A Survey on m-Asynchronous Cellular Automata*

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Abstract. The paper after briefly surveying main asynchronous models in cellular automata will report recent developments in the study of m-ACA, a new general framework for studying asynchrony in cellular spaces.

1 Introduction

Last twenty years witnessed the rapid growth of the domain of complex systems both from the theoretical and the application point of view. In parallel, a number of formal models were developed in order to reproduce their behavior and, possibly, deduce general properties from the formal models using a number of knowledge domains going from theoretical computer science to mathematics.

Cellular automata play a central role in this context because of their three main characteristics: locality, uniformity, synchronicity. (which is present in most complex systems at the point that is often taken as a definition of complex system): the emergence of a complex collective behavior starting from local interactions between simple individuals.

A cellular automaton consists in an infinite set of identical finite automata arranged on a regular lattice ($\mathbb{Z}$ in this article). Each finite automaton takes its state from a finite set S, called the set of *states* or the *alphabet*. The state is updated according to a *local rule* λ which take into account the state of the automaton and the one of a fixed finite neighborhood of neighboring automata. All automata in the lattice are updated in parallel.

This simple definition of the model contrasts the huge variety of long-term dynamical behaviors which has attracted the attention of many researchers (see [25, 13, 17, 12, 1] for recent results and a comprehensive bibliography). At the same time, this great variety of behaviors qualifies CA as very useful models in applications [26, 11, 10, 19, 3, 6, 32, 18].

Applications also motivated the introduction of a number of variants of CA model. Each new model is meant to highlight peculiar properties. For example, this paper focus on asynchrony. This last property turns out to be interesting in

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a number of different context ranging from modeling chemical reactions in living cells, to asynchronous computation and communication in distributed systems, and so on. As non-uniformity [16], asynchrony can also be useful to introduce inside CA more realistic features as noise [31].

The first part of the paper briefly survey the most known manners of dealing with asynchrony in CA. We do absolutely not pretend to be exhaustive or complete. We just review those models that motivate the studies reported in the rest of the paper. This second part of the paper introduces a new model for asynchrony which aims at generalizing existing ones (at some extent) and at the same time to offer enough theoretical "hooks" to be able to significantly analyze the long-term behavior. The basic idea is to augment the classical CA model with a measure μ over the integers. At each time step, a set of integers τ (finite or infinite) is extracted according to μ. The elements of τ are the indexes of the sites that are allowed to be updated, the cells with index in $\mathbb{Z} \setminus \tau$ leave their state unchanged. We call this new model m-ACA. Clearly, some work has to be done at the formal level to adapt the existing definitions to take into account the fact that in the new situation one works with family of functions and not with a single function (*the global function*) like in the classical setting. This have also to be combined with the fact of drawing sites to be updated using the measure μ.

After briefly surveying models related to m-ACA that can be found in literature, the paper reviews main results and ideas about m-ACA exploring both the dynamics and some set theoretic properties. The final section contains the seeds for a new research program which we believe will illustrate the "usefulness" of this new model in the study of the asynchrony in cellular automata.

2 The General Framework

Notation. For all $i, j \in \mathbb{Z}$ with $i \leq j$ (resp., $i < j$) let $[i, j] = \{k \in \mathbb{N} | \, i \leq k \leq j\}$ (resp., $[i, j) = \{k \in \mathbb{N} | \, i \leq k < j\}$). The set of positive integers (resp., reals) is denoted by $\mathbb{N}_+$ (resp., $\mathbb{R}_+$). Given a set X, $\mathcal{P}(X)$ denotes the collection of subsets of all X.

Let S be a finite alphabet. A *configuration* is a function from $\mathbb{Z}$ to S. The *configuration set* $S^{\mathbb{Z}}$ is usually equipped with the metric d defined as follows:

$$\forall x, y \in S^{\mathbb{Z}} \; d(x, y) = 2^{-n} \text{ , where } n = \min\{i \in \mathbb{N} \mid x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$$

The set $S^{\mathbb{Z}}$ is a Cantor space i.e., a compact, totally disconnected and perfect topological space. For any pair $i, j \in \mathbb{Z}$, with $i \leq j$, and any configuration $x \in S^{\mathbb{Z}}$ we denote by $x_{[i,j]}$ the word $x_i \cdots x_j \in S^{j-i+1}$. Similarly, for every $u \in S^{\ell}$ and for every $i, j \in [0, \ell)$, $u_{[i,j]} = u_i \ldots u_j$ is the portion of a word inside $[i, j]$. In both the previous notations, $[i, j]$ can be replaced by $[i, j)$, with the obvious meaning. A configuration x is said to be a-finite for some $a \in S$ if the number of positions i with $x_i \neq a$ is finite.

Formally, a (one-dimensional) CA is a structure (S, λ, r) where S is the *alphabet* or *set of states*, $r \in \mathbb{N}$ is the *radius*, and $\lambda : S^{2r+1} \to S$ is the *local rule* of the automaton. The local rule λ induces a *global rule* $F : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ which describes the new global state of the CA after one time step

$$\forall x \in S^{\mathbb{Z}},\ \forall i \in \mathbb{Z}, \qquad F(x)_i = \lambda(x_{i-r}, \ldots, x_i, \ldots, x_{i+r}) \ .$$

The *space-time diagram* of initial configuration c can be represented by a bi-infinite figure, where for the sake of simplicity we set $c^t(j) := F^t(c)_j$:

$$\left|\begin{array}{c|cccccccc|l} t=0 & \ldots & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & \ldots & = c \\ t=1 & \ldots & c^1_{-2} & c^1_{-1} & c^1_0 & c^1_1 & c^1_2 & \ldots & = F(c) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t & \ldots & c^t_{-2} & c^t_{-1} & c^t_0 & c^t_1 & c^t_2 & \ldots & = F^t(c) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right|$$

Space-time diagrams are a nice visual tool that might sometimes provide intuitions on the dynamical behavior or emergent phenomena. In Section 4.4 we will see that this is exactly the case.

An *activation function* υ i.e., a function from $\mathbb{N}$ to $\mathcal{P}(\mathbb{Z})$ is our main tool to control synchronicity. Indeed, at any time step $t \in \mathbb{N}$, $\upsilon(t)$ tells which sites are active and must be updated; all other cells are left unchanged. Therefore, one can redefine the global function at time $t \in \mathbb{N}$ using υ as follows.
For time step $t = 0$,

$$\forall x \in S^{\mathbb{Z}},\ \forall i \in \mathbb{Z},\ F_\upsilon(x)_{0,i} = x_i \ .$$

For time step $t > 0$, define $\forall x \in S^{\mathbb{Z}}$, $\forall i \in \mathbb{Z}$

$$F_\upsilon(x)_{t+1,i} = \begin{cases} \lambda(F_\upsilon(x)_{t,i-r}, \ldots, F_\upsilon(x)_{t,i}, \ldots, F_\upsilon(x)_{t,i+r}) & \text{if } i \in \upsilon(t+1) \ , \\ F_\upsilon(x)_{t,i} & \text{otherwise} \ . \end{cases}$$

Remark that choosing υ such that $\forall t \in \mathbb{N}, \upsilon(t) = \mathbb{Z}$, it means that all cells are updated at each time step i.e., we recover the classical CA setting. Summing up, one can give the following formal definition.

