldsymbol{K}) \cdot (\boldsymbol{I} - \boldsymbol{K})^{\mathrm{T}} = \boldsymbol{I} - \boldsymbol{K} - \boldsymbol{K}^{\mathrm{T}} + \boldsymbol{K} \cdot \boldsymbol{K}^{\mathrm{T}}, \\ \boldsymbol{B} &= (\boldsymbol{I} + \boldsymbol{J}) \cdot (\boldsymbol{I} + \boldsymbol{J})^{\mathrm{T}} = \boldsymbol{I} + \boldsymbol{J} + \boldsymbol{J}^{\mathrm{T}} + \boldsymbol{J} \cdot \boldsymbol{J}^{\mathrm{T}}, \\ \boldsymbol{B}^{-1} &= (\boldsymbol{I} - \boldsymbol{K})^{\mathrm{T}} \cdot (\boldsymbol{I} - \boldsymbol{K}) = \boldsymbol{I} - \boldsymbol{K} - \boldsymbol{K}^{\mathrm{T}} + \boldsymbol{K}^{\mathrm{T}} \cdot \boldsymbol{K}, \\ \boldsymbol{G} &= \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{J} + \boldsymbol{J}^{\mathrm{T}} + \boldsymbol{J} \cdot \boldsymbol{J}^{\mathrm{T}}), \\ \boldsymbol{A} &= \frac{1}{2}(\boldsymbol{I} - \boldsymbol{B}^{-1}) = \frac{1}{2}(\boldsymbol{K} + \boldsymbol{K}^{\mathrm{T}} - \boldsymbol{K}^{\mathrm{T}} \cdot \boldsymbol{K}) \end{aligned}$$

It is easy to show that $\boldsymbol{G}$ and $\boldsymbol{A}$ are nonlinear

$$\boldsymbol{G} = \frac{1}{2}\left[(\nabla_{\boldsymbol{X}}\boldsymbol{u})^{\mathrm{T}} + (\nabla_{\boldsymbol{X}}\boldsymbol{u}) + (\nabla_{\boldsymbol{X}}\boldsymbol{u}) \cdot (\nabla_{\boldsymbol{X}}\boldsymbol{u})^{\mathrm{T}}\right] = G_{ij}\boldsymbol{e}_i\boldsymbol{e}_j,$$

$$\boldsymbol{A} = \frac{1}{2}\left[(\nabla_{\boldsymbol{x}}\boldsymbol{u})^{\mathrm{T}} + (\nabla_{\boldsymbol{x}}\boldsymbol{u}) - (\nabla_{\boldsymbol{x}}\boldsymbol{u}) \cdot (\nabla_{\boldsymbol{x}}\boldsymbol{u})^{\mathrm{T}}\right] = A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$$

Obviously, $\boldsymbol{G}$ and $\boldsymbol{A}$ contain quadratic terms w.r.t. $\boldsymbol{u}$. This is the so-called geometrical nonlinearity. It is easy to deduce the consistent geometrical linear relations (see, for example, [1]).

2 Stress State

2.1 Classification of External Actions

The actions on a body can by classified as volume or surface actions. The following actions are known: pure mechanical, thermal, electromagnetic, etc. Here we focus our attention on mechanical actions, which can split into forces and moments as known from general mechanics (part statics). Then we can introduce:

- mass or volume forces and moments and
- surface forces and moments.

In general, the actions are continuously defined in the volume or on the surface functions. They are introduced as models since they cannot be observed directly (only the response of the actions can be measured). It is easy to show that line and concentrated single actions are limit cases of the volume and surface actions. These limit cases are the result of the different order of the three spatial dimensions or of the two dimensions of the surface.

Any material body is characterized by a continuous mass density distribution $\rho(\boldsymbol{x})$. The mass or volume actions are also continuous functions applied to any material point of the body. Examples of volume forces are the gravitational force, the force of inertia and the Coriolis force among others. The sources of these forces are out of the body, that means they are external volume forces. By analogy one can introduce sources for volume moments.

Volume forces are related to volume or mass. By $\boldsymbol{k}^{\mathrm{V}}$ the volume force density and by $\boldsymbol{k}^{\mathrm{m}} \equiv \boldsymbol{k}$ the mass force density are denoted (in what is following $\boldsymbol{k}$ is used instead of $\boldsymbol{k}^{\mathrm{m}}$). It holds

$$\rho(\boldsymbol{x}, t)\boldsymbol{k}(\boldsymbol{x}, t) = \boldsymbol{k}^{\mathrm{V}} \tag{21}$$

with the mass force density $\boldsymbol{k}(\boldsymbol{x}, t)$, the volume force density $\boldsymbol{k}^{\mathrm{V}}(\boldsymbol{x}, t)$ and the mass density $\rho(\boldsymbol{x}, t)$. Examples of the volume force density are the weight, the centrifugal force or in general the potential forces:

- the weight
$$\rho\boldsymbol{k} = -\rho g\boldsymbol{e}_3,$$
where g is the gravity acceleration, $\boldsymbol{e}_3$ is the basis vector in the opposite direction to the gravity acceleration.
- the centrifugal force
$$\rho\boldsymbol{k} = -\rho\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}),$$
where $\boldsymbol{\omega}$ is the angular velocity
- general potential force
$$\rho\boldsymbol{k} = -\rho\nabla_{\boldsymbol{x}}\Pi$$

The force potential Π in the case of weight or centrifugal force can be expressed as

$$\Pi = \boldsymbol{e}_3 \cdot \boldsymbol{x} g \qquad \text{or} \qquad \Pi = -\frac{1}{2}|\boldsymbol{\omega} \times \boldsymbol{x}|^2$$

In the case of volume moments the analogous equation is valid

$$\rho(\boldsymbol{x}, t)\boldsymbol{l}^{\mathrm{m}}(\boldsymbol{x}, t) = \rho(\boldsymbol{x}, t)\boldsymbol{l}(\boldsymbol{x}, t) = \boldsymbol{l}^{\mathrm{V}} \tag{22}$$

with $\boldsymbol{l}^{\mathrm{m}}(\boldsymbol{x}, t)$ as the mass moment density and $\boldsymbol{l}^{\mathrm{V}}(\boldsymbol{x}, t)$ as the volume moment density.

External surface loads are acting on surfaces. Such loads are named contact loads. The surface can be the surface of a material body with a volume $A(V)$, but also common interfaces between the parts of the body or between two different bodies. External surface loads are existing also between solids and fluids, for example, the hydrostatic pressure of the fluid on a solid surrounded by the fluid. The surface loads can be split again in surface forces and surface moments. The surface forces are related to the surface and result in the stress vector $\boldsymbol{t}$, the surface moments by analogy result in moment stress vectors $\boldsymbol{\mu}$. The following limits can be introduced [2, 3]

$$\boldsymbol{t} = \lim_{\triangle A \to 0} \frac{\triangle \boldsymbol{f}}{\triangle A}, \qquad \boldsymbol{\mu} = \lim_{\triangle A \to 0} \frac{\triangle \boldsymbol{m}}{\triangle A}, \tag{23}$$

where $\triangle \boldsymbol{f}$ and $\triangle \boldsymbol{m}$ are the resulting force and moment vectors acting on the surface $\triangle A$. Note that $\triangle A$ is oriented that means $\triangle \boldsymbol{A} = \boldsymbol{n} \triangle A$. The vectors depend on the position on the surface and the orientation of the surface ($\mathrm{d}\boldsymbol{A} = \boldsymbol{n}\mathrm{d}A$)

$$\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}, t), \qquad \boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{x}, \boldsymbol{n}, t) \tag{24}$$

One gets by integration of the external volume and surface forces the resulting external force $\boldsymbol{f}^{\mathrm{R}}$ acting on the body

$$\boldsymbol{f}^{\mathrm{R}} = \int_V \rho \boldsymbol{k} \, \mathrm{d}V + \int_A \boldsymbol{t} \, \mathrm{d}A \tag{25}$$

The resulting external moment can be introduced in the same manner

$$\boldsymbol{m}_0^{\mathrm{R}} = \int_V \rho(\boldsymbol{l} + \boldsymbol{x} \times \boldsymbol{k}) \, \mathrm{d}V + \int_A (\boldsymbol{\mu} + \boldsymbol{x} \times \boldsymbol{t}) \, \mathrm{d}A \tag{26}$$

In the classical mechanics the moment vectors are ignored and we get the following expression

$$\boldsymbol{m}_0^{\mathrm{R}} = \int_V \rho(\boldsymbol{x} \times \boldsymbol{k}) \, \mathrm{d}V + \int_A (\boldsymbol{x} \times \boldsymbol{t}) \, \mathrm{d}A \tag{27}$$

2.2 Cauchy's Stress Vector and Tensor

As the result of the external action on the body one obtains stresses in the body. Let us introduce the Euler-Cauchy stress principle.

Definition 1.13 (Euler-Cauchy stress principle) The external forces result in a vector field of stress vectors $\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}, t)$ acting on a surface A with the normal $\boldsymbol{n}(\boldsymbol{x}, t)$. In the case that the surface is the body surface the stress vectors are resulting from the surface forces and named traction.

The stresses in the body can be defined using the method of sections (Fig. 2). From the Statics follows that we have some actions in the body (Fig. 3). $\mathrm{d}\boldsymbol{f}$ is the resulting force vector and $\mathrm{d}\boldsymbol{m}$ is the resulting moment vector on the surface element $\mathrm{d}A$, $\boldsymbol{n}$ is the unit normal vector on the surface. With respect to Eq. (23) one gets the stress and the couple stress vector

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}, t) = \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}A}, \qquad \boldsymbol{\mu}(\boldsymbol{x}, \boldsymbol{n}, t) = \frac{\mathrm{d}\boldsymbol{m}}{\mathrm{d}A}$$

Ignoring the inner moments, we get the classical continuum (Fig. 4). In this case we have no surface moments and $\boldsymbol{\mu} \equiv \boldsymbol{0}$ is valid. For polar continua that is when $\boldsymbol{\mu} \neq \boldsymbol{0}$ we refer to [4–8].

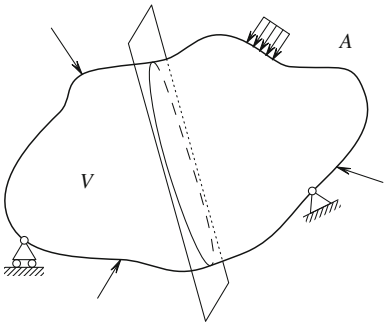

Fig. 2 Method of sections (arbitrary section in body under external loadings)

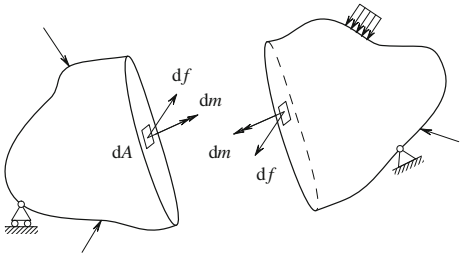

Fig. 3 Actions in the body (polar continuum)

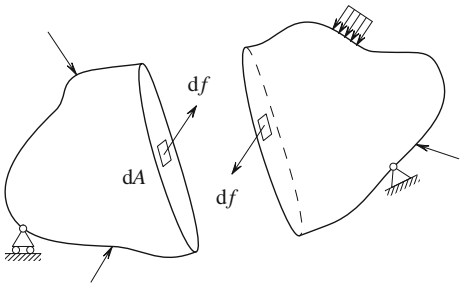

Fig. 4 Actions in the body (classical continuum)

Note 1.1 As a measure of the inner force in point P of the body we introduce the stress vector

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}, t) = \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}A}$$

In general, $\boldsymbol{t}$ depends on the position, time and orientation of the surface. In each point of the body $\boldsymbol{t}(\boldsymbol{n}) = -\boldsymbol{t}(-\boldsymbol{n})$ holds (Cauchy's lemma).

Definition 1.14 (Stress state) All possible stress vectors in a material point P define the stress state in this point.

From material's testing we know two definitions for the stresses.

Definition 1.15 (Engineering stresses) The acting force is related to the surface in the reference configuration.

Definition 1.16 (True stresses) The acting force is related to the surface in the actual configuration.

In continuum mechanics we have more possibilities since we can define the force vector in both configurations, the surface orientation in two configurations and in addition we can introduce intermediate configurations.

Definition 1.17 (Cauchy's stress vector) The Cauchy stress vector is called true stress vector. The actual force is related to the actual section.

The stress tensor follows as

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}, t) = \boldsymbol{n} \cdot \boldsymbol{T}(\boldsymbol{x}, t)$$

with the Cauchy stress vector $\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}, t)$, the normal $\boldsymbol{n}$ and the Cauchy stress tensor $\boldsymbol{T}(\boldsymbol{x}, t)$.

2.3 Equilibrium Equations, Equations of Motion

For the body under surface forces $\boldsymbol{t}\mathrm{d}A$ and volume forces $\rho\boldsymbol{k}\mathrm{d}V$, which is in an equilibrium state, the following equations considering Eqs. (25) and (27) hold

$$\int\limits_V \rho\boldsymbol{k}\,\mathrm{d}V + \int\limits_A \boldsymbol{t}\,\mathrm{d}A = \boldsymbol{0}, \qquad \int\limits_V (\boldsymbol{x}\times\rho\boldsymbol{k})\,\mathrm{d}V + \int\limits_A (\boldsymbol{x}\times\boldsymbol{t})\,\mathrm{d}A = \boldsymbol{0} \tag{28}$$

With

$$\boldsymbol{t} = \boldsymbol{n}\cdot\boldsymbol{T}$$

and applying the divergence theorem it follows

$$\int\limits_A \boldsymbol{t}\,\mathrm{d}A = \int\limits_A \boldsymbol{n}\cdot\boldsymbol{T}\,\mathrm{d}A = \int\limits_V \nabla_{\boldsymbol{x}}\cdot\boldsymbol{T}\,\mathrm{d}V \tag{29}$$

and

$$\int\limits_V (\rho\boldsymbol{k} + \nabla_{\boldsymbol{x}}\cdot\boldsymbol{T})\,\mathrm{d}V = \boldsymbol{0} \tag{30}$$

This is the integral equilibrium. If the volume is arbitrary and all fields are smooth the local equilibrium can be expressed as

$$\nabla_{\boldsymbol{x}}\cdot\boldsymbol{T} + \rho\boldsymbol{k} = \boldsymbol{0} \tag{31}$$

Adding the inertial force $-\ddot{\boldsymbol{x}}\mathrm{d}M = -\ddot{\boldsymbol{x}}\rho\mathrm{d}V$ in the sense of the Newton/d'Alembert principle, one obtains

$$\int\limits_V \rho\boldsymbol{k}\,\mathrm{d}V + \int\limits_A \boldsymbol{T}\,\mathrm{d}A - \int\limits_V \ddot{\boldsymbol{x}}\rho\,\mathrm{d}V = \boldsymbol{0}$$

or

$$\int\limits_V (\rho\boldsymbol{k} + \nabla_{\boldsymbol{x}}\cdot\boldsymbol{T} - \rho\ddot{\boldsymbol{x}})\,\mathrm{d}V = \boldsymbol{0}, \tag{32}$$

The local equation of motion holds

$$\rho\ddot{\boldsymbol{x}} = \nabla_{\boldsymbol{x}}\cdot\boldsymbol{T} + \rho\boldsymbol{k} \tag{33}$$

The second equation of (28) results in the symmetry of the stress tensor ($\boldsymbol{T} = \boldsymbol{T}^{\mathrm{T}}$).

The stress tensor $\boldsymbol{T}$ can be split in a spherical and a deviatoric part

$$\boldsymbol{T} = \frac{1}{3}\boldsymbol{T} \cdot\cdot \boldsymbol{II} + \left(\boldsymbol{T} - \frac{1}{3}\boldsymbol{T} \cdot\cdot \boldsymbol{II}\right) = \frac{1}{3}(\mathrm{tr}\boldsymbol{T})\boldsymbol{I} + \left[\boldsymbol{T} - \frac{1}{3}(\mathrm{tr}\boldsymbol{T})\boldsymbol{I}\right] = \boldsymbol{T}^{\mathrm{K}} + \boldsymbol{T}^{\mathrm{D}}$$

2.4 Stress Vectors and Tensors After Piola-Kirchhoff

Up to now the stress vector and the stress tensor are presented in the Eulerian coordinates. The Cauchy tensor of true stresses is defined by the actual force vector and the actual oriented surface element. In many applications it is better to use the Lagrangian description and as the minimum to relate the volume and surface elements to the reference configuration. The transforms (8) and (9) are valid.

Definition 1.18 (First Piola-Kirchhoff stress tensor) If we relate the actual force vector $\mathrm{d}\boldsymbol{f}$ to the oriented surface element $\mathrm{d}\boldsymbol{A}_0 = \boldsymbol{n}_0 \mathrm{d}A_0$ in the reference configuration we get the following stress vector

$$^{I}\boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}A_0}$$

The respective tensor $^{I}\boldsymbol{P}$, which describes the stress state in a material point of the reference configuration is named first Piola-Kirchhoff or Lagrangian stress tensor.

With

$$\boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}A}, \qquad ^{I}\boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}A_0}$$

one obtains

$$\boldsymbol{t}\mathrm{d}A = ^{I}\boldsymbol{t}\mathrm{d}A_0 = \mathrm{d}\boldsymbol{f}, \qquad \boldsymbol{t} = ^{I}\boldsymbol{t} \cdot (\det\boldsymbol{F})^{-1}\boldsymbol{F}^{\mathrm{T}}, \qquad ^{I}\boldsymbol{t} = \boldsymbol{t} \cdot \det\boldsymbol{F}\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}} \tag{34}$$

$^{I}\boldsymbol{P}$ is in general a non-symmetric tensor. With

$$\mathrm{d}\boldsymbol{A}_0 = (\det\boldsymbol{F})^{-1}\boldsymbol{F}^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}$$

one gets the relations between the Cauchy and the first Piola-Kirchhoff stress tensors

$$\boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{T}, \qquad ^{I}\boldsymbol{t} = \boldsymbol{n}_0 \cdot ^{I}\boldsymbol{P}, \tag{35}$$

$$\boldsymbol{T} = (\det\boldsymbol{F})^{-1}\boldsymbol{F} \cdot ^{I}\boldsymbol{P}, \qquad ^{I}\boldsymbol{P} = (\det\boldsymbol{F})\boldsymbol{F}^{-1} \cdot \boldsymbol{T} \tag{36}$$

Assuming the force and the moment equilibrium in the reference configuration, we get the equations of motion in Lagrangian coordinates

$$\int\limits_{A_0} \boldsymbol{n}_0 \cdot {}^{I}\boldsymbol{P}\, \mathrm{d}A_0 + \int\limits_{V_0} \rho_0 \boldsymbol{k}\, \mathrm{d}V_0 - \int\limits_{V_0} \ddot{\boldsymbol{x}} \rho_0\, \mathrm{d}V_0 = \boldsymbol{0}, \tag{37}$$

$$\int\limits_{A_0} \left[\boldsymbol{x} \times (\boldsymbol{n}_0 \cdot {}^{I}\boldsymbol{P})\right] \mathrm{d}A_0 + \int\limits_{V_0} (\boldsymbol{x} \times \rho_0 \boldsymbol{k})\, \mathrm{d}V_0 = \boldsymbol{0}, \tag{38}$$

$$\ddot{\boldsymbol{x}} \rho_0 = \nabla_X \cdot {}^{I}\boldsymbol{P} + \rho \boldsymbol{k}, \qquad {}^{I}\boldsymbol{P} \cdot \boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F} \cdot {}^{I}\boldsymbol{P}^{\mathrm{T}} \tag{39}$$

From the last equation results that ${}^{I}\boldsymbol{P}$ is a non-symmetrical tensor.

The non-symmetric stress tensor ${}^{I}\boldsymbol{P}$ is not convenient for the formulation of the constitutive equations. ${}^{I}\boldsymbol{P}$ should be modified in such a manner that we get a symmetric tensor again for the reference configuration. Let us introduce a "fictitious force vector"

$$\mathrm{d}\boldsymbol{f}_0 = \boldsymbol{F}^{-1} \cdot \mathrm{d}\boldsymbol{f} \tag{40}$$

Since

$$d\boldsymbol{f}_0 = {}^{II}\boldsymbol{P}^{\mathrm{T}} \cdot d\boldsymbol{A}_0 \tag{41}$$

the stress tensor ${}^{II}\boldsymbol{P}$ can be introduced.

Definition 1.19 (Second Piola-Kirchhoff stress tensor) If we relate the force vector $\mathrm{d}\boldsymbol{f}_0 = \boldsymbol{F}^{-1} \cdot \mathrm{d}\boldsymbol{f}$ to the oriented surface element $\mathrm{d}\boldsymbol{A}_0$ of the reference configuration, one gets the pseudo stress vector ${}^{II}\boldsymbol{t}$ with the respective pseudo stress tensor

$${}^{II}\boldsymbol{P} = {}^{I}\boldsymbol{P} \cdot \left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}$$

This is the second Piola-Kirchhoff stress tensor. $\mathrm{d}\boldsymbol{f} = \boldsymbol{T} \cdot \mathrm{d}\boldsymbol{A}$ is the analogous relation in the actual configuration to $\mathrm{d}\boldsymbol{f}_0 = {}^{II}\boldsymbol{P} \cdot \mathrm{d}\boldsymbol{A}_0$ in the reference configuration. ${}^{II}\boldsymbol{P}$ is a symmetric tensor since

$${}^{II}\boldsymbol{P} = (\det \boldsymbol{F}) \boldsymbol{F}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{F}^{-\mathrm{T}}$$

3 Balance Equations

3.1 General Formulation of Balance Equations

Here we present a brief description of the balance equations. They are material independent. We focus our attention to smooth fields. For further discussions and extensions we recommend the special literature, for example, [1, 9–12].

The problems of two- and one-dimensional or generalized continua are presented in [13–19] among others.

The task of continuum mechanics is the estimation the density $\rho = \rho(\boldsymbol{X}, t)$, motion $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$ and, if thermodynamics is taken into account, the temperature $\vartheta = \vartheta(\boldsymbol{X}, t)$ for all material points $\boldsymbol{X}$ as a function of the time t. It can be shown that the material independent statements can be given by balance equations.

Definition 1.20 (Balance equation) Balance equations are empirical judgements basic in the continuum mechanics, which express the relationships between the variables describing the state of the continuum and the external loadings on the body.

The general structure of a balance equation can be given as follows. $\Psi(\boldsymbol{x}, t)$ and $\Psi_0(\boldsymbol{X}, t)$ are distributions of a scalar mechanical variable w.r.t. the volume elements $\mathrm{d}V$ and $\mathrm{d}V_0$ in the actual and the reference configuration. The integration over the volume results in an additive (extensive) variable $Y(t)$

$$Y(t) = \int_V \Psi(\boldsymbol{x}, t)\, \mathrm{d}V = \int_{V_0} \Psi_0(\boldsymbol{X}, t)\, \mathrm{d}V_0 \tag{42}$$

With $\mathrm{d}V = (\det \boldsymbol{F})\mathrm{d}V_0$ holds $\Psi_0(\boldsymbol{X}, t) = (\det \boldsymbol{F})\Psi(\boldsymbol{x}, t)$. The material time derivative of $Y(t)$ is the rate of changes of the state of the system expressed by $\Psi(\boldsymbol{x}, t)$. This rate is balanced with the action of the surrounding on the body. In the actual configuration we have

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \Psi(\boldsymbol{x}, t)\, \mathrm{d}V = \int_A \Phi(\boldsymbol{x}, t)\, \mathrm{d}A + \int_V \Xi(\boldsymbol{x}, t)\, \mathrm{d}V \tag{43}$$

and in the reference configuration

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{V_0} \Psi_0(\boldsymbol{X}, t)\, \mathrm{d}V_0 = \int_{A_0} \Phi_0(\boldsymbol{X}, t)\, \mathrm{d}A_0 + \int_{V_0} \Xi_0(\boldsymbol{X}, t)\, \mathrm{d}V_0 \tag{44}$$

Φ and Φ_0 are the external fluxes through the surface in both configurations, $\boldsymbol{\Xi}$ and $\boldsymbol{\Xi}_0$ are the source (production) terms in the volume. Such balance equation for scalar fields can be extended to vectorial and tensorial fields.

Remark 1.1 Φ in the actual configuration is a function of $\boldsymbol{x}$ and t, but also of the orientation of the surface element $d\boldsymbol{A} = \boldsymbol{n}(\boldsymbol{x}, t)\mathrm{d}A$ (with other words dependence of the normal $\boldsymbol{n}$). This statement is valid for arbitrary tensor fields ${}^{(n)}\boldsymbol{\Phi} = {}^{(n)}\boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{n}, t)$ of order $n \geq 0$. For the reference configuration we get ${}^{(n)}\boldsymbol{\Phi}_0 = {}^{(n)}\boldsymbol{\Phi}_0(\boldsymbol{X}, \boldsymbol{n}_0, t)$ and $d\boldsymbol{A}_0 = \boldsymbol{n}_0(\boldsymbol{X}, t)\mathrm{d}A_0$.

Remark 1.2 For the $\boldsymbol{n}-$or $\boldsymbol{n}_0$ dependency of the surface functions $\boldsymbol{\Phi}$ or $\boldsymbol{\Phi}_0$ the Cauchy's lemma is valid

$$ {}^{(n)}\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{n},t)=\boldsymbol{n}\cdot{}^{(n+1)}\tilde{\boldsymbol{\Phi}}(\boldsymbol{x},t),\quad {}^{(n)}\boldsymbol{\Phi}_0(\boldsymbol{X},\boldsymbol{n}_0,t)=\boldsymbol{n}_0\cdot{}^{(n+1)}\tilde{\boldsymbol{\Phi}}_0(\boldsymbol{X},t), \tag{45} $$

Remark 1.3 For the fluxes the third Newton's law (actio = reactio) is valid. Two fluxes acting on the surface in a common material point and characterized by the normals $\boldsymbol{n}$ and $-\boldsymbol{n}$ or $\boldsymbol{n}_0$ and $-\boldsymbol{n}_0$ have the same value, but opposite sign

$$ \boldsymbol{\Phi}(\boldsymbol{n})=-\boldsymbol{\Phi}(-\boldsymbol{n}),\qquad \boldsymbol{\Phi}_0(\boldsymbol{n}_0)=-\boldsymbol{\Phi}_0(-\boldsymbol{n}_0) \tag{46} $$

The balance equations stating the equilibrium between the changes of the state of the body and the fluxes on the surface and the production in the volume have in the actual configuration the following structure

$$ \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_V {}^{(n)}\boldsymbol{\Psi}(\boldsymbol{x},t)\,\mathrm{d}V=\int\limits_A \boldsymbol{n}(\boldsymbol{x},t)\cdot{}^{(n+1)}\boldsymbol{\Phi}(\boldsymbol{x},t)\,\mathrm{d}A+\int\limits_V {}^{(n)}\boldsymbol{\Xi}(\boldsymbol{x},t)\,\mathrm{d}V \tag{47} $$

In the reference configuration one has

$$ \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_0} {}^{(n)}\boldsymbol{\Psi}_0(\boldsymbol{X},t)\,\mathrm{d}V_0 &\equiv \frac{\partial}{\partial t}\int\limits_{V_0} {}^{(n)}\boldsymbol{\Psi}_0(\boldsymbol{X},t)\,\mathrm{d}V_0 \\ &=\int\limits_{A_0}\boldsymbol{n}_0(\boldsymbol{X},t)\cdot{}^{(n+1)}\boldsymbol{\Phi}_0(\boldsymbol{X},t)\,\mathrm{d}A_0+\int\limits_{V_0} {}^{(n)}\boldsymbol{\Xi}_0(\boldsymbol{X},t)\,\mathrm{d}V_0 \end{aligned} \tag{48} $$

${}^{(n)}\boldsymbol{\Psi}$ and ${}^{(n)}\boldsymbol{\Psi}_0$ or ${}^{(n)}\boldsymbol{\Xi}$ and ${}^{(n)}\boldsymbol{\Xi}_0$ are tensorial fields of the order n ($n\geq 0$), ${}^{(n+1)}\boldsymbol{\Phi}$ and ${}^{(n+1)}\boldsymbol{\Phi}_0$ are tensorial fields of order $(n+1)$.