Definition 1 (ACA). *An Asynchronous Cellular Automaton (ACA) is structure $(S, \lambda, r, \upsilon)$ where (S, λ, r) is a CA and υ is an activation function.*

The main novelty with ACA vs. CA is that in ACA one does no more have a single global function but there is a family of global functions. This implies that all notions concerning dynamical behavior have to be adapted to work with family of functions.

Let $\mathcal{T}$ be a *monoid of continuous functions* from $S^{\mathbb{Z}}$ to $S^{\mathbb{Z}}$ where Id denotes the *identity* map on $S^{\mathbb{Z}}$. The family $\mathcal{T}$ is said to be *sensitive to initial conditions* (or, simply, *sensitive*) if there exists $\varepsilon > 0$ such that for any $x \in S^{\mathbb{Z}}$ and any $\delta > 0$, there is an element $y \in S^{\mathbb{Z}}$ with $d(x, y) < \delta$ such that $d(T(x), T(y)) \geq \varepsilon$ for some $T \in \mathcal{T}$. Furthermore, the family $\mathcal{T}$ is said to be *positively expansive* (or, briefly, *expansive*) if there exists a constant $\varepsilon > 0$ such that for every pair of distinct elements $x, y \in S^{\mathbb{Z}}$, we have $d(T(x), T(y)) \geq \varepsilon$ for some $T \in \mathcal{T}$.

Sensitivity and expansivity are elements of instability for a system whose dynamics are described by the family $\mathcal{T}$. The following notions instead refer to elements of stability for $\mathcal{T}$.

A configuration $x \in S^{\mathbb{Z}}$ is said to be an *equicontinuity point* for $\mathcal{T}$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall y \in S^{\mathbb{Z}}$, $d(x, y) < \delta$ implies that $\forall T \in \mathcal{T}$, $d(T(x), T(y)) < \varepsilon$. The family $\mathcal{T}$ is *equicontinuous* if every configuration is an equicontinuity point for $\mathcal{T}$ or, equivalently, $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall x, y \in S^{\mathbb{Z}}$, $d(x, y) < \delta$ implies that $\forall T \in \mathcal{T}$, $d(T(x), T(y)) < \varepsilon$. The family $\mathcal{T}$ is said to be *almost equicontinuous* if the set E of all equicontinuity points for $\mathcal{T}$ is residual (i.e., E contains a countable intersection of open dense subsets).

The family of functions $\mathcal{T}_\nu$ induced by an ACA with activation function ν defined as follows

$$\mathcal{T}_\nu = \bigcup_{t \in \mathbb{N}} \{F_\nu(\cdot)_t\}$$

An ACA is sensitive (resp. expansive) (resp. equicontinuous) (resp., almost equicontinuous) iff its induced family of functions is sensitive (resp. expansive) (resp. equicontinuous) (resp., almost equicontinuous).

Modifying the activation function one can also introduce in a natural way the notion of non-determinism, simply saying that υ is defined from $\mathbb{N}$ to $\mathcal{P}(\mathcal{P}(\mathbb{Z}))$. However, even if this subject will not be developed in this paper, it suggests the idea that different choices of ν may bring to distinct models of asynchronism. The next sections, briefly review the most known ones that can be found in literature.

3 Asynchrony in Cellular Automata Literature

This section briefly surveys models and results from recent. Literature is really huge and it cannot be exhaustively reported in these few pages. We have chosen to survey only those models that inspired us directly the pathway to m-ACA model.

3.1 Fully Asynchronous CA

One of the possible approaches to asynchrony is to assume that *two updates never happen at the same time*. This means that only one cell updates at every time step. The resulting dynamics is determined not only by the local rule of the automaton, but also from an updating function whose image contains only singletons.

Definition 2. *A* fully-ACA *is a quadruple* $C_\upsilon = (S, \lambda, r, \upsilon)$ *where* S *is a finite set called the* alphabet*,* $r \in \mathbb{N}$ *is the* radius*,* $\lambda : S^{2r+1} \to S$ *is the* local rule*, and* υ *is the activation function. For fully-ACA the function* υ *respect the following property:*

$$\forall t \in \mathbb{N} \qquad \upsilon(t) = \{i\}$$

That is, exactly one cell is updated at every time step.

It is interesting to consider the behaviour of a fully-ACA when the sequence is not fixed. In fact, we can consider a family of fully-ACA defined in the following way:

$$C = \{C_\upsilon \mid \forall t \in \mathbb{N}\ |\upsilon(t)| = 1\}$$

A family of fully-ACA is characterized by the triple (S, λ, r) of the alphabet, local rule, and radius in common between all the fully-ACA of the family. In this way we can differentiate between properties that holds only when a particular updating function is selected from the ones that can hold independently from the particular choice of cells to be updated.

A first property that was studied is the relation between injectivity and surjectivity. A family C of fully-ACA is α-injective (resp. α-surjective) when every fully-ACA $C_\upsilon \in C$ is injective (resp. surjective). Contrarily to classical CA, those properties are equivalent for fully-ACA.

Proposition 1 ([28]). *Let* $C = (S, \lambda, r)$ *be a family of fully-ACA. Then the following statements are equivalent:*

1. *C is α-injective;*
2. *C is α-surjective;*
3. *λ is center-permutative.*

The equivalence of two "global" properties with a "local" property is a recurring theme for fully-ACA. In fact permutativity appears to be a sufficient condition for obtaining many interesting dynamical behaviours.

Dynamical Properties. Classical properties that are interesting to study in this new setting are sensitivity, expansivity and transitivity. It is easy to see that for any class C of fully-ACA there exists at least one activation function υ whose corresponding fully-ACA C_υ is not sensitive (resp. expansive) (resp. transitive). Hence, we say that a family of fully-ACA $C = (S, \lambda, r)$ is α-sensitive (resp. α-expansive) (resp. α-transitive) it there exists an activation function υ for which the fully-ACA $C_\upsilon = (S, \lambda, r, \upsilon)$ is sensitive (resp. expansive) (resp. transitive).

A first link was found between the presence of a leftmost or rightmost local rule and sensitivity.