Using the following transforms

$$ \begin{aligned} \boldsymbol{n}&=\det\boldsymbol{F}\frac{\mathrm{d}A_0}{\mathrm{d}A}\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\cdot\boldsymbol{n}_0\Longleftrightarrow\boldsymbol{n}_0=(\det\boldsymbol{F})^{-1}\frac{\mathrm{d}A}{\mathrm{d}A_0}\boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{n},\\ \mathrm{d}\boldsymbol{A}&=\det\boldsymbol{F}\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_0\Longleftrightarrow\mathrm{d}\boldsymbol{A}_0=(\det\boldsymbol{F})^{-1}\boldsymbol{F}^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A},\\ \mathrm{d}V&=\det\boldsymbol{F}\mathrm{d}V_0\Longleftrightarrow\mathrm{d}V_0=(\det\boldsymbol{F})^{-1}\mathrm{d}V \end{aligned} \tag{49} $$

we get for example from

$$ \boldsymbol{\Phi}_0\cdot\mathrm{d}\boldsymbol{A}_0=\boldsymbol{\Phi}\cdot\mathrm{d}\boldsymbol{A}=\boldsymbol{\Phi}\cdot\det\boldsymbol{F}\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_0 \tag{50} $$

the relationship between $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Phi}$

$$ \boldsymbol{\Phi}_0=(\det\boldsymbol{F})\boldsymbol{\Phi}\cdot\left(\boldsymbol{F}^{-1}\right)^{\mathrm{T}},\qquad \boldsymbol{\Phi}_0=\boldsymbol{\Phi}_0(\boldsymbol{X},\boldsymbol{n}_0,t),\qquad \boldsymbol{\Phi}=\boldsymbol{\Phi}(\boldsymbol{x},\boldsymbol{n},t) \tag{51} $$

and from $\boldsymbol{\Xi}_0\mathrm{d}V_0=\boldsymbol{\Xi}\mathrm{d}V=\boldsymbol{\Xi}(\det\boldsymbol{F})\mathrm{d}V_0$ the relationship between $\boldsymbol{\Xi}_0$ and $\boldsymbol{\Xi}$

$$\boldsymbol{\Xi}_0 = (\det \boldsymbol{F})\,\boldsymbol{\Xi}, \qquad \boldsymbol{\Xi}_0 = \boldsymbol{\Xi}_0(\boldsymbol{X}, t), \qquad \boldsymbol{\Xi} = \boldsymbol{\Xi}(\boldsymbol{x}, t) \tag{52}$$

Sometimes it is more useful to apply the mass integrals. Using the same variables for the distribution functions $\boldsymbol{\Psi}$ and $\boldsymbol{\Xi}$i the global mechanical balance equation in the actual configuration can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_m \boldsymbol{\Psi}(\boldsymbol{x}, t)\,\mathrm{d}m \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_V \boldsymbol{\Psi}(\boldsymbol{x}, t)\rho\,\mathrm{d}V = \int\limits_A \boldsymbol{n}\cdot\boldsymbol{\Phi}(\boldsymbol{x}, t)\,\mathrm{d}A + \int\limits_V \boldsymbol{\Xi}(\boldsymbol{x}, t)\rho\,\mathrm{d}V \tag{53}$$

In Eq. (53) $\boldsymbol{\Psi}(\boldsymbol{x}, t)$, $\boldsymbol{\Xi}(\boldsymbol{x}, t)$ are tensorial fields of the same order n ($n \geq 0$), $\boldsymbol{\Phi}(\boldsymbol{x}, t)$ is a tensorial field of the order $(n + 1)$, $\boldsymbol{n}(\boldsymbol{x}, t)$ is the outer normal on A, $m(\boldsymbol{x}, t)$ is the mass as a continues function of the volume.

If the continuity in the sense of the divergence theorem is fulfilled for $\boldsymbol{\Psi}$ after application of the theorem to Eq. (53) one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_V \boldsymbol{\Psi}(\boldsymbol{x}, t)\rho\,\mathrm{d}V = \int\limits_V \nabla_{\boldsymbol{x}}\cdot\boldsymbol{\Phi}(\boldsymbol{x}, t)\,\mathrm{d}V + \int\limits_V \boldsymbol{\Xi}(\boldsymbol{x}, t)\rho\,\mathrm{d}V \tag{54}$$

and with $\mathrm{d}V \to 0$ the local formulation of the general balance equation

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\Psi}(\boldsymbol{x}, t)] = \nabla_{\boldsymbol{x}}\cdot\boldsymbol{\Phi}(\boldsymbol{x}, t) + \boldsymbol{\Xi}(\boldsymbol{x}, t)\rho \tag{55}$$

If we transform the equations into the reference configuration one gets

$$\begin{aligned}\frac{\partial}{\partial t}\int\limits_{V_0} \boldsymbol{\Psi}_0(\boldsymbol{X}, t)\rho_0\,\mathrm{d}V_0 &= \int\limits_{A_0} \boldsymbol{n}_0\cdot\boldsymbol{\Phi}_0(\boldsymbol{X}, t)\,\mathrm{d}A_0 + \int\limits_{V_0} \boldsymbol{\Xi}_0(\boldsymbol{X}, t)\rho_0\,\mathrm{d}V_0 \\ &= \int\limits_{V_0} [\nabla_{\boldsymbol{X}}\cdot\boldsymbol{\Phi}_0(\boldsymbol{X}, t) + \boldsymbol{\Xi}_0(\boldsymbol{X}, t)\rho_0]\,\mathrm{d}V_0\end{aligned}$$

or locally

$$\rho_0\frac{\partial}{\partial t}[\boldsymbol{\Psi}_0(\boldsymbol{X}, t)] = \nabla_{\boldsymbol{X}}\cdot\boldsymbol{\Phi}_0(\boldsymbol{X}, t) + \boldsymbol{\Xi}(\boldsymbol{X}, t)\rho_0 \tag{56}$$

3.2 Mass Balance and Mass Conservation

The mass is one of the main characteristics of a material body. The mass of the body can be estimated as the volume integral over the density field

$$m = \int\limits_V \rho(\boldsymbol{x}, t)\, \mathrm{d}V = \int\limits_{V_0} \rho_0(\boldsymbol{X})\, \mathrm{d}V_0 \tag{57}$$

This equation contains the global law of mass conservation.

Definition 1.21 (Mass conservation) If we have no mass exchange through the surface and no mass production in the volume the mass of the body is always constant that means independent of the time.

$\rho \mathrm{d}V$ and $\rho_0 \mathrm{d}V_0$ are the mass of a material point before and after the deformation. If they are the same one gets $\rho \det \boldsymbol{F} = \rho_0$ and finally

$$\frac{\rho_0}{\rho} = \det \boldsymbol{F},$$

The mass conservation is valid locally.

Theorem 1.2 (Law of mass conservation) *The mass* $\mathrm{d}m = \rho(\boldsymbol{x}, t)\mathrm{d}V$ *of a material volume* $\mathrm{d}V$ *is always constant*

$$\mathrm{d}m = \rho(\boldsymbol{x}, t)\mathrm{d}V = \rho_0(\boldsymbol{X})\mathrm{d}V_0 = const$$

The law of mass conservation is equivalent to the continuity of the mass distribution for the continuum with continuous placement of the material points.

Applying the general balance Eq. (53) with $\boldsymbol{\Psi} \to 1$ (scalar), $\boldsymbol{\Phi} = 0$ (no mass exchange through the surface A) and $\boldsymbol{\Xi} \to 0$ (no mass production in the volume) follows

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_V \rho(\boldsymbol{x}, t)\, \mathrm{d}V = \frac{\partial}{\partial t} \int\limits_{V_0} \rho_0(\boldsymbol{X})\, \mathrm{d}V_0 = 0 \tag{58}$$

or locally

$$\frac{\mathrm{d}}{\mathrm{d}t}(dm) = \frac{\mathrm{d}}{\mathrm{d}t}(\rho \mathrm{d}V) = \frac{\partial}{\partial t}(\rho_0 \mathrm{d}V_0) = 0 \tag{59}$$

The global law of mass conservation in the Eulerian formulation can be given as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_V \rho(\boldsymbol{x}, t)\, \mathrm{d}V &= \int\limits_V \left[\dot{\rho}(\boldsymbol{x}, t) + \rho(\boldsymbol{x}, t) \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}\right]\, \mathrm{d}V \\ &= \int\limits_V \left\{ \frac{\partial}{\partial t} \rho(\boldsymbol{x}, t) + \nabla_{\boldsymbol{x}} \cdot [\rho(\boldsymbol{x}, t) \boldsymbol{v}] \right\}\, \mathrm{d}V \end{aligned} \tag{60}$$

The local conservation law is

$$\frac{\mathrm{d}}{\mathrm{d}t} \rho(\boldsymbol{x}, t) + \rho(\boldsymbol{x}, t) \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}(\boldsymbol{x}, t) = \frac{\partial}{\partial t} \rho(\boldsymbol{x}, t) + \nabla_{\boldsymbol{x}} \cdot [\rho(\boldsymbol{x}, t) \boldsymbol{v}] = 0 \tag{61}$$

The equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0 \tag{62}$$

is the continuity equation, which can be given also as

$$\frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v}) = 0 \tag{63}$$

From this equation can be made the following conclusions. If $\rho \mathrm{d}V = \rho_0 \mathrm{d}V_0$ one gets

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}t} = \frac{1}{\mathrm{d}V_0} (\rho \mathrm{d}V)^{\cdot} = \frac{1}{\mathrm{d}V_0} \left[\dot{\rho} \mathrm{d}V + \rho (\mathrm{d}V)^{\cdot}\right] = \frac{1}{\mathrm{d}V_0} [\dot{\rho} + \rho \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}] \mathrm{d}V = 0$$

For $\rho_0 = \rho \det \boldsymbol{F}$ we can compute $(\rho \det \boldsymbol{F})^{\cdot} = 0$. $\dot{\rho} = 0$ results in div $\boldsymbol{v} = \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0$. In Eq. (61) the volume integral over $\nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v})$ can be transformed into a surface integral. The global mass balance is in this case

$$\int_V \frac{\partial \rho}{\partial t} \mathrm{d}V + \int_A \boldsymbol{n} \cdot (\rho \boldsymbol{v}) \mathrm{d}A = 0 \tag{64}$$

3.2.1 Balance of Momentum

The momentum vector $\boldsymbol{p}$ of the body is defined by

$$\boldsymbol{p}(\boldsymbol{x}, t) = \int_m \boldsymbol{v}(\boldsymbol{x}, t) \mathrm{d}m = \int_V \boldsymbol{v}(\boldsymbol{x}, t) \rho(\boldsymbol{x}, t) \mathrm{d}V \tag{65}$$

The global balance of momentum is named first Euler-Cauchy law of motion and can be related to the second Newton's axiom specified for continua.

Theorem 1.3 (Balance of momentum) *The rate of changes of the momentum* $\boldsymbol{p}(\boldsymbol{x}, t)$ *during the deformation of the body is equal to the sum of all on the acting surface and volume forces.*

The balance of momentum in the Eulerian description is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \boldsymbol{v}(\boldsymbol{x}, t) \rho(\boldsymbol{x}, t) \mathrm{d}V = \int_A \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}, t) \mathrm{d}A + \int_V \boldsymbol{k}(\boldsymbol{x}, t) \rho(\boldsymbol{x}, t) \mathrm{d}V \tag{66}$$

Equation (66) follows from the general balance Eq. (53) with $\boldsymbol{\Psi} = \boldsymbol{v}$, $\boldsymbol{\Phi} = \boldsymbol{T}$ and $\boldsymbol{\Xi} = \boldsymbol{k}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_V \boldsymbol{v}\rho \,\mathrm{d}V = \int\limits_A \boldsymbol{n}\cdot\boldsymbol{T}\,\mathrm{d}A + \int\limits_V \boldsymbol{k}\rho\,\mathrm{d}V \tag{67}$$

In the reference configuration we have

$$\frac{\partial}{\partial t}\int\limits_{V_0} \boldsymbol{v}(\boldsymbol{X},t)\rho_0(\boldsymbol{X})\,\mathrm{d}V_0 = \int\limits_{A_0} {}^{I}\boldsymbol{t}(\boldsymbol{X},\boldsymbol{n}_0,t)\,\mathrm{d}A_0 + \int\limits_{V_0} \boldsymbol{k}(\boldsymbol{X},t)\rho_0(\boldsymbol{X})\,\mathrm{d}V_0, \tag{68}$$

and with

$${}^{I}\boldsymbol{t} = \boldsymbol{n}_0\cdot{}^{I}\boldsymbol{P}$$

one gets

$$\frac{\partial}{\partial t}\int\limits_{V_0} \boldsymbol{v}(\boldsymbol{X},t)\rho_0(\boldsymbol{X})\,\mathrm{d}V_0 = \int\limits_{A_0} \boldsymbol{n}_0(\boldsymbol{X})\cdot{}^{I}\boldsymbol{P}(\boldsymbol{X},t)\,\mathrm{d}A_0 + \int\limits_{V_0} \boldsymbol{k}(\boldsymbol{X},t)\rho_0(\boldsymbol{X})\,\mathrm{d}V_0 \tag{69}$$

Applying the divergence theorem to Eqs. (67) and (69) we get

$$\frac{\mathrm{d}}{Dt}\int\limits_V \boldsymbol{v}(\boldsymbol{x},t)\rho(\boldsymbol{x},t)\,\mathrm{d}V = \int\limits_V \Big[\nabla_{\boldsymbol{x}}\cdot\boldsymbol{T}(\boldsymbol{x},t) + \boldsymbol{k}(\boldsymbol{x},t)\rho(\boldsymbol{x},t)\Big]\,\mathrm{d}V, \tag{70}$$

$$\frac{\partial}{\partial t}\int\limits_{V_0} \boldsymbol{v}(\boldsymbol{X},t)\rho_0(\boldsymbol{X})\,\mathrm{d}V_0 = \int\limits_{V_0} [\nabla_{\boldsymbol{X}}\cdot{}^{I}\boldsymbol{P}(\boldsymbol{X},t) + \boldsymbol{k}(\boldsymbol{X},t)\rho_0(\boldsymbol{X})]\,\mathrm{d}V_0 \tag{71}$$

The local formulations are

$$\nabla_{\boldsymbol{x}}\cdot\boldsymbol{T}(\boldsymbol{x},t) + \rho(\boldsymbol{x},t)\boldsymbol{k}(\boldsymbol{x},t) = \rho(\boldsymbol{x},t)\frac{D\boldsymbol{v}(\boldsymbol{x},t)}{Dt}, \tag{72}$$

$$\nabla_{\boldsymbol{X}}\cdot{}^{I}\boldsymbol{P}(\boldsymbol{X},t) + \rho_0(\boldsymbol{X})\boldsymbol{k}(\boldsymbol{X},t) = \rho_0(\boldsymbol{X})\frac{\partial\boldsymbol{v}(\boldsymbol{X},t)}{\partial t} \tag{73}$$

3.3 Balance of Moment of Momentum

Let us introduce the global vector of the moment of momentum

$$\boldsymbol{l}_O(\boldsymbol{x},t) = \int\limits_V \boldsymbol{x}\times\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\,\mathrm{d}V \tag{74}$$

The respective balance equations result in the second Euler-Cauchy equation of motion.

Theorem 1.4 (Balance of moment of momentum) *The rate of changes of the moment of momentum of a body $\boldsymbol{l}_O(\boldsymbol{x}, t)$ w.r.t. to the arbitrary point O is equal to the moments of all on the body acting surface and volume forces.*

The Eulerian balance of the moment of momentum is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V [\boldsymbol{x} \times \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)]\,\mathrm{d}V = \int_V [\boldsymbol{x} \times \rho(\boldsymbol{x}, t)\boldsymbol{k}(\boldsymbol{x}, t)]\,\mathrm{d}V + \int_A [\boldsymbol{x} \times \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}, t)]\,\mathrm{d}A \tag{75}$$

With $\boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{T}$ and $\boldsymbol{x} \times \boldsymbol{n} \cdot \boldsymbol{T} = -\boldsymbol{n} \cdot \boldsymbol{T} \times \boldsymbol{x}$ one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{x} \times \rho\boldsymbol{v}\,\mathrm{d}V = -\int_A \boldsymbol{n} \cdot \boldsymbol{T} \times \boldsymbol{x}\,\mathrm{d}A + \int_V \boldsymbol{x} \times \rho\boldsymbol{k}\,\mathrm{d}V \tag{76}$$

Equation (76) results from the general balance Eq. (53), if $\boldsymbol{\Psi} = (\boldsymbol{x} \times \boldsymbol{v})$, $\boldsymbol{\Phi} = -(\boldsymbol{T} \times \boldsymbol{x})$ and $\boldsymbol{\Xi} = (\boldsymbol{x} \times \boldsymbol{k})$. After some algebra it can be shown that for classical continua the balance of moment of momentum results in the symmetry condition for the Cauchy stress tensor. It can be shown that we have for the first and second Piola-Kirchhoff tensors different conclusions. The symmetry for the first Piola-Kirchhoff tensor cannot be established since we obtain

$$^I\boldsymbol{P} \cdot \boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F} \cdot {}^I\boldsymbol{P}^{\mathrm{T}}$$

3.4 Balance of Energy

Now we are prescribing two properties in each material point: the density ρ and the temperature θ. Both are non-negative ($\rho \geq 0, \theta \geq 0$). The foundation of any thermomechanical analysis is given by the first and the second law of thermodynamics. The first law is the balance of energy, the second one of entropy.

Within the framework of continuum thermodynamics we have to introduce at first suitable variables for the description of the macroscopic properties of the continuum. Let us assume macroscopic measurable, independent of each other parameters, which describe the state of the continuum in a unique manner. These parameters are named state variables. We have to distinguish extensive (additive) and intensive variables. The additive variables are proportional to the amount of continuum, expressed for example by the mass. The inner energy of the system is a typical extensive state variable. It depends only on the kinematical variables and the temperature: $\mathscr{U} = \mathscr{U}$ (kinematic variables, θ). If we divide a homogeneous system with the mass m into n homogeneous subsystems with the masses m_i the following statement is valid

$$\mathscr{U}_i = \left(\frac{m_i}{m}\right)\mathscr{U}, \quad i = 1, \ldots, n, \qquad \sum_{i=1}^{n} \mathscr{U}_i = \mathscr{U}, \quad \sum_{i=1}^{n} m_i = m \tag{77}$$

Intensive variables are independent of the amount of the continuum. If we divide now the system which is in an equilibrium state, into n subsystems, than the intensive state variable in each subsystem has the same value. Examples are the density and temperature.

Let us introduce the following limitations for the further developments:

- The continuum is assumed to be homogeneous that means in each material point we have the same properties.
- We assume that there is no mass exchange with the surrounding. The mass conservation is valid.
- We are assuming only mechanical and thermal actions.

Let us introduce the first law of thermodynamics.

Definition 1.22 (Thermomechanical balance of energy) The rate of changes of the total energy W within the volume is equal to the sum of the rate of the external heat supply Q and the power of all external forces P_a

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{W} = \mathscr{P}_{\mathrm{a}} + \mathscr{Q} \tag{78}$$

The total energy $\mathscr{W}$ consists of the inner energy $\mathscr{U}$ and the kinetic energy $\mathscr{K}$

$$\mathscr{W} = \mathscr{U} + \mathscr{K} \tag{79}$$

The kinetic energy is given as

$$\mathscr{K} = \frac{1}{2}\int_V \boldsymbol{v}\cdot\boldsymbol{v}\rho \,\mathrm{d}V$$

The inner energy is an additive function of the mass

$$\mathscr{U} = \int_m u \,\mathrm{d}m = \int_V \rho u \,\mathrm{d}V$$

with u as the inner energy density. The power of the external forces $\mathscr{P}_{\mathrm{a}}$ is based on the introduced volume and surface forces

$$\mathscr{P}_{\mathrm{a}} = \int_A \boldsymbol{t}\cdot\boldsymbol{v} \,\mathrm{d}A + \int_V \boldsymbol{k}\cdot\boldsymbol{v}\rho \,\mathrm{d}V \tag{80}$$

The rate of heat supply consists of two parts: contribution of the heat sources r in the volume and the heat flux through A

$$\mathscr{Q} = \int_V \rho r \, \mathrm{d}V - \int_A \boldsymbol{n} \cdot \boldsymbol{h} \, \mathrm{d}A \tag{81}$$

$\boldsymbol{h}$ is the heat flux vector.

Now we have instead of Eq. (78)

$$\dot{\mathscr{U}} + \dot{\mathscr{K}} = \mathscr{P}_\mathrm{a} + \mathscr{Q} \tag{82}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \left(u + \frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v} \right) \rho \, \mathrm{d}V = \int_A \boldsymbol{t} \cdot \boldsymbol{v} \, \mathrm{d}A + \int_V \boldsymbol{k} \cdot \boldsymbol{v} \rho \, \mathrm{d}V - \int_A \boldsymbol{n} \cdot \boldsymbol{h} \, \mathrm{d}A + \int_V r \rho \, \mathrm{d}V \tag{83}$$

Taking into account $\boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{T}$ the first law of thermodynamics can be obtained from the general balance Eq. (53) with

$$\boldsymbol{\Psi} \to u + \frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v}, \quad \boldsymbol{\Phi} = \boldsymbol{T} \cdot \boldsymbol{v} - \boldsymbol{h}, \quad \boldsymbol{\Xi} \to \boldsymbol{k} \cdot \boldsymbol{v} + r$$

In the reference configuration one gets

$$\begin{aligned} \frac{\partial}{\partial t} \int_{V_0} \left(u + \frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v} \right) \rho_0 \, \mathrm{d}V_0 = \int_{A_0} {}^I\boldsymbol{t} \cdot \boldsymbol{v} \, \mathrm{d}A_0 + \int_{V_0} \boldsymbol{k} \cdot \boldsymbol{v} \rho_0 \, \mathrm{d}V_0 \\ - \int_{A_0} \boldsymbol{n}_0 \cdot \boldsymbol{h}_0 \, \mathrm{d}A_0 + \int_{V_0} r \rho_0 \, \mathrm{d}V_0 \end{aligned} \tag{84}$$

With respect to Eq. (34) for the relationship between $\boldsymbol{t}$ and ${}^I\boldsymbol{t}$ we have the relationship between $\boldsymbol{h}$ and $\boldsymbol{h}_0$

$$\boldsymbol{h}_0 = (\det \boldsymbol{F}) \boldsymbol{F}^{-1} \cdot \boldsymbol{h}, \quad \boldsymbol{h} = (\det \boldsymbol{F})^{-1} \boldsymbol{F} \cdot \boldsymbol{h}_0 \tag{85}$$

With

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v} \right) = \frac{1}{2} \dot{\boldsymbol{v}} \cdot \boldsymbol{v} + \frac{1}{2} \boldsymbol{v} \cdot \dot{\boldsymbol{v}} = \dot{\boldsymbol{v}} \cdot \boldsymbol{v},$$

$$\int_A \boldsymbol{n} \cdot (\boldsymbol{T} \cdot \boldsymbol{v} - \boldsymbol{h}) \, \mathrm{d}A = \int_V [\nabla_{\boldsymbol{x}} \cdot (\boldsymbol{T} \cdot \boldsymbol{v}) - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{h}] \, \mathrm{d}V$$

$$\nabla_{\boldsymbol{x}} \cdot (\boldsymbol{T} \cdot \boldsymbol{v}) = (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{T}) \cdot \boldsymbol{v} + \boldsymbol{T} \cdot\cdot (\nabla_{\boldsymbol{x}} \boldsymbol{v})^\mathrm{T} = (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{T}) \cdot \boldsymbol{v} + \boldsymbol{T} \cdot\cdot \boldsymbol{D}$$

from Eq. (83) it follows

$$\int_V \left(\frac{\mathrm{d}u}{\mathrm{d}t} + \underline{\dot{\boldsymbol{v}} \cdot \boldsymbol{v}} \right) \rho \, \mathrm{d}V = \int_V (\boldsymbol{T} \cdot\cdot \boldsymbol{D} - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{h} + \rho r) \, \mathrm{d}V + \int_V [\underline{(\nabla_{\boldsymbol{x}} \cdot \boldsymbol{T}) \cdot \boldsymbol{v} + \rho \boldsymbol{k} \cdot \boldsymbol{v}}] \, \mathrm{d}V \tag{86}$$

The underlined terms are the balance of momentum and we can simplify Eq. (86)

$$\int_V (\rho \dot{u} - \boldsymbol{T} \cdot\cdot \boldsymbol{D} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{h} - \rho r) \, \mathrm{d}V = 0 \tag{87}$$

The local form holds

$$\rho \dot{u} = \boldsymbol{T} \cdot\cdot \boldsymbol{D} - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{h} + \rho r \tag{88}$$

The balance of energy in the reference configuration is

$$\int_{V_0} \rho_0 \frac{\partial u}{\partial t} \, \mathrm{d}V_0 = \int_{V_0} ({}^{II}\boldsymbol{P} \cdot\cdot \dot{\boldsymbol{G}} - \nabla_{\boldsymbol{X}} \cdot \boldsymbol{h}_0 + \rho_0 r) \, \mathrm{d}V_0 \tag{89}$$

or in the local form

$$\rho_0 \frac{\partial u}{\partial t} = {}^{II}\boldsymbol{P} \cdot\cdot \dot{\boldsymbol{G}} - \nabla_{\boldsymbol{X}} \cdot \boldsymbol{h}_0 + \rho_0 r \tag{90}$$

In Eq. (90) the conjugated pair $({}^{II}\boldsymbol{P}, \dot{\boldsymbol{G}})$ can be substituted by $({}^{I}\boldsymbol{P}, \dot{\boldsymbol{F}})$.