Proposition 2 ([28]). *Let* $C = (S, \lambda, r)$ *be a family of fully-ACA with* $r > 0$*. If* λ *is either leftmost or rightmost permutative then* C *is* α*-sensitive.*

It is important to point out that, like for classical CA [7], leftmost and rightmost permutativity are only sufficient but not necessary conditions to obtain α-sensitivity. When both leftmost and rightmost permutativity are present the dynamical behavior changes and expansivity is obtained.

Proposition 3 ([28]). *Let $C = (S, \lambda, r)$ be a family of fully-ACA with $r > 0$. If λ is both leftmost and rightmost permutative then C is α-expansive.*

It is interesting to note that there exist rightmost permutative local rules that give α-sensitivity but not α-expansivity.

Example 1. Let $C = (\{0, 1\}, \lambda, 1)$ be a family of fully-ACA with λ the shift rule (i.e., $\forall a, b, c \in \{0, 1\}$, $\lambda(a, b, c) = c$. This rule is rightmost permutative and, by Proposition 2, C is α-sensitive. However, given two configurations $x, y \in S^{\mathbb{Z}}$ with $d(x, y) = \delta$ and with all the differences between x and y in negative position, independently of the updating function chosen the distance between the orbits of the two configurations cannot grow larger than δ.

Transitivity, like sensitivity, only requires permutativity either in the leftmost or the rightmost position.

Proposition 4 ([28]). *Let $C = (S, \lambda, r)$ be a family of fully-ACA with $r > 0$. If λ is either leftmost or rightmost permutative then C is α-transitive.*

Remark 1. For sensitivity and transitivity there is an easy necessary condition that can be used to decide quickly if a fully-ACA $C_\upsilon = (S, \lambda, r, \upsilon)$ is not sensitive and transitive. To obtain both sensitivity and transitivity the updating function must be such that $\bigcup_{i\in\mathbb{N}} \upsilon(i)$ is infinite (i.e., it is a subset of $\mathbb{Z}$ that is unbounded either in the positive or in the negative values).

To obtain expansivity the necessary condition is more stringent: $\bigcup_{i\in\mathbb{N}} \upsilon(i)$ must be unbounded in both the negative and the positive values.

A first idea to reduce the dependence of the dynamics of the particular sequence chosen was to find if the property defined by an activation function was *stable*. A property was defined to be stable for an activation function υ if the property also holds for all activation function υ' such that $\upsilon(t) = \upsilon'(t)$ on all but a finite number of elements of $\mathbb{N}$.

In [28], it has been proved that there exists a sequence for which sensitivity is a stable property. Similar results holds for expansivity and transitivity.

Turing Completeness. Another question that was investigated in the fully-ACA setting was to establish whenever the model allows computations to be performed and the power of such a computation both in term of Turing-completeness [29] and in term of slowdown of the simulation of a Turing Machine.

It is possible to use the lattice $S^{\mathbb{Z}}$ as the tape of a Turing machine in which, in addition to the symbol written in a cell of the tape, the state of the machine and some control information are encoded.

An activation function v respect is called *universal* when:

$$\forall i \in \mathbb{N} \qquad |\{t \in \mathbb{N} \mid v(t) = i\}| = \infty$$

That is, when every cell is updated infinitely many time. When v satisfies such a property, it is possible to simulate a Turing Machine. However, depending on the particular activation function, the simulation can be arbitrarily slow. In [14], it has been proved that the simulation of a Turing Machine working in time $T(n)$ can be simulated in time $O(T(n)^2)$, hence, differently from other models like register machines, the simulation is not exponentially slower but, in some sense, it is "fast". For further computation aspects of asynchronous CA we address the reader to [27, 33].

3.2 Stochastic Fully Asynchronous CA

Even if it is interesting to study the dynamics of fully-ACA when the updating function is fixed, when modeling real-life processes there is almost always a stochastic component involved. Therefore, the model of fully-ACA has been extended by choosing the cell to be updated by means of a stochastic process. An *Elementary Cellular Automaton* (ECA) is a (one-dimensional) CA with radius 1 and set of states $S = \{0, 1\}$. An ECA of local rule λ is *doubly quiescent* if $\lambda(0, 0, 0) = 0$ and $\lambda(1, 1, 1) = 1$. In [20], Fates *et al.* studied doubly quiescent ECA over finite rings of size n (with periodic boundary conditions). The authors devised the following update policy. At each time step $t \in \mathbb{N}$ an integer $i \in \{0, \ldots, n-1\}$ is drawn with uniform probability and they set $\nu(t) = i$. A complete classification of the expected convergence time (when a convergence to a configuration was possible) towards a fixed point (i.e., a configuration consisting of either all 0 or of all 1). The classification consists in the following seven classes.

Class	**Behavior**	**Conv. time**
I	Identity	0
II	Coupon collector	$\Theta(n \ln n)$
III	Monotonic	$\Theta(n^2)$
IV	Biased random walk	$\Theta(n^2)$
V	Random walk	$\Theta(n^3)$
VI	Biased random walk	$\Theta(n2^n)$
VII	Divergent	Divergent

3.3 α-Asynchronous CA

A slight relaxation of the asynchrony condition of fully-ACA lead to the notion of α-asynchronous CA (α-ACA). Every cell has a (not necessarily fair) coin that is tossed at every time step to decide if the cell has to update or not. The type of coin is fixed for every cell. That is, a certain value $\alpha \in (0, 1)$ is chosen and every cell updates with probability α (and remain unchanged with probability $1 - \alpha$).

Like in the case of stochastic ACA, the study of α-ACA has been carried on the restricted class of doubly quiescent ECA with focus on convergence time towards a fixed point.

Theorem 1 ([21]). *The behaviour of* 52 *of the* 64 *different doubly quiescent ECA under* α*-asynchronous dynamics is the following:*

- 48 *ECA converge to a random fixed point from any initial configuration with a time dependent from* α *that is one of the following:* 0, $\Theta\left(\frac{\log n}{\log(1-\alpha)}\right)$, $\Theta\left(\frac{n}{\alpha} + \frac{1}{\alpha(1-\alpha)}\right)$, $O\left(\frac{n}{\alpha(1-\alpha)}\right)$, $O\left(\frac{n}{\alpha^2(1-\alpha)}\right)$, $\Theta\left(\frac{n^2}{\alpha(1-\alpha)}\right)$;
- *Two of them diverges;*
- *Two of them converges with a small probability if the length of the automaton is even and diverges otherwise.*

The classification of the behavior of the last 12 doubly quiescent ECA remains an open problem. We found this work interesting and deep. What follows in this article is an attempt to give a more general setting and provide new tools in order to solve some of these open questions.

Among the open questions raised in [21], one concerns the analysis of the time needed for a finite configuration to converge to a stable configuration (i.e., a fixed point) and more in particular, if there exists an α-asynchronous CA with a *phase transition* between a polynomial and an exponential convergence time. This question has been solved very recently in [30].