3.5 Balance of Entropy

One of the possible formulations of the second law of thermodynamics is the following.

Theorem 1.5 (Balance of entropy) *The rate of changes of the entropy S within the volume is not less than the rate of the external entropy supply.*

The entropy is an additive function that means

$$\mathscr{S} = \int_m s \, \mathrm{d}m = \int_V \rho s \, \mathrm{d}V \tag{91}$$

with s as the inner entropy density. Now the second law of thermodynamics can be stated in the global form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_V \rho s \,\mathrm{d}V \geq \int\limits_V \frac{r}{\theta}\rho \,\mathrm{d}V - \int\limits_A \frac{\boldsymbol{n}\cdot\boldsymbol{h}}{\theta}\,\mathrm{d}A \tag{92}$$

For all real processes (92) is an inequality ($>$) or with other words real processes are always irreversible.

With the transform

$$\int\limits_A \frac{\boldsymbol{n}\cdot\boldsymbol{h}}{\theta}\,\mathrm{d}A = \int\limits_V \nabla_{\boldsymbol{x}}\cdot\left(\frac{\boldsymbol{h}}{\theta}\right)\mathrm{d}V = \int\limits_V \left(\frac{\nabla_{\boldsymbol{x}}\cdot\boldsymbol{h}}{\theta} - \frac{\boldsymbol{h}\cdot\nabla_{\boldsymbol{x}}\theta}{\theta^2}\right)\mathrm{d}V, \tag{93}$$

we get the local formulation

$$\rho\theta\dot{s} \geq \rho r - \nabla_{\boldsymbol{x}}\cdot\boldsymbol{h} + \frac{1}{\theta}\boldsymbol{h}\cdot\nabla_{\boldsymbol{x}}\theta \tag{94}$$

or

$$\rho\theta\dot{s} - (\rho r - \nabla_{\boldsymbol{x}}\cdot\boldsymbol{h}) - \boldsymbol{h}\cdot\nabla_{\boldsymbol{x}}\ln\theta \geq 0 \tag{95}$$

The term in brackets can be substituted by Eq. (88)

$$\rho\theta\dot{s} + \boldsymbol{T}\cdot\cdot\,\boldsymbol{D} - \rho\dot{u} - \boldsymbol{h}\cdot\nabla_{\boldsymbol{x}}\ln\theta \geq 0 \tag{96}$$

and with

$$\rho\theta\dot{s} = \rho(\theta s)^{\cdot} - \rho s\dot{\theta}$$

it follows

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}(\theta s - u) - \rho s\frac{\mathrm{d}\theta}{\mathrm{d}t} + \boldsymbol{T}\cdot\cdot\,\boldsymbol{D} - \boldsymbol{h}\cdot\nabla_{\boldsymbol{x}}\ln\theta \geq 0 \tag{97}$$

The term

$$(u - \theta s) = f \tag{98}$$

is the Helmholtz free energy.

All entropy balances can be presented in the reference configuration, for example the global balance

$$\frac{\partial}{\partial t}\int\limits_{V_0} \rho_0 s \,\mathrm{d}V_0 \geq \int\limits_{V_0} \frac{r}{\theta}\rho_0 \,\mathrm{d}V_0 - \int\limits_{A_0} \frac{\boldsymbol{n}_0\cdot\boldsymbol{h}_0}{\theta}\,\mathrm{d}A_0$$

or the local one

$$\rho_0\theta\frac{\partial s}{\partial t} \geq (\rho_0 r - \nabla_{\boldsymbol{X}}\cdot\boldsymbol{h}_0) + \frac{1}{\theta}\boldsymbol{h}_0\cdot\nabla_{\boldsymbol{X}}\theta$$

4 Constitutive Modeling

Equations describing the specific behavior of the continua are named constitutive, physical or state equations. Let us introduce according to [20] the following definition.

Definition 1.23 (Constitutive equations) Constitutive equations link all phenomenological variables describing the macroscopic behavior of the continuum.

Among such variables are: stresses, strains, temperature, heat flux, etc.

The number of necessary constitutive equations depends on the problem. In the case of thermomechanical problems we have the following balance equations: mass (1 scalar equation), momentum (1 vectorial equation or for three-dimensional problems 3 scalar equations), moment of momentum (1 vectorial or 3 scalar equations), energy (1 scalar equation) and entropy (1 scalar inequality). Since the entropy inequality yields only in the determination of the process direction we have only 8 scalar equations to estimate 19 variables: density ρ (1 scalar variable), displacements $\boldsymbol{u}$ or velocities $\boldsymbol{v}$ (1 vector or its 3 coordinates), stress tensor $\boldsymbol{T}$ (1 second-order tensor or 9 coordinates, inner energy u (1 variable), entropy s (1 variable), temperature θ (1 variable) and heat flux $\boldsymbol{h}$ (1 vector or 3 variables). In this case we have to add 11 constitutive equations otherwise the the system of governing equations is underestimated and we cannot solve problems.

Before we start the formulation of constitutive equations let us introduce some restrictions and definitions. We limit ourselves by the consideration of classical continua. Other continua are discussed in the literature, for example, micropolar continua are presented in [4]. In addition, we assume that we have only simple materials of first order [21].

Definition 1.24 (Simple material of first order) A simple material of first order defined by constitutive equations which contains only local variables, for example the local strain tensor and the locale heat flux vector with the local stress tensor and the local temperature gradient. All conclusions are made for the same material point and its differential surrounding of first order.

Since the constitutive equations present the individual response of the material universal constitutive equations cannot be established [22]. That means we are discussing always special cases. The constitutive equations can be formulated using the top-down approach, the bottom-up approach or rheological models. Here we present only the top-down approach that we start with some mathematical and physical statements. At first we assume that the balances are valid. The mathematical structure of the constitutive equations can be established with the axioms of the theory of materials.

Let us start with ordering the variables which are included in the set of constitutive equations. We assume that all variables depend on the position ($\boldsymbol{x}$ or $\boldsymbol{X}$) and the

time t. The material behavior can be presented by functional relationships between the constitutive variables and the constitutive parameters. For example, within the framework of thermomechanics the temperature θ is a constitutive parameter and the heat flux vector $\boldsymbol{h}$ is a constitutive variable. The choice of the constitutive parameters and the constitutive variables is arbitrary.

If we want to describe the processes realized in the material points of the continuum history of changes of the constitutive parameters should be given.

Definition 1.25 (Process) The time changes of the constitutive parameters in the material points is named process.

Definition 1.26 (Constitutive variable) The behavior of the continuum in each material point can be presented by a set of constitutive variables. They can be time operators of the processes in the points.

The respective functional relationships are the constitutive equations. Below we discuss only constitutive equations for solids. With respect to the second axiom of rheology [23, 24] all real bodies show solid and fluid properties. So we consider that the solid behavior is dominant.

Definition 1.27 (Solid) The acting on the body loads results in the stress deviator to non-zero elements that means it resists to the changes of the shape of the body.

In this sense we distinguish time-independent and time-dependent material behavior [1]. Elastic and plastic materials belong to the first group, viscoelastic and viscoplastic materials—to the second one.

4.1 Basics of Material Theory

Let us discuss some basic tools in material modeling based on the top-down approach. As usual one should focus on the attention to the following three questions [1]:

- formulation of constitutive equations,
- consideration of symmetries of the material behavior and
- consideration of kinematic constraints.

The last one is important w.r.t. models for compressible and incompressible materials.

The systematical deduction of constitutive equations can be realized with the help of some principal axioms (constitutive principles). The main axioms are:

- causality,
- determinism,
- equipresence,
- materielle objectivity,
- locale action,

- memory and
- physical consistency.

A detailed discussion of the physical meaning and the mathematical consequences for the aforementioned axiom is given in [25].

4.2 General Constitutive Equations of Thermo-Mechanical Materials

The thermodynamical state of the continuum is defined by the motion $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$ and the temperature $\theta = \theta(\boldsymbol{X}, t)$ of the material points $\boldsymbol{X}$ at the instant time t. $\boldsymbol{x}$ and θ are independent variables. As dependent variables (constitutive variables) we postulate the stress tensor, the heat flux vector, the free energy and the entropy.

For general models of the material behavior we should assume that the actual state not only depends on the actual loading, but also from the loading history $t_0 < \tau \leq t$. In addition, the behavior of the given material point $\boldsymbol{X}$ also depends on the behavior of all other points of the body $\tilde{\boldsymbol{X}}$. If we assume for the functions $\boldsymbol{x}(\tilde{\boldsymbol{X}}, \tau)$ and $\theta(\tilde{\boldsymbol{X}}, \tau)$ the continuity w.r.t. $\tilde{\boldsymbol{X}}$ and τ, we can represent the behavior by Taylor series for the points $\boldsymbol{X}$ by powers of $(\tilde{\boldsymbol{X}} - \boldsymbol{X})$ and for the time τ by powers $(\tau - t)$. Applying the axiom of local action and memory axiom (in special situations fading memory the power series for $\boldsymbol{x}(\tilde{\boldsymbol{X}}, \tau)$ and $\theta(\tilde{\boldsymbol{X}}, \tau)$ can be limited by the first derivative with respect to $\tilde{\boldsymbol{X}}$ and τ, respectively. The constitutive variables depend in this case on $\boldsymbol{X}$ and θ, but also from $\nabla_{\boldsymbol{X}}\boldsymbol{x}$, $\nabla_{\boldsymbol{X}}\theta$ and $\dot{\theta}$. Other models take into account second gradients (see [26–29] among others).

Assuming the axiom of material objectivity there is no explicit dependency of $\boldsymbol{x}$ or $\dot{\boldsymbol{x}}$ since only the strains or the strain rates and not the rigid body motions define the material behavior. For many real materials the gradients $\nabla_{\boldsymbol{X}}\boldsymbol{x}$ and $\nabla_{\boldsymbol{X}}\theta$ are influenced by the loading history. Taking into account the first derivatives only in this case $\nabla_{\boldsymbol{X}}\dot{\boldsymbol{x}}$ and $\nabla_{\boldsymbol{X}}\dot{\theta}$ act as constitutive variables.

The constitutive equations of a simple thermomechanical material take the form

$$\begin{aligned}
\boldsymbol{P}(\boldsymbol{X}, t) &= \boldsymbol{P} \left\{\boldsymbol{X}, \theta(\boldsymbol{X}, t), \dot{\theta}(\boldsymbol{X}, t), \nabla_{\boldsymbol{X}}\theta(\boldsymbol{X}, t), \nabla_{\boldsymbol{X}}\dot{\theta}(\boldsymbol{X}, t), \boldsymbol{\Gamma}(\boldsymbol{X}, t)\right\}, \\
\boldsymbol{h}_0(\boldsymbol{X}, t) &= \boldsymbol{h}_0 \left\{\boldsymbol{X}, \theta(\boldsymbol{X}, t), \dot{\theta}(\boldsymbol{X}, t), \nabla_{\boldsymbol{X}}\theta(\boldsymbol{X}, t), \nabla_{\boldsymbol{X}}\dot{\theta}(\boldsymbol{X}, t), \boldsymbol{\Gamma}(\boldsymbol{X}, t)\right\}, \\
f(\boldsymbol{X}, t) &= f \left\{\boldsymbol{X}, \theta(\boldsymbol{X}, t), \dot{\theta}(\boldsymbol{X}, t), \nabla_{\boldsymbol{X}}\theta(\boldsymbol{X}, t), \nabla_{\boldsymbol{X}}\dot{\theta}(\boldsymbol{X}, t), \boldsymbol{\Gamma}(\boldsymbol{X}, t)\right\}, \\
s(\boldsymbol{X}, t) &= s \left\{\boldsymbol{X}, \theta(\boldsymbol{X}, t), \dot{\theta}(\boldsymbol{X}, t), \nabla_{\boldsymbol{X}}\theta(\boldsymbol{X}, t), \nabla_{\boldsymbol{X}}\dot{\theta}(\boldsymbol{X}, t), \boldsymbol{\Gamma}(\boldsymbol{X}, t)\right\}
\end{aligned} \tag{99}$$

The set $\boldsymbol{\Gamma}$ includes all mechanical variables $\nabla_{\boldsymbol{X}}\boldsymbol{x}(\boldsymbol{X}, t)$, $\nabla_{\boldsymbol{X}}\dot{\boldsymbol{x}}(\boldsymbol{X}, t)$ characterizing the deformation. The stress tensor $\boldsymbol{P}$ can be the first or the second Piola-Kirchhoff tensor. The explicit dependency of the constitutive equations on $\boldsymbol{X}$ is equivalent to the statement that in each point of the body one can have different kinds of material behavior otherwise the body is homogeneous. In general the constitutive equations are functionals (operators). This is shown by the symbol $\{\ldots\}$. If the prehistory has

no influence on the actual material behavior, the constitutive equations are functions and the symbol (...) is used.

The axiom of physical consistency axiom allows a further specification of Eq. (99). In the case of simple thermoviscoelastic materials (the time dependency is related to the initial state, but not to the prehistory) the elastic strains and the strain rates can be expressed by $\boldsymbol{C}, \dot{\boldsymbol{C}}$. If an independent relation to parameters ρ, $\dot{\rho}$ cannot be assumed, these parameters can be ignored and we have for simple thermoviscoelastic solids (no influence of the prehistory)

$$\begin{aligned} \boldsymbol{P}(\boldsymbol{X},t) &= \boldsymbol{P}\left\{\boldsymbol{X},\theta,\dot{\theta},\nabla_{\boldsymbol{X}}\theta,\nabla_{\boldsymbol{X}}\dot{\theta},\boldsymbol{C},\dot{\boldsymbol{C}}\right\},\\ \boldsymbol{h}_0(\boldsymbol{X},t) &= \boldsymbol{h}\left\{\boldsymbol{X},\theta,\dot{\theta},\nabla_{\boldsymbol{X}}\theta,\nabla_{\boldsymbol{X}}\dot{\theta},\boldsymbol{C},\dot{\boldsymbol{C}}\right\},\\ f(\boldsymbol{X},t) &= f\left\{\boldsymbol{X},\theta,\dot{\theta},\nabla_{\boldsymbol{X}}\theta,\nabla_{\boldsymbol{X}}\dot{\theta},\boldsymbol{C},\dot{\boldsymbol{C}}\right\},\\ s(\boldsymbol{X},t) &= s\left\{\boldsymbol{X},\theta,\dot{\theta},\nabla_{\boldsymbol{X}}\theta,\nabla_{\boldsymbol{X}}\dot{\theta},\boldsymbol{C},\dot{\boldsymbol{C}}\right\} \end{aligned} \tag{100}$$

For pure thermoelastische solids the time derivatives can be ignored

$$\begin{aligned} \boldsymbol{P}(\boldsymbol{X},t) &= \boldsymbol{P}\left\{\boldsymbol{X},\theta,\nabla_{\boldsymbol{X}}\theta,\boldsymbol{C}\right\},\\ \boldsymbol{h}_0(\boldsymbol{X},t) &= \boldsymbol{h}\left\{\boldsymbol{X},\theta,\nabla_{\boldsymbol{X}}\theta,\boldsymbol{C}\right\},\\ f(\boldsymbol{X},t) &= f\left\{\boldsymbol{X},\theta,\nabla_{\boldsymbol{X}}\theta,\boldsymbol{C}\right\},\\ s(\boldsymbol{X},t) &= s\left\{\boldsymbol{X},\theta,\nabla_{\boldsymbol{X}}\theta,\boldsymbol{C}\right\} \end{aligned} \tag{101}$$

4.3 Elastic Simple Material

As a first example let us discuss the ideal-elastic material behavior. In this case we have constitutive equations which are functions. In the pure mechanical case one has to present a relation between the stress and the strain tensors. Considering the material objectivity in this relation the following equations are valid

$${}^{II}\boldsymbol{P}(\boldsymbol{X},t) = \boldsymbol{f}(\boldsymbol{C},\boldsymbol{X},t) \qquad \text{or} \qquad {}^{II}\boldsymbol{P}(\boldsymbol{X},t) = \boldsymbol{g}(\boldsymbol{G},\boldsymbol{X},t)$$

Here the assumption of simple material is included since only the gradient of $\boldsymbol{x}(\boldsymbol{X},t)$ (deformation gradient tensor $\boldsymbol{F}(\boldsymbol{X},t)$) is taken into account.

The specific elementary work in the reference configuration is the starting point for the further discussions. The constitutive Eq. (101) have a very simple form

$${}^{I}\boldsymbol{P}(\boldsymbol{X},t) = {}^{I}\boldsymbol{P}(\boldsymbol{F}),$$

and the variation of the elementary work W_i gives

$$\delta W_i = \frac{1}{\rho_0}\,{}^{I}\boldsymbol{P} \cdot\cdot\, \delta\boldsymbol{F}^{\mathrm{T}}$$

The work depends on the strains at current time t. This work is stored as the strain energy of the body

$$\delta W_i = \delta \mathscr{U} = \frac{1}{\rho_0}\, {}^{I}\boldsymbol{P} \cdot\cdot\, \delta \boldsymbol{F}$$

with $u = u(\boldsymbol{F})$ as the specific strain energy density function. This function should not be influenced by rigid motions of the body (material objectivity axiom). From this it follows for arbitrary rotations $\boldsymbol{Q}$

$$u(\boldsymbol{F}) = u(\boldsymbol{Q}\cdot\boldsymbol{F}) = u\left(\sqrt{(\boldsymbol{Q}\cdot\boldsymbol{F})^{\mathrm{T}}\cdot(\boldsymbol{Q}\cdot\boldsymbol{F})}\right) = u\left(\sqrt{\boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{F}}\right) = u(\boldsymbol{U}) \tag{102}$$

With $\boldsymbol{U}^2 = \boldsymbol{C}$ and $\boldsymbol{G} = \dfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$ we have $u(\boldsymbol{U}) = \hat{u}(\boldsymbol{C})$ or $u(\boldsymbol{U}) = \check{u}(\boldsymbol{G})$. From

$$\delta u(\boldsymbol{F}) = \frac{\partial u(\boldsymbol{F})}{\partial \boldsymbol{F}} \cdot\cdot\, \delta \boldsymbol{F}^{\mathrm{T}} = \left[\frac{\partial u(\boldsymbol{F})}{\partial \boldsymbol{F}}\right] \cdot\cdot\, \delta \boldsymbol{F} = \left[u(\boldsymbol{F})_{,\boldsymbol{F}}\right]^{\mathrm{T}} \cdot\cdot\, \delta \boldsymbol{F}$$

one gets

$$\frac{1}{\rho_0}\, {}^{I}\boldsymbol{P} = \left[\frac{\partial u(\boldsymbol{F})}{\partial \boldsymbol{F}}\right] = \left[\frac{\partial \hat{u}(\boldsymbol{C})}{\partial \boldsymbol{F}}\right] \tag{103}$$

or after some manipulations

$${}^{I}\boldsymbol{P} = 2\rho_0 \boldsymbol{F}\cdot\frac{\partial \hat{u}(\boldsymbol{C})}{\partial \boldsymbol{C}} \tag{104}$$

With respect to the transform rules

$$\boldsymbol{T} = (\det \boldsymbol{F})^{-1}\, {}^{I}\boldsymbol{P}\cdot\boldsymbol{F}^{\mathrm{T}}, \qquad {}^{II}\boldsymbol{P} = \boldsymbol{F}^{-1}\cdot{}^{I}\boldsymbol{P}$$

we can formulate the constitutive equation for the Cauchy stress tensor $\boldsymbol{T}$ and the second Piola-Kirchhoff stress tensor ${}^{II}\boldsymbol{P}$

$$\boldsymbol{T} = 2\rho \boldsymbol{F}\cdot\frac{\partial \hat{u}}{\partial \boldsymbol{C}}\cdot\boldsymbol{F}^{\mathrm{T}},$$

$${}^{II}\boldsymbol{P} = 2\rho_0\frac{\partial \hat{u}(\boldsymbol{C})}{\partial \boldsymbol{C}} = \boldsymbol{f}(\boldsymbol{C}) \qquad \text{or} \qquad {}^{II}\boldsymbol{P} = 2\rho_0\frac{\partial \check{u}(\boldsymbol{G})}{\partial \boldsymbol{G}} = \boldsymbol{g}(\boldsymbol{G})$$

This is the general constitutive equation in the non-linear elasticity assuming large deformations and isothermal behavior. If we have special cases of anisotropy than further simplifications are possible. In the simplest case (isotropy) the energy density function $u = u(\boldsymbol{C})$ depends only on the invariants of the tensor $\boldsymbol{C}$

$$u = \hat{u}(\boldsymbol{C}) = \hat{u}[I_1(\boldsymbol{C}), I_2(\boldsymbol{C}), I_3(\boldsymbol{C})]$$

At first we consider the chain rule

$$\frac{\partial \hat{u}(\boldsymbol{C})}{\partial \boldsymbol{C}} = \frac{\partial \hat{u}}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{C}} + \frac{\partial \hat{u}}{\partial I_2}\frac{\partial I_2}{\partial \boldsymbol{C}} + \frac{\partial \hat{u}}{\partial I_3}\frac{\partial I_3}{\partial \boldsymbol{C}}$$

With

$$\begin{aligned} I_1(\boldsymbol{C}) &= \operatorname{tr}\boldsymbol{C}, \\ I_2(\boldsymbol{C}) &= \frac{1}{2}\left[I_1^2(\boldsymbol{C}) - I_1(\boldsymbol{C}^2)\right], \\ I_3(\boldsymbol{C}) &= \frac{1}{3}\left[I_1(\boldsymbol{C}^3) + 3I_1(\boldsymbol{C})I_2(\boldsymbol{C}) - I_1^3(\boldsymbol{C})\right] \end{aligned}$$

one gets

$$\begin{aligned} \frac{\partial I_1}{\partial \boldsymbol{C}} &= \boldsymbol{I}, \\ \frac{\partial I_2}{\partial \boldsymbol{C}} &= I_1\boldsymbol{I} - \boldsymbol{C}, \\ \frac{\partial I_3}{\partial \boldsymbol{C}} &= \boldsymbol{C}^2 + \boldsymbol{I}I_2(\boldsymbol{C}) + I_1(\boldsymbol{C})[I_1(\boldsymbol{C})\boldsymbol{I} - \boldsymbol{C}] - I_1^2(\boldsymbol{C})\boldsymbol{C} \end{aligned}$$

and

$$\begin{aligned} \hat{u}_{,\boldsymbol{C}} &= \left(\frac{\partial \hat{u}}{\partial I_1} + I_1\frac{\partial \hat{u}}{\partial I_2} + I_2\frac{\partial \hat{u}}{\partial I_3}\right)\boldsymbol{I} - \left(\frac{\partial \hat{u}}{\partial I_2} + I_1\frac{\partial \hat{u}}{\partial I_3}\right)\boldsymbol{C} + \frac{\partial \hat{u}}{\partial I_3}\boldsymbol{C}^2 \\ &= \phi_0\boldsymbol{I} + \phi_1\boldsymbol{C} + \phi_2\boldsymbol{C}^2, \end{aligned}$$

Here $\phi_i = \phi_i(I_1, I_2, I_3)$. Finally we obtain

$${}^{I}\boldsymbol{P} = 2\rho_0\boldsymbol{F}\cdot(\phi_0\boldsymbol{I} + \phi_1\boldsymbol{C} + \phi_2\boldsymbol{C}^2)$$

For an arbitrary isotropic tensor function $\boldsymbol{f}(\boldsymbol{A})$ one gets in the case of orthogonal tensors $\boldsymbol{Q}$

$$\boldsymbol{Q}\cdot\boldsymbol{f}(\boldsymbol{A})\cdot\boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{f}(\boldsymbol{Q}\cdot\boldsymbol{A}\cdot\boldsymbol{Q}^{\mathrm{T}})$$

and the following representation

$$\boldsymbol{f}(\boldsymbol{A}) = \phi_0\boldsymbol{I} + \phi_1\boldsymbol{A} + \phi_2\boldsymbol{A}^2$$

Then for the isotropic elastic body we can write the constitutive equation

$${}^{II}\boldsymbol{P} = \psi_0\boldsymbol{I} + \psi_1\boldsymbol{G} + \psi_2\boldsymbol{G}^2,$$

where ψ_i is now a function of the invariants of $\boldsymbol{G}$.

Introducing kinematic restrictions another special case can be deduced. For example, for the incompressibility (isochoric motions) the following equations are valid

$$\det \boldsymbol{F} = 1 \qquad \det \boldsymbol{C} = 1 \qquad \text{or} \qquad \sqrt{\det(2\boldsymbol{G} - \boldsymbol{I})} - 1 = 0,$$

The kinematic constraint can be expressed as follows

$$\lambda(\boldsymbol{C}) = \det \boldsymbol{C} - 1 = 0$$

That means $I_3(\boldsymbol{C}) = 1$ and instead of $\hat{u} = \hat{u}(I_1, I_2, I_3)$ we have $\hat{u} = \hat{u}(I_1, I_2)$ or

$$\frac{\partial \hat{u}}{\partial I_3} = 0$$

With the help of the method of Lagrangian multipliers one gets the constitutive equation

$${}^{I}\boldsymbol{P} = 2\rho_0 \boldsymbol{F} \cdot \left[\left(\frac{\partial \hat{u}}{\partial I_1} + I_1 \frac{\partial \hat{u}}{\partial I_2} \right) \boldsymbol{I} - \frac{\partial \hat{u}}{\partial I_2} \boldsymbol{C} \right] - p\boldsymbol{F}^{-1},$$

where p is an a priori unknown function (called hydrostatic pressure), which can be estimated from the equilibrium/motion equation.

4.4 Models with Internal Variables

Dissipative effects can be modeled with the help of different concepts and we have various possibilities to formulate the constitutive equations. One possibility is the introduction of viscose stresses depending on the strain rates. Another possibility is given using the fading memory principle. Here we apply a third approach: at first some inner variables are postulated, which have an influence on the free energy. In addition, for the inner variable we need evolution equations. Examples of inner variables are creep strains, plastic strains or damage variables.