4 m-ACA

More general forms of asynchrony should involve more complex updating sequences in which possibly infinite sets of cells are updated at each time step and the there are correlations between updated sites. If the formal modeling of such general systems is easy, one cannot say the same thing about the analysis of the long-term behavior. Therefore, one should tradeoff between full generality, maximal non-determinism and capability of analysis. In this section, we propose to constrain non-determinism using probability measures over the set of integers (i.e. over the set of cells that should be updated in parallel at each time step). Therefore, the new model is nothing but a classical CA with the addition of a probability measure μ which is used to extract the set of cells to update. More formally,

Definition 3 (m-ACA). *An m-ACA* $\mathcal{C}$ *is a quadruple* (S, r, λ, μ) *where* S *is a finite alphabet,* $r > 0$ *is the* radius, $\lambda : S^{2r+1} \to S$ *is the* local rule *and* μ *is a probability measure on the Borel* σ*-algebra on* $\mathcal{P}(\mathbb{Z})$.

Given the measure μ, we say that the *activation function* ν *is generated by* μ if for all $t \in \mathbb{N}$, the set $\nu(t) \in \mathcal{P}(\mathbb{Z})$ is extracted using μ. Therefore, F_ν is the global function of the m-ACA.

Denote by $\mathcal{S}$ the set of all activation functions. In the model proposed in this paper, μ is used to extract the subset of $\mathbb{Z}$ indicating which cells are allowed to be updated. At each time step, a new extraction is performed and we made the hypothesis that extractions are independent. Therefore, it is natural to consider the product measure μ_s of the measure μ to measure sets of activation functions i.e., subsets of $\mathcal{S}$ (μ_s always exists and is unique, see [22, Thm. B, pag. 157]).

Consider the power set of integers $\mathcal{P}(\mathbb{Z})$ ordered by set inclusion. Then, a *filter* on $\mathcal{U}$ on $\mathcal{P}(\mathbb{Z})$ is a subset of $\mathcal{P}(\mathbb{Z})$ such that $\mathbb{Z} \in \mathcal{U}$, $A \cap B \in \mathcal{U}$ for any $A, B \in \mathcal{U}$ and $\emptyset \notin \mathcal{U}$; moreover it has the upward closure property i.e., if $A \in \mathcal{U}$ and $A \subseteq B$, then $B \in \mathcal{U}$. A set $\mathcal{U} \in \mathcal{P}(\mathbb{Z})$ is an *ultrafilter* if it is a filter and for any $A \subseteq \mathcal{P}(\mathbb{Z})$ it follows that either $A \in \mathcal{U}$ or $(\mathcal{P}(\mathbb{Z}) \setminus A) \in \mathcal{U}$. For $i \in \mathbb{Z}$, denote $\mathcal{U}_i$ the *principal ultrafilter* of element i i.e., the collection of all subset of $\mathcal{P}(\mathbb{Z})$ containing the integer i.

We stress that each cell $i \in \mathbb{Z}$ is updated with a probability given by $\mu(\mathcal{U}_i)$.

4.1 Fair and Quasi-Fair Measures

As we have already said, we are interested in studying m-ACA where the probability measure associated has some interesting properties. First of all, all cell should have the a non-zero probability of being updated. This is a necessary requirement for allowing the information exchange within region of the cellular space and hence allow the computing (in the Turing sense) capabilities [14]. For similar reasons, no cell should be updated with probability 1. Moreover, the event "update all cells" should have zero probability since, more or less, this corresponds to turn back to the classical CA model. Indeed, in order to totally avoid mimicking the classical model, we require even more, all events concerning an infinite number of cells should have zero probability. More formally,

Definition 4. *A probability measure μ over a σ-algebra of $\mathcal{P}(\mathbb{Z})$ is* fair *if it satisfies the following properties:*

1. $\forall i \in \mathbb{Z}$, $0 < \mu(U_i) < 1$,
2. $\forall I \in \mathcal{P}(\mathbb{Z})\ (|I| < \infty) \Rightarrow \mu\left(\bigcap_{i\in I} U_i\right) = \prod_{i\in I} \mu(\mathcal{U}_i)$,
3. $\forall I \in \mathcal{P}(\mathbb{Z})\ (|I| = +\infty) \Rightarrow \mu\left(\bigcap_{i\in I} U_i\right) = 0$,

where U_i is the ultrafilter $\mathcal{U}_i$ or its complement.

The second condition in the definition of fair measure simply tells that cells update independently. Remark that this condition is not sufficient to avoid the extremal cases discussed in the introduction to this section. The following example shows that the necessity of the third requirement on fair measures to avoid the possibility of having infinite sets of integers with positive measure.

Example 2. Consider the measure $\mu : \mathcal{P}(\mathbb{Z}) \to [0, 1]$ defined as follows:

$$\forall i \in \mathbb{Z},\ \mu(\mathcal{U}_i) = \frac{i}{c(i)}$$

where $c(i) = i + 1$ if $i \equiv 3 \mod 4$, $i - 1$ otherwise. Consider the set $I \subseteq \mathbb{N}$ of odd prime integers. Then,

$$\mu\left(\bigcap_{i\in I}\mathcal{U}_i\right) = \prod_{i\in I}\mu\left(\mathcal{U}_i\right) = \prod_{i\in I}\frac{i}{c(i)} = \frac{\pi}{4}.$$

Remark that for any set $S \in \mathcal{P}(\mathbb{Z})$, if the cardinality of S is finite then

$$0 < \mu\left(\bigcap_{i\in I\setminus S}\mathcal{U}_i\right) < 1 \ .$$

The following shows how to deduce a fair measure from a Bernoulli measure over $\{0, 1\}$.

Example 3. Consider the Bernoulli measure β over $\{0, 1\}$ such that $\beta(1) = a$ and $\beta(0) = 1 - a$. It is not difficult to verify that the measure μ_β defined as $\forall i \in \mathbb{Z}$, $\mu_\beta(\mathcal{U}_i) = a$ is fair. We call μ_β the *Bernoulli fair measure* induced by β.

Clearly, any m-ACA induced by a Bernoulli fair measure is an α-asynchronous CA and vice versa. It is also clear that the class of m-ACA is strictly bigger than the one of α-asynchronous CA since not all fair measures are Bernoulli fair measures.

Fair measures have some interesting properties which reveal very useful when studying m-ACA dynamics. First of all, the measure of an ultrafilters or of complements of ultrafilters is uniformly bounded. More formally, in [15, Lemma 1 and Remark 5], it is proved the following.

Proposition 5. *For any fair measure μ there exist two constants ϵ, ξ such that*

$$\forall I \in \mathcal{P}(\mathbb{Z})\,,\ 0 < \epsilon < \mu\left(\bigcap_{i\in I}U_i\right) < \xi < 1$$

where U_i is the ultrafilter $\mathcal{U}_i$ or its complement and $|I| < \infty$.