The starting point are the balances and the constitutive equations for homogeneous materials. The last one depends now additionally on $\boldsymbol{\Upsilon}_i(\boldsymbol{X}, t)$ $(i = 1, \ldots, n)$—the inner variables. The inner variables are tensor-valued variables of different order. For example, in the case of isotropic damage a scalar is used, for isotropic hardening—scalar, kinematic hardening-second-order tensor or anisotropic damage-fourth-order tensor. The evolution equations can be postulated in the following form

$$\frac{D\boldsymbol{\Upsilon}_i}{Dt} = \boldsymbol{Y}_i(\theta, \nabla_{\boldsymbol{x}}\theta, \boldsymbol{g}, \boldsymbol{\Upsilon}_1, \ldots, \boldsymbol{\Upsilon}_n) \tag{105}$$

Finally, dissipative materials can be presented by the following set of constitutive and evolution equations

$$\begin{aligned} {}^{I}\boldsymbol{P}(\boldsymbol{X},t) &= {}^{I}\boldsymbol{P}\ (\theta, \nabla_{\boldsymbol{x}}\theta, \boldsymbol{g}, \boldsymbol{\Upsilon}_i), \\ \boldsymbol{h}_0(\boldsymbol{X},t) &= \boldsymbol{h}_0\ (\theta, \nabla_{\boldsymbol{x}}\theta, \boldsymbol{g}, \boldsymbol{\Upsilon}_i), \\ f(\boldsymbol{X},t) &= f\ \ (\theta, \nabla_{\boldsymbol{x}}\theta, \boldsymbol{g}, \boldsymbol{\Upsilon}_i), \\ s(\boldsymbol{X},t) &= s\ \ (\theta, \nabla_{\boldsymbol{x}}\theta, \boldsymbol{g}, \boldsymbol{\Upsilon}_i), \\ \dot{\boldsymbol{\Upsilon}}_i(\boldsymbol{X},t) &= \boldsymbol{Y}_i\ (\theta, \nabla_{\boldsymbol{x}}\theta, \boldsymbol{g}, \boldsymbol{\Upsilon}_1, \ \ldots, \ \boldsymbol{\Upsilon}_n) \end{aligned} \tag{106}$$

They should be completed by initial conditions

$$\boldsymbol{\Upsilon}_i(\boldsymbol{X}, t_0) = \boldsymbol{\Upsilon}_i^0(\boldsymbol{X}) \tag{107}$$

The problems of the formulation of constitutive equations incorporating inner variables and large deformations are widely discussed in the literature, see [30] among others. Assuming geometrical linearity it will be much more simpler. Now we present the strains by $\boldsymbol{\varepsilon}$ (Cauchy or small strain tensor) and the stresses by $\boldsymbol{\sigma}$. It is not necessary to distinguish two configurations. In addition, we assume that in analogy to Sect. 4.3 the heat flux vector is given by the anisotropic Fourier's law. For the other constitutive equations we suggest the independence of the temperature gradient. Finally we have the constitutive and evolution Eq. (106)

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}(\theta, \boldsymbol{\varepsilon}, \boldsymbol{\Upsilon}_i), \\ \boldsymbol{h} &= -\kappa \cdot \boldsymbol{\nabla}\theta, \\ f &= f(\theta, \boldsymbol{\varepsilon}, \boldsymbol{\Upsilon}_i), \\ s &= s(\theta, \boldsymbol{\varepsilon}, \boldsymbol{\Upsilon}_i), \\ \dot{\boldsymbol{\Upsilon}}_i &= \boldsymbol{Y}_i(\theta, \nabla_{\boldsymbol{X}}\theta, \boldsymbol{\varepsilon}, \boldsymbol{\Upsilon}_i) \end{aligned} \tag{108}$$

The analysis will be performed as shown in Sect. 4.3. The staring point is the free energy f. Note that the strains consist of an elastic and an inelastic part

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{el}} + \boldsymbol{\varepsilon}^{\mathrm{inel}} = \boldsymbol{\varepsilon}^{\mathrm{el}} + \boldsymbol{\varepsilon}^{\mathrm{pl}} \tag{109}$$

with $\boldsymbol{\varepsilon}^{\mathrm{el}}$ as thermoelastic strains, $\boldsymbol{\varepsilon}^{\mathrm{inel}}$ as inelastic strains and $\boldsymbol{\varepsilon}^{\mathrm{pl}}$ as plastic strains.

For the free energy w.r.t. Eq. (109) we can assume

$$f = f(\theta, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{\mathrm{el}}, \boldsymbol{\varepsilon}^{\mathrm{pl}}, \boldsymbol{\Upsilon}_i) \tag{110}$$

The total strains $\boldsymbol{\varepsilon}$, the elastic strains $\boldsymbol{\varepsilon}^{\mathrm{el}}$ and the plastic strains $\boldsymbol{\varepsilon}^{\mathrm{pl}}$ are connected by one equation. Following [31] the form of the free energy considering the decomposition of the total strains can be obtained

$$f = f(\theta, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{pl}}, \boldsymbol{\Upsilon}_i) = f(\theta, \boldsymbol{\varepsilon}^{\mathrm{el}}, \boldsymbol{\Upsilon}_i)$$

The derivative of the free energy is

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\varepsilon}^{\mathrm{el}}} \cdot\cdot\ \dot{\boldsymbol{\varepsilon}}^{\mathrm{el}} + \frac{\partial f}{\partial \theta}\dot{\theta} + \frac{\partial f}{\partial \boldsymbol{\Upsilon}_i} \odot \dot{\boldsymbol{\Upsilon}}_i$$

and finally we get the following dissipative inequality

$$\underline{\left(\boldsymbol{\sigma} - \rho\frac{\partial f}{\partial \boldsymbol{\varepsilon}^{\mathrm{el}}}\right) \cdot\cdot\, \dot{\boldsymbol{\varepsilon}}^{\mathrm{el}}} + \boldsymbol{\sigma} \cdot\cdot\, \dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}} - \underline{\rho\left(s + \frac{\partial f}{\partial \theta}\right)\dot{\theta}} - \rho\frac{\partial f}{\partial \boldsymbol{\Upsilon}_i} \odot \dot{\boldsymbol{\Upsilon}}_i + \frac{1}{\theta}(\boldsymbol{\kappa} \cdot \boldsymbol{\nabla}\theta) \cdot \boldsymbol{\nabla}\theta \geq 0 \tag{111}$$

$\odot$ is the scalar product of tensors of an arbitrary order. The underlined terms in the inequality (111) are discussed in Sect. 4.3. If the thermoelastic strains are independent, for the stresses can be assumed

$$\boldsymbol{\sigma} = \rho\frac{\partial f}{\partial \boldsymbol{\varepsilon}^{\mathrm{el}}} \tag{112}$$

This yields

$$s = -\frac{\partial f}{\partial \theta} \tag{113}$$

Equations (112) and (113) describe the thermoelastic state of the materials. This state is free from dissipation. From the dissipative inequality follows

$$\boldsymbol{\sigma} \cdot\cdot\, \dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}} - \rho\frac{\partial f}{\partial \boldsymbol{\Upsilon}_i} \odot \dot{\boldsymbol{\Upsilon}}_i + \frac{1}{\theta}(\boldsymbol{\kappa} \cdot \boldsymbol{\nabla}\theta) \cdot \boldsymbol{\nabla}\theta \geq 0 \tag{114}$$

The first two terms describe the mechanical dissipation, the last one—the thermal.

Let us assume the existence of a scalar dissipation potential and the thermal and mechanical dissipation can be decoupled. We suggest that the dissipation potential must be convex. For the mechanical dissipation we obtain

$$\chi = \chi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}}, \dot{\boldsymbol{\Upsilon}}_i)$$

If we can assume an associated law (normality rule)

$$\boldsymbol{\sigma} = \frac{\partial \chi}{\partial \dot{\boldsymbol{\varepsilon}}^{\mathrm{pl}}}$$

and

$$\boldsymbol{\Lambda}_i = \frac{\partial \chi}{\partial \dot{\boldsymbol{\Upsilon}}_i}$$

$\boldsymbol{\Lambda}_i$ are the associated to the inner variables functions.

5 Governing Equations of Mechanics of Hyperelastic Materials

A material is named hyperelastic, if an elastic potential u exists and u is a scalar function of the strain tensor. This energy is preserved as the result of the loading process. If we have only elastic behavior (we ignore, for example, viscoelastic behavior, creep

or damage) this energy is reservable. In addition in the further developments we ignore the influence of the temperature. For rubber-like materials the elastic strains are sometimes more than 1,000 %. Limiting ourselves by the isotropy assumption, the nonlinear elasticity theory should be applied.

Note that in the case of finite strains we must distinguish the reference and the actual configurations. That means the constitutive equations can be presented in both configurations. Below we present all equations and variables in the reference configuration (Lagrangian description).

The starting point for the kinematics are the considerations of Sect. 1. The deformable body is a part of the three-dimensional space. The position of the material points in the reference configuration is given by the radius-vector $\boldsymbol{X}$, in the actual configuration by the radius-vector $\boldsymbol{x}$. The displacement vector can be presented by $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$. The deformation gradient $\boldsymbol{F}$ is defined by Eq. (7). In the nonlinear elasticity the right Cauchy-Green tensor $\boldsymbol{C}$ (15) can be used as strain measure and the respective strain tensor $\boldsymbol{G}$ is the Green-Lagrange tensor (16).

In the case of rubber-like materials the incompressibility assumption often is applied. The physical meaning of this assumption is that the volumetric strains can be neglected in comparison with the strains responsible for the shape changes. The incompressibility condition is equivalent to the following equation

$$J \equiv \det \boldsymbol{F} = 1. \tag{115}$$

Let us introduce the following invariants

$$I_1 = \operatorname{tr} \boldsymbol{C}, \quad I_2 = \frac{1}{2}\left[\operatorname{tr}^2 \boldsymbol{C} - \operatorname{tr} \boldsymbol{C}^2\right], \quad I_3 = \det \boldsymbol{C} = J^2. \tag{116}$$

For incompressible materials we get $I_3 = 1$. The invariants I_k can be expressed with the help of the eigen-values λ_i^2 of the Cauchy-Green strain measure $\boldsymbol{C}$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{117}$$

In linear elasticity the split of the strain tensor into spherical and deviatoric parts is used. The first one is responsible for the volumetric strains, the second one—for the deviatoric (isochoric) strains, which have no relations to the changes of the volume. In non-linear elasticity if we have finite strains the isochoric strains can be presented by the normalized strain gradient $\bar{\boldsymbol{F}} = J^{-1/3}\boldsymbol{F}$, which results in $\det \bar{\boldsymbol{F}} = 1$. In addition, one can introduce the normalized Cauchy-Green strain measure

$$\bar{\boldsymbol{C}} = \bar{\boldsymbol{F}}^{\mathrm{T}} \cdot \bar{\boldsymbol{F}}. \tag{118}$$

and instead of the invariants I_1, I_2, I_3-the normalized invariants of $\bar{\boldsymbol{F}}$

$$\bar{I}_1 = J^{-1/3} I_1, \quad \bar{I}_2 = J^{-2/3} I_2, \quad J. \tag{119}$$

The invariants (119) are used in formulation of constitutive equations for weak compressible materials. They can be expressed by the eigen-values $\bar{\lambda}_i^2$ of the strain measure of $\bar{\boldsymbol{C}}$

$$\bar{I}_1 = \bar{\lambda}_1^2 + \bar{\lambda}_2^2 + \bar{\lambda}_3^2, \quad I_2 = \bar{\lambda}_1^2\bar{\lambda}_2^2 + \bar{\lambda}_2^2\bar{\lambda}_3^2 + \bar{\lambda}_1^2\bar{\lambda}_3^2. \tag{120}$$

The constitutive equations of a non-linear elastic body can be introduced with the help of the specific strain energy (related to the volume in the reference configuration) W as a function of the Cauchy-Green strain measure or tensor

$$u = u(\boldsymbol{C}) = u(\boldsymbol{G}) \tag{121}$$

For finite strains the $\boldsymbol{C}$ is more used than $\boldsymbol{G}$. In the case of isotropic materials W is a function of the invariants of $\boldsymbol{C}$ and we have the representation by I_k or λ_k

$$u = u(I_1, I_2, I_3) = u(\lambda_1, \lambda_2, \lambda_3) = u(\bar{I}_1, \bar{I}_2, J) \tag{122}$$

Introducing u the stress tensors can be computed as

- the Piola stress tensor

$$\boldsymbol{P} = 2\boldsymbol{F} \cdot \frac{\partial u}{\partial \boldsymbol{C}}, \tag{123}$$

- or the Cauchy stress tensor (tensor of true stresses)

$$\boldsymbol{T} = 2J^{-1}\boldsymbol{F} \cdot \frac{\partial u}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^{\mathrm{T}}. \tag{124}$$

In the case of incompressible materials the stress tensors can be estimated from the strain energy only for terms which are related to isochoric strains. That means the following expressions are valid for

- the Piola stress tensor

$$\boldsymbol{P} = -p\boldsymbol{F}^{-1} + 2\boldsymbol{F} \cdot \frac{\partial u}{\partial \boldsymbol{C}}, \tag{125}$$

- for the Cauchy stress tensor

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\boldsymbol{F} \cdot \frac{\partial u}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^{\mathrm{T}}. \tag{126}$$

Here $\boldsymbol{I}$ is the unit tensor and p is a unknown function. The physical meaning of the last one is similar to the pressure in an incompressible fluid. From the point of view of mathematics this is a Lagrangian multiplier related to Eq. (115).

Taking into account the equation for the derivatives of the invariants I_k

$$\frac{\partial I_1}{\partial \boldsymbol{C}} = \boldsymbol{I}, \quad \frac{\partial I_2}{\partial \boldsymbol{C}} = \boldsymbol{I}\, I_1 - \boldsymbol{C}, \quad \frac{\partial I_3}{\partial \boldsymbol{C}} = I_3\boldsymbol{C}^{-1}, \tag{127}$$

we get

$$\frac{\partial u}{\partial \boldsymbol{C}} = (u_{,1} + I_1 u_{,2})\boldsymbol{I} - u_{,2}\boldsymbol{C} + I_3 u_{,3}\boldsymbol{C}^{-1}, \quad u_{,k} \equiv \frac{\partial u}{\partial I_k} \tag{128}$$

Finally, for the isotropic incompressible material the Cauchy stress tensor (124) can be expressed by

$$\boldsymbol{T} = 2J^{-1}\left[(u_{,1} + I_1 u_{,2})\boldsymbol{B} - u_{,2}\boldsymbol{B}^2 + I_3 u_{,3}\boldsymbol{I}\right], \tag{129}$$

where $\boldsymbol{B} = \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F}$ is the left Cauchy-Green tensor or Finger strain measure (15).

It can be shown [32] that the principal stresses σ_k (eigen-values of the Cauchy stress tensor $\boldsymbol{T}$) for isotropic materials can be estimated by derivatives of u w.r.t. λ_k

$$\sigma_k = J^{-1}\lambda_k \frac{\partial u}{\partial \lambda_k}. \tag{130}$$

Note that sometimes the principal forces t_k are applied. They are related to σ_k by the following equation

$$t_k = \frac{J}{\lambda_k}\sigma_k \tag{131}$$

and we get a very simple relation between t_k and u

$$t_k = \frac{\partial u}{\partial \lambda_k} \tag{132}$$

The physical meaning of σ_k and t_k are as follows. Let us introduce a infinitesimal cube in the volume, which is oriented in such a manner that the normals to surfaces have the same direction as the eigen-vectors of $\boldsymbol{T}$ (principal axes). Than σ_k are forces related to the surfaces in the actual configuration and t_k are forces related to surfaces in the reference configuration. In the case of finite strains both quantities can differ significantly. In the case of isotropic incompressible materials the Cauchy stress tensor (126) can be computed as

$$\boldsymbol{T} = -p\boldsymbol{I} + 2(u_{,1} + I_1 u_{,2})\boldsymbol{B} - 2u_{,2}\boldsymbol{B}^2, \tag{133}$$

the principal stresses σ_k and principal forces t_k as

$$\sigma_k = -p + \lambda_k \frac{\partial u}{\partial \lambda_k}, \quad t_k = -\frac{p}{\lambda_k} + \frac{\partial u}{\partial \lambda_k}. \tag{134}$$

6 Constitutive Equations of Incompressible Materials

The incompressibility assumption is often used in modeling of highly elastic materials. The strain should be a function of the first and the second invariant of the strain measure $\boldsymbol{C}$

$$u = u(I_1, I_2). \tag{135}$$

6.1 Polynomial Approximation

This model suggested in [33] is based on the Taylor series expansion of the strain energy u in the neighborhood of the undeformed state

$$u = \sum_{m=0}^{M} \sum_{n=0}^{N} C_{mn} (I_1 - 3)^m (I_2 - 3)^n \tag{136}$$

C_{mn} are material parameters with $C_{00} = 0$.

6.2 Treloar (Neo-Hookean) Material

The Treloar material [34] is an example of Eq. (136). This model is based on the Gaussian kinetic theory for rubber-like materials. The following constitutive equation is suggested

$$u = C_1(I_1 - 3), \quad C_1 > 0. \tag{137}$$

The model is very simple and some vulcanized rubbers with organic fillers can be presented at moderate strains.

6.3 Mooney Material

The model of the Mooney material [35] can be deduced from (136) if one preserves only the linear term

$$u = C_1(I_1 - 3) + C_2(I_2 - 3), \quad C_1 > 0, \quad C_2 \geq 0. \tag{138}$$

It is easy to show that the Treloar model is a special case of the Mooney material. Equation (138) allows to describe the behavior of some natural and vulcanized rubbers in a wide range. But in the case of strains higher 400–500 % the description is

dissatisfying. Together with the Ogden material the Mooney material is mostly used in applications. Some generalizations of Eq. (138) are known.

6.4 Rivlin-Saunders Material

Rivlin and Saunders [36] generalized the Mooney material

$$u = C_1(I_1 - 3) + F(I_2 - 3), \tag{139}$$

where $F(I_2 - 3)$ is an arbitrary function, which should be estimated for the given material. As usual for $F(I_2 - 3)$ are proposed polynomial approximations, for example, Klosner and Segal introduced a cubic polynomial for $(I_2 - 3)$

$$u = C_1(I_1 - 3) + C_2(I_2 - 3) + C_3(I_2 - 3)^2 + C_4(I_2 - 3)^3. \tag{140}$$

6.5 Biderman Model

Biderman [37] suggested the following function for the strain energy for gray rubber

$$u = C_1(I_1 - 3) + B_1(I_1 - 3)^2 + B_2(I_1 - 3)^3 + C_2(I_2 - 3). \tag{141}$$

6.6 Non-Polynomial Approximation

Not in all notations for u the function $F(I_2 - 3)$ in (139) is included as a polynomial of $(I_2 - 3)$. For example, in [38] an exponential-hyperbolic constitutive equation is suggested

$$u = C\left[\int e^{k_1(I_1-3)^2}\mathrm{d}I_1 + k_2 \ln\frac{I_2}{3}\right]. \tag{142}$$

Alexander introduced [39]

$$u = C_1(I_1 - 3) + C_2(I_2 - 3) + C_3 \ln\frac{I_2 - 3 + k}{k} \tag{143}$$

and

$$u = C_1\int e^{k_1(I_1-3)^2}\mathrm{d}I_1 + C_2(I_2 - 3) + C_3 \ln\frac{I_2 - 3 + k_2}{k_2} \tag{144}$$

with $C, C_1, C_2, C_3, k, k_1, k_2$ as material parameters. Equation (143) is equivalent to the Rivlin-Saunders Eq. (139) and (144) summarizes elements of (139) and (142). The last one describes the behavior of natural rubber. The last example is the strain function proposed by Hutchinson, Becker and Landel [40]

$$u = C_1(I_1 - 3) + B_1(I_1 - 3)^2 + B_2(1 - e^{k_1(I_2-3)}) + B_3(1 - e^{k_2(I_2-3)}). \quad (145)$$

describing the behavior of silicon rubber in one-and two-axial tests.

6.7 Incompressible Ogden Material

Ogden [41] suggested a constitutive equation for the strain energy u, based on λ_1, λ_2, λ_3

$$u = \frac{2\mu}{\alpha}(\lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha - 3) = \frac{2\mu}{\alpha}\left[\operatorname{tr} \boldsymbol{C}^{\alpha/2} - 3\right]. \quad (146)$$

The generalization of this two-parameter model (146)

$$u = \sum_{n=1}^{N} \frac{2\mu_n}{\alpha_n}(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3) = \sum_{n=1}^{N} \frac{2\mu_n}{\alpha_n}(\operatorname{tr} \boldsymbol{C}^{\alpha_n/2} - 3). \quad (147)$$

is also called Ogden material. Similar to the polynomial approximation (136) model (147) contains enough material parameters for the fitting of experimental data. At the same time it is difficult to perform the large number of tests.

6.8 Chernykh-Shubina Material

Independent from the Ogden material model the two-parameter model of Chernykh and Shubina [42, 43] was created

$$\begin{aligned} u &= \mu\left[(1+\beta)(\lambda_1 + \lambda_2 + \lambda_3 - 3) + (1-\beta)(\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} - 3)\right] \\ &= \mu\left[(1+\beta)(\operatorname{tr} \boldsymbol{C}^{1/2} - 3) + (1-\beta)(\operatorname{tr} \boldsymbol{C}^{-1/2} - 3)\right]. \end{aligned} \quad (148)$$

6.9 Bartenev-Khazanovich Material

In the mechanics of polymers the one-parameter model of Bartenev-Khazanovich [44], which is a special case of the Ogden material or the Chernykh-Shubina model,

is established

$$u = 2\mu(\lambda_1 + \lambda_2 + \lambda_3 - 3) = 2\mu(\mathrm{tr}\, \boldsymbol{C}^{-1/2} - 3). \tag{149}$$

In the literature the constitutive Eq. (149) is sometimes named Varga material [45].

7 Constitutive Equations of Compressible Materials

The incompressibility condition is often assumed in modeling of rubber-like materials. In some cases this approximation is not acceptable. For example, the deformation problem of high-elastic damping material between two rigid plates can be solved only if the weak compressibility of the material is taken into account. In the literature such models for weak compressible materials are introduced which are based on the aforementioned constitutive equations for incompressible materials adding terms accounting the changes of the specific volume of the material. In addition, there are models for compressible materials based on experimental data and not using the hypothesis on weak compressibility.

7.1 Compressible Neo-Hookean Material

This model for weak compressible materials is a generalization of the Treloar material. The constitutive equation has the form

$$u = C_1(\bar{I}_1 - 3) + \frac{1}{d}(J - 1)^2, \tag{150}$$

where d is a material parameter characterizing the compressibility. In the case of small strains the modulus of compressibility k for the material (150) is related to d by the equation $k = 2/d$. For incompressible materials $d = 0$ is valid and the last term in (150) is the penalty, presenting the contribution of the volumetric strains to the strain energy.

7.2 Mooney-Rivlin Material

The polynomial approximation (136) can be generalized for the case of materials with small compressibility, for example elastomers. The generalization is given as additional terms depending on J in the strain energy

$$u = \sum_{m=0}^{M} \sum_{n=0}^{N} C_{mn} (\bar{I}_1 - 3)^m (\bar{I}_2 - 3)^n + \sum_{n=1}^{N} \frac{1}{d_n} (J-1)^{2n}. \tag{151}$$

The special case of (151) is the two-parameter Mooney-Rivlin material

$$u = C_{10}(\bar{I}_1 - 3) + C_{01}(\bar{I}_2 - 3) + \frac{1}{d}(J-1)^2. \tag{152}$$

In FE codes polynomial approximations are implemented, which contain a big amount of material parameters, for example the three-parameter Mooney-Rivlin material

$$u = C_{10}(\bar{I}_1 - 3) + C_{01}(\bar{I}_2 - 3) + C_{11}(\bar{I}_1 - 3)(\bar{I}_2 - 3) + \frac{1}{d}(J-1)^2, \tag{153}$$

the five-parameter Mooney-Rivlin material

$$\begin{aligned} u = {} & C_{10}(\bar{I}_1 - 3) + C_{01}(\bar{I}_2 - 3) + C_{20}(\bar{I}_1 - 3)^2 \\ & + C_{02}(\bar{I}_2 - 3)^2 + C_{11}(\bar{I}_1 - 3)(\bar{I}_2 - 3) + \tfrac{1}{d}(J-1)^2, \end{aligned} \tag{154}$$

or the nine-parameter Mooney-Rivlin material

$$\begin{aligned} u = {} & C_{10}(\bar{I}_1 - 3) + C_{01}(\bar{I}_2 - 3) + C_{20}(\bar{I}_1 - 3)^2 + C_{02}(\bar{I}_2 - 3)^2 \\ & + C_{11}(\bar{I}_1 - 3)(\bar{I}_2 - 3) + C_{30}(\bar{I}_1 - 3)^3 + C_{21}(\bar{I}_1 - 3)^2(\bar{I}_2 - 3) \\ & + C_{12}(\bar{I}_1 - 3)(\bar{I}_2 - 3)^2 + C_{03}(\bar{I}_2 - 3)^3 + \frac{1}{d}(J-1)^2 \end{aligned} \tag{155}$$

7.3 Compressible Ogden Material

Model (147) can be used for materials with small incompressibility

$$u = \sum_{n=1}^{N} \frac{2\mu_n}{\alpha_n} (\bar{\lambda}_1^{\alpha_n} + \bar{\lambda}_2^{\alpha_n} + \bar{\lambda}_3^{\alpha_n} - 3) + \sum_{n=1}^{N} \frac{1}{d_n} (J-1)^{2n} \tag{156}$$

7.4 Arruda-Boyce Material

The potential in this case takes the form [46]

$$u = \mu \left[\frac{1}{2}(\bar{I}_1 - 3) + \frac{1}{20\lambda_L^2}(\bar{I}_1^2 - 9) + \frac{11}{1050\lambda_L^4}(\bar{I}_1^3 - 279) + \frac{19}{7000\lambda_L^6}(\bar{I}_1^4 - 81) + \frac{519}{673750\lambda_L^8}(\bar{I}_1^5 - 243) \right] + \frac{1}{d}\left(\frac{J^2 - 1}{2} - \ln J \right) \tag{157}$$

Here μ, λ_L, d are the material parameters. With $\lambda_L \to \infty$ we get again the Neo-Hookean material.

7.5 Gent Material

For this material the following potential holds

$$u = -\frac{E I_m}{6} \log\left(1 - \frac{\bar{I}_1 - 3}{I_m}\right) \tag{158}$$

Here E, I_m are material parameters.

7.6 Blatz-Ko Material

This model was suggested by Blatz and Ko [47] for the case of modeling of several rubbers (porous materials—foams—made of polyurethane (PU)). The potential has the form

$$u = \frac{1}{2}\mu\beta\left[I_1 + \frac{1}{\alpha}(I_3^{-\alpha} - 1) - 3\right] \frac{1}{2}\mu(1-\beta)\left[I_2 I_3^{-1} + \frac{1}{\alpha}(I_3^{\alpha} - 1) - 3\right],$$
$$\alpha = \frac{\lambda}{2\mu} \tag{159}$$

This model contains 3 parameters: μ, α and β. In FE codes a simpler potential is implemented (159)

$$u = \frac{1}{2}\mu\left[\frac{I_2}{I_3} + 2\sqrt{I_3} - 5\right] \tag{160}$$

7.7 Other Material Models

In the literature there are given references to other compressible non-linear elastic materials (Seth, Signorini [48], Murnaghan [49], harmonic (semi-linear), etc.)

[32, 49]. The material parameters for these models are known. Applications of these models are related, for example to acoustics.

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