Fair measures are a pretty large class but it is still not clear how large it is the class of measures that respects the design principle that we have discussed at the beginning of the section. A first step in this direction consists in considering measures that behave much like a fair measure in the sense that they have the same set of null measure. Before giving the formal definition we need some preliminary definition.

Given a sigma-algebra Σ over $\mathbb{Z}$, $\mathcal{M}_\Sigma$ denotes the set of all measures over Σ.

Definition 5. *A function $f : \mathcal{M}_\Sigma \to \mathcal{M}_\Sigma$ is* zero-preserving *if*

$$\forall \mu \in \mathcal{M}_\Sigma,\ \forall A \in \Sigma \qquad \mu(A) = 0 \iff \left(f(\mu)\right)(A) = 0 \ .$$

Denote by $\mathfrak{Z}$ the set of all zero-preserving functions from $\mathcal{M}_\Sigma$ to itself.

In other words, a zero-preserving function takes a measure μ over Σ into another measure μ' over Σ such they have the same sets of null measure (and of course the same sets of full measure). The idea is that if μ is fair then μ' is not far from being fair since it will satisfy at least conditions 1 and 3 of fair measures. Indeed, we can give the following

Definition 6. *A* quasi-fair measure *is the image of a fair measure under a zero-preserving function. Let* $\mathfrak{A}_{\text{QFAIR}}$ *be the set of all quasi-fair measures.*

Given a set of measures $M \subseteq \mathcal{M}_\Sigma$, the *zeta-closure* $\mathfrak{Z}(M)$ of M is the set of measures which are image of some measure in M via a zero-preserving function, more formally

$$\mathfrak{Z}(M) = \bigcup_{f\in\mathfrak{Z}} \bigcup_{\mu\in M} \{f(\mu)\} \ .$$

Since the composition of two zero-preserving functions is a zero-preserving function, the class of quasi-fair measures is closed under composition by zero-preserving functions.

Proposition 6. $\mathfrak{Z}(\mathfrak{A}_{\text{QFAIR}}) = \mathfrak{A}_{\text{QFAIR}}$.

The above proposition tells us that the class of $\mathfrak{A}_{\text{QFAIR}}$ measures is the largest class of measures that one can obtain by using zero-preserving functions but it does say anything if there are other possibilities for extending the class of measures that respect the defining principles discussed so far. However, $\mathfrak{A}_{\text{QFAIR}}$ is large enough to allow to take into account complex dependencies between cells update since the condition 2 of fair measures is no more necessarily satisfied. The following example shows that there exists a quasi-fair measure which is not fair and hence $\mathfrak{A}_{\text{QFAIR}}$ is a real extension of $\mathfrak{A}_{\text{FAIR}}$.

Example 4. Choose $\epsilon \in (0,1)$ and define μ as follows:

$$\forall I \in \mathcal{P}(\mathbb{Z})\,,\ \mu\left(\bigcap_{i\in I} U_i\right) = \begin{cases} \frac{\epsilon^n}{n} & \text{if } n = |I| < \infty \text{ and } I \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where U_i is the ultrafilter $\mathcal{U}_i$ or its complement. Clearly, μ is not fair since it does not satisfy condition 2 of fair measures. Consider the Bernoulli fair measure μ_ϵ and define a function f as follows

$$\forall \xi \in \mathcal{M}_\Sigma,\ f(\xi) = \begin{cases} \mu & \text{if } \xi = \mu_\epsilon \\ \xi & \text{otherwise} \end{cases}$$

Since f is zero-preserving μ is a quasi-fair measure.

4.2 Set Theoretic Properties

Set theoretic properties like surjectivity and injectivity are very important when studying the dynamics since they are often necessary condition for the presence of a given dynamical behavior. For example, injectivity is a necessary and sufficient condition for reversibility [23], while surjectivity is often necessary for the presence of chaotic behavior [8, 1, 13] and strictly related to other dynamical properties [9, 2]. Moreover, both surjectivity and injectivity are one of the most well-known of dimension sensitive properties. Indeed, they are decidable in dimension one [4] and undecidable for dimension two or grater [24].

Definition 7. *An m-ACA $C = (S, \lambda, r, \mu)$ is surjective (resp. injective) iff for all activation functions ν, $F_\nu(\cdot)_1$ is surjective (resp. injective). C is μ-*almost surely *surjective (resp., injective) iff*

$$\mu\left(\{\nu(1) \mid F_\nu(\cdot)_1 \textit{ is surjective (resp. injective)}\}\right) = 1 \ .$$

Permutativity is an easy-to-verify combinatorial property which is strictly connected with surjectivity and injectivity.

Definition 8. *A CA local rule $\lambda : \{0,1\}^{2r+1} \to \{0,1\}$ is center-permutative if and only if for all $(x_1, \ldots, x_r), (y_1, \ldots, y_r) \in \{0,1\}^r$ it holds*

$$\lambda(x_1, \ldots, x_r, 0, y_1, \ldots, y_n) \neq \lambda(x_1, \ldots, x_r, 1, y_1, \ldots, y_n) \ .$$

An m-ACA is center-permutative if its local rule is center-permutative.

The following results links all the three notions introduced above.

Theorem 2 ([15]). *For any measure $\mu \in \mathfrak{A}_{\mathrm{QFAIR}}$ and for any m-ACA $\mathcal{C} = (S, \lambda, r, \mu)$, the following statements are equivalent:*

1. *$\mathcal{C}$ is μ-almost surely surjective;*
2. *$\mathcal{C}$ is μ-almost surely injective;*
3. *$\mathcal{C}$ is center-permutative.*

Example 5. The shift map σ is a bijective CA. For any measure $\mu \in \mathfrak{A}_{\mathrm{QFAIR}}$, the m-ACA version $(\{0,1\}, 1, \sigma, \mu)$ is not surjective. Indeed, consider an activation function ν such that $\nu(1) = \{i\}$ for some $i \in \mathbb{Z}$ and a configuration $y \in \{0,1\}^{\mathbb{Z}}$ such that $y_i = 0$ and $y_{i+1} = 1$. Then, any possible pre-image x should have $x_{i+1} = 1$ since the site $i+1$ is not updated but this implies $y_i = 1$ contradicting the former hypothesis. By Theorem 2, the shift map is not even μ-almost surely surjective since it is not center-permutative.

Example 6. Consider the xor CA $\mathcal{C} = (\{0,1\}, 1, \lambda)$ with local rule defined as follows

$$\forall x, y, z \in \{0,1\},\ \lambda(x, y, z) = y \oplus z$$

where $\oplus$ is the usual xor operation. It is well-known that $\mathcal{C}$ is surjective but not injective. Given a measure $\mu \in \mathfrak{A}_{\mathrm{QFAIR}}$, its m-ACA version $\mathcal{C} = (\{0,1\}, 1, \lambda, \mu)$ is μ-almost surely surjective since λ is center-permutative. Indeed, this m-ACA is surjective. Given any activation function ν and any configuration y, let us build a pre-image x such that $F_\nu(x)_1 = y$. We build the part with positive index, the negative one is very similar. At stage $i = 0$, define $x_0 = y_0$ if $\nu(1)_0 = 0$, otherwise let $x_0 = a$ and $x_1 = (1-a) \cdot y_0 + a \cdot (1 - y_0)$ for $a \in \{0,1\}$. At stage n, we have two cases

1. x_n has been defined at the previous stage. If $\nu(1)_n = 0$ then leave x_n unchanged. If $\nu(1)_n = 1$ then let $x_{n+1} = (1 - x_n) \cdot y_n + x_n \cdot (1 - y_n)$.
2. x_n has not been defined at the previous stage. Define $x_n = y_n$ if $\nu(1)_n = 0$, otherwise let $x_n = a$ and $x_{n+1} = (1 - a) \cdot y_n + a \cdot (1 - y_n)$ for $a \in \{0, 1\}$.

By compactness, the process described above completely constructs the pre-image x.

From the previous simple examples we deduce that the situation about surjectivity and injectivity is quite different from classical CA. However, from the decidability point of view nothing changes like it is stated by the following.

Proposition 7 ([15]). *For any measure $\mu \in \mathfrak{A}_{\text{QFAIR}}$, μ-almost surely surjectivity is decidable for one-dimensional CA and undecidable in greater dimensions.*

4.3 About the Dynamical Behavior

This section surveys the (still few) knonw dynamical properties of m-ACA. Some of the notions concerning families of global functions have been introduced in the previous sections, here also the measure of the set of activation functions giving rise to a certain behavior is taken into account. Results and examples in this section are taken directly from [15].

Definition 9. *Consider an m-ACA $\mathcal{C} = (S, \lambda, r, \mu)$, a real number $p \in [0, 1]$, and an activation function $\upsilon \in \mathcal{P}(\mathbb{Z})^{\mathbb{N}}$ generated by μ. The m-ACA $\mathcal{C}$ is said to be either p–*equicontinuous *or p–*almost equicontinuous *or p–*sensitive *or p–*expansive *if $\mu_s(\Upsilon) = p$, where Υ is the set of all sequence υ with respect to which $\mathcal{C}$ has that behavior.*

The remainder of the this section focuses on the situations when the above dynamical properties happen *almost surely*, i.e., when $p = 1$.

Example 7. Let λ_σ be the local rule of the classical CA shift map σ. The m-ACA $\mathcal{C} = (\{0, 1\}, \lambda_\sigma, 1, \mu)$ is almost surely sensitive. Indeed, consider any sequence υ with the following property: for all $n \in \mathbb{N}$ there exists a time t such that $n - i \in \upsilon_{t+i}$ for each $i \in [0, n]$, i.e., each cell of position $n, n-1, \ldots, 0$ is updated respectively at time $t, t+1, \ldots, t+n$. Now, for any such a sequence υ, any configuration x, and any integer $n \in \mathbb{N}$, consider the configuration y such that $y_{[-n,n]} = x_{[-n,n]}$ and $y_i \neq x_i$ for every $i > n$. So, the $(n+t)$–th element T of the family $\mathcal{T}_\upsilon$ is such that $T(y)_0 \neq T(x)_0$. Thus, $\mathcal{T}_\upsilon$ is sensitive or, in other words, $\mathcal{C}$ is sensitive w.r.t. υ. Furthermore, by the second Borel-Cantelli Lemma, one finds that the set of all the updating sequences υ with the above property has measure equal to one. Therefore, $\mathcal{C}$ is almost surely sensitive.

Definition 10. *Consider an m-ACA $\mathcal{C} = (S, \lambda, r, \mu)$ and let υ be an activation function generated by μ. A word $w \in S^k$ is s-*blocking w.r.t. *υ for some integer $s \in [1, k]$ if there exists an offset $j \in [0, k - s]$ such that*

$$\forall i \in \mathbb{Z},\ \forall x, y \in [w]_i,\ \forall T \in \mathcal{T}_\upsilon, \quad T(x)_{[i+j,i+j+s)} = T(y)_{[i+j,i+j+s)} \tag{1}$$

Let $p \in \mathbb{R}$ with $0 \le p \le 1$. A word $w \in S^k$ is said to be (p, s)-blocking *for some integer $s \in [1, k]$ if $\mu_s(\Upsilon) = p$, where Υ is the set of activation functions w.r.t. which w is s-blocking.*

Example 8. Let $\mathcal{C} = (\{0, 1\}, \lambda, 1, \mu)$ be an m-ACA where λ is the majority rule, i.e., $\lambda(a, b, c) = \lfloor (a+b+c)/2 \rfloor$. The word $w = 00$ is 2-blocking w.r.t. any sequence $\upsilon \in \mathcal{P}(\mathbb{Z})^{\mathbb{N}}$. In fact, we have that for all $a \in \{0, 1\}$, $\lambda(a, 0, 0) = \lambda(0, 0, a) = 0$. That is, w remains unchanged w.r.t. all possible activation functions and then it is a $(1, 2)$-blocking word.

In order to state that a word w is blocking w.r.t. a given activation function υ, Condition (1) from Definition 10 prescribes that the equality holds independently of where w is placed inside configurations. The fact that the equality holds for some positions does not imply that it is also true for all other positions as it is illustrated by the following example.

Example 9. Let $\mathcal{C}$ be the m-ACA of Example 8. Consider the activation function $\upsilon = (\mathbb{Z} \setminus \{0, 1\}, \mathbb{Z} \setminus \{0, 1\}, \ldots)$. The word $w = 01$ satisfies (1) only for $i = 0$.

Given an m-ACA and a word $w \in \{0, 1\}^*$, there can be activation functions w.r.t. which w is blocking and others w.r.t. which w is not.

Example 10. Let $\mathcal{C} = (S, \lambda, r, \mu)$ be any m-ACA which is υ-sensitive for some $\upsilon \in \mathcal{P}(\mathbb{Z})^{\mathbb{N}}$. Clearly $\mathcal{C}$ admits no blocking word w.r.t. υ. However, any word is blocking w.r.t. the activation function $(\emptyset, \emptyset, \ldots)$.

Given an m-ACA $\mathcal{C} = (S, \lambda, r, \mu)$ and an activation function $\upsilon \in \mathcal{P}(\mathbb{Z})^{\mathbb{N}}$, denote by E_υ the set of all equicontinuity points for the family $\mathcal{T}_\upsilon$. Recall that in the classical CA setting, almost equicontinuity is characterized by the presence of a r-blocking word. Concerning the m-ACA context, this can be rephrased as follows: there exists a (p, r)-blocking word $\Leftrightarrow$ the m-ACA is p-almost equicontinuous. Proposition 8 shows a strong result for the left-to-right implication, namely, there exists a residual set of points which are equicontinuity points w.r.t. all activation functions in a set of μ_s-measure p. The opposite implication is not yet completely understood.

Proposition 8. *Consider an m-ACA $\mathcal{C} = (S, \lambda, r, \mu)$ with $\mu \in \mathcal{M}_\Sigma$. If $\mathcal{C}$ admits a (p, r)-blocking word for some $p \in [0, 1]$, then there exists a subset $\Upsilon \subseteq \mathcal{P}(\mathbb{Z})^{\mathbb{N}}$ with $\mu_s(\Upsilon) = p$ such that the set $\bigcap_{\upsilon \in \Upsilon} E_\upsilon$ is residual.*

Corollary 1. *Consider an m-ACA $\mathcal{C} = (S, \lambda, r, \mu)$ with $\mu \in \mathcal{M}_\Sigma$. If $\mathcal{C}$ admits a (p, r)-blocking word w for some $p \in [0, 1]$, then $\mathcal{C}$ is p-almost equicontinuous.*

The following result is a first step towards a possible proof that p-almost equicontinuous m-ACA admit a (p, r)-blocking word.

Proposition 9. *Consider an m-ACA $\mathcal{C} = (S, \lambda, r, \mu)$ with $\mu \in \mathcal{M}_\Sigma$. Consider the set Υ of all sequences $\upsilon \in \mathcal{P}(\mathbb{Z})^{\mathbb{N}}$ w.r.t. which $\mathcal{C}$ admits an r-blocking word. Then, the set $\bigcap_{\upsilon \in \Upsilon} E_\upsilon$ is residual.*

The following proposition is a further witness that the new setting gives a new genuine model which is quite different from classical CA. Indeed, in classical CA, expansive CA are surjective but not injective [5].

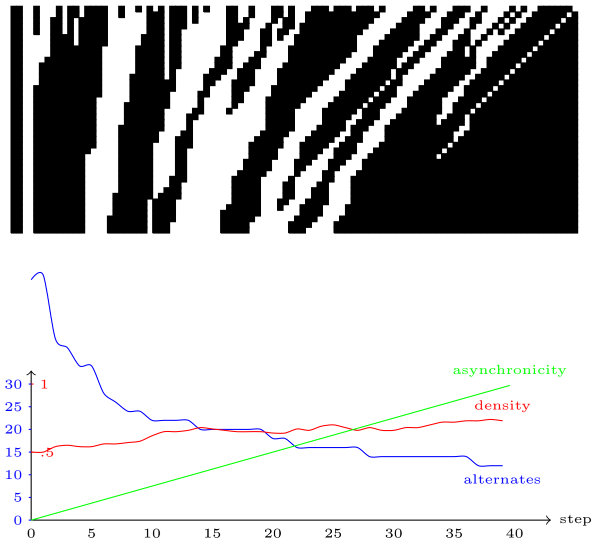
Proposition 10. *Consider an m-ACA* $\mathcal{C} = (S, \lambda, r, \mu)$ *with* $\mu \in \mathfrak{A}_{\text{QFAIR}}$. *If* $\mathcal{C}$ *is almost surely expansive then* $\mathcal{C}$ *is* μ*-almost surely injective.*

4.4 Experiments

In order to explore m-ACA dynamics further one can turn to experiments in the hope to have new intuitions. At present we are going to experiment with only one m-ACA, namely, the shift map σ and try to analyze both the effect of different measures μ and of initial measures with which the initial configuration is extracted. Three classes of experiments are reported in this section. Each class of experiments is illustrated by two figures, a space-time diagram and a quantitative diagram. In the space-time diagram time goes downward, the state 1 is represented by a black box, 0 by a white one. The quantitative diagram reports the value of 3 curves. The density of ones w.r.t. zeroes (colored in red) and the number of alternates (colored in blue) during the evolution of the m-ACA. An alternate is the boundary between a sequence of cells in state 1 and a sequence of cells in state 0, or viceversa. This can be quickly computed counting the number of patterns 01 or 10 in the current configuration. The third curve is the value of the measure μ (colored in green). Remark that this time the a value i on the x axis represents the ultrafilter $\mathcal{U}_i$ and the value on the y axis is the measure of $\mathcal{U}_i$. The value of μ are then repeated periodically with period 100 and rescaled to fit the same range with the other two curves.

From the first experiment we can see that the m-ACA behaves more or less like a shift CA on the right part of the space-time diagram and like the identity in the leftmost part of the space-time diagram. This agrees with the distribution of values of μ.

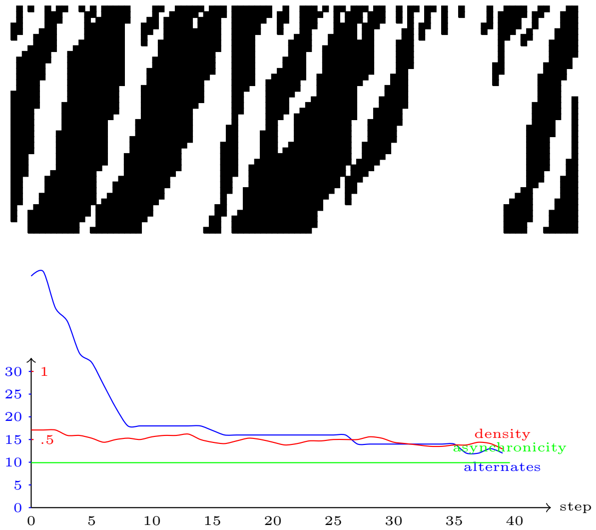
From the second experiment, we deduce that now the m-ACA behaves more like a shift. Sort initial segments of zeroes or ones tend to disappear rather quickly and a (slow) process of homogenization seems to start. This is confirmed by the curve of alternates. Remark that this curve seems to stabilize. Indeed, more experiments (not reported here for lack of space) confirmed that the curve continue decreasing during time although very slowly. In the third experiment, the initial measure with which the initial configuration is extracted has been changed so to produce a large majority of ones, μ is the same as in the first experiments. Again, we experience the rapid decrease of alternates in the first part of the evolution and it become slower and slower when time grows. Summing up all the three experiments showed that there is some kind of homogenization process that takes place. The curve of alternates seem to confirm it and to illustrate that the speed of homogenization depends more on μ than on the measure with which the initial configuration is extracted.



Experiment 1. An evolution of the shift m-ACA. The measure μ takes values in $[0.01, 0.99]$, grows linearly with i between 0 and 100 and then it repeats periodically. The initial configuration is extracted using a uniform measure over $\{0, 1\}$.

4.5 Exploring a New Research Direction

This section is going to explore an new research direction suggested from previous experiments. We shall concentrate on the evolutions of alternates. Assume that the current configuration contains the pattern $x_1 x_2 \dots x_6 = 111000$ which contains one alternate and consider a generic local rule λ of a CA of radius 1 such that $\lambda(111) = 1$ and $\lambda(000) = 0$ (i.e., it is a doubly quiescent ECA). Let us try to understand what can be the possible images of 111000 under each possible updating policy. Assume that x_2 as to be updated then, according to λ, its image is 1. Remark that if x_2 is not updated, its image is also going to be 1. The same reasoning can be applied to x_5, this cell is going to conserve the state 0 independently of the updating policy. Consider now to update x_3, its new value depends on $\lambda(110)$. While it would have been 1 in case of no update. Similarly, the new value of x_4 is zero if there are no updates, $\lambda(100)$ otherwise. Therefore according the to the values of λ and to the update policy, the alternate can change of position, stay or even give birth to a new alternate. Figure 1, illustrates all the possible cases according to the update policy and the probability with which they might happen if one assumes that $\alpha = \mu\left(\mathcal{U}_{x_3}\right)$ and $\beta = \mu\left(\mathcal{U}_{x_4}\right)$



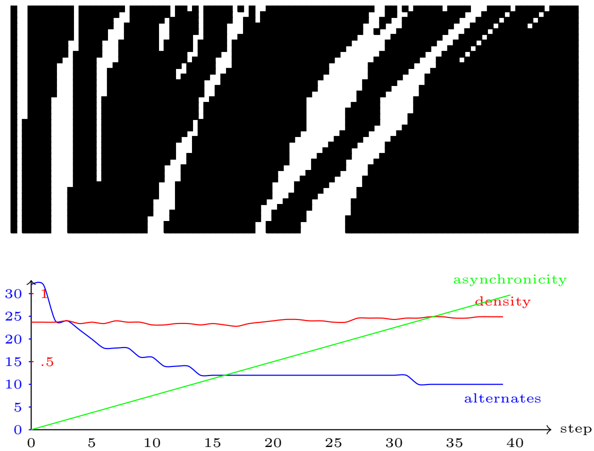
Experiment 2. Another evolution of the shift m-ACA. The measure μ takes constant value .33. The initial configuration is extracted using a uniform measure over $\{0, 1\}$.

Figure 2 specializes Figure 1 to the case of the shift map and shows how alternates have moved. Since all updates are independent it is easy to see that in this case

$$\begin{aligned} \mathbb{P}\,(\text{the alternate moves to the left}) &= \alpha \\ \mathbb{P}\,(\text{the alternate does not move}) &= 1-\alpha \\ \mathbb{P}\,(\text{the alternate moves to the right}) &= 0 \end{aligned}$$

Therefore, in this case, the probability of moving for an alternate does not depend on β but only on α. A *segment* is the set of cells between two successive alternates. Assume that the successive alternate to the right w.r.t. to the one we have considered above is between sites y and $y+1$. Moreover, assume that $\gamma = \mu\,(\mathcal{U}_y)$. Clearly, the length of the segment makes a biased random walk according to the updating probabilities of the alternates by which it is defined.

Figure 3 illustrates the state graph of the random-walk with the corresponding transitions probabilities which can be easily computed from α and γ. Remark that the state 0 is absorbing, indeed, the shift map cannot create new alternates and therefore when a segment disappears, it is forever. Disappearance of short segments is precisely what we have remarked in the experiments of the previous section. At this point one might try to observe the density of segments to determine if the homogenization process that seems to take place in experiments can be expressed formally.



Experiment 3. Another evolution of the shift m-ACA. The measure μ is like in the first experiment. The initial configuration is extracted using a Bernoulli measure over $\{0, 1\}$ of ratio 0.8.

Denote $X_n(c)$ the random variable representing the number of alternates in the configuration c for cells with index between $-n$ and n divided by $2n + 1$. The density of alternates for a configuration c is given by

$$\delta(c) = \limsup_{n\to\infty} X_n(c)$$

Of course, what interests us is understanding the behavior of δ along orbits of the shift map, in other words we need to study the random process $\{X_n^n(c)\}_{t\geq 0}$ in which X_n^n represents the value of X_n after n iterations of the shift m-ACA started on the configuration c. Since there are no alternates creation, it is clear that

$$\mathbb{E}\left[X_{n+1}^{n+1} \mid X_n^n, X_{n-1}^{n-1}, \ldots, X_0^0\right] \leq X_n^n$$

and hence $\{X_n^n(c)\}_{t\geq 0}$ is a super-martingale. Remark that for all $n \in \mathbb{N}$, $\mathbb{E}\left[X_n^n\right] \leq 1$. Then, by the convergence theorem for super-martingales one concludes that $\lim_{n\to\infty} X_n^n(c) = k$ for some real $k > 0$ for μ-almost all updating functions ν extracted using μ. Let us prove that $k = 0$. Indeed, since the process converges to k, for any $\epsilon > 0$ there must be a large enough $n \in \mathbb{N}$ such that $|X_n^n - k| < \epsilon$. Consider now all the segments of size $\ell < n$ in between the cells of index $-n$ and n. We have seen that the length of these segments perform a random walk with 0 as an absorbing state. It means that after a time large enough they will have disappeared with non-zero probability. Since the shift map cannot create new segments we have that the density must have decreased to some $k' < k$.

The long analysis above proves that for μ-almost all activation functions generated by μ, for η-almost all initial configurations c, there is a long-term homogenization process which turns c into a "mono-chromatic" configuration i.e.,

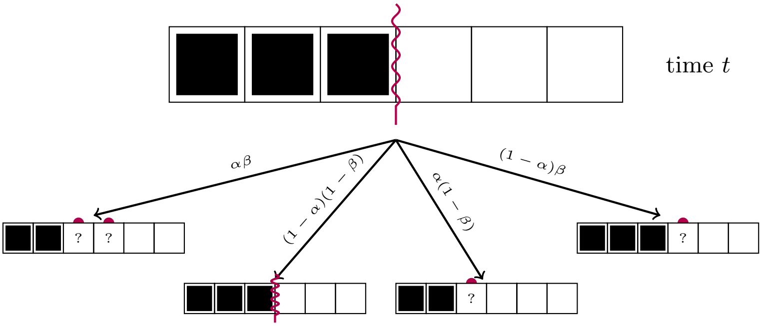


Fig. 1. Alternate dynamics according to a generic doubly quiescent ECA local rule λ. A question mark indicates that the new value depends on the local rule.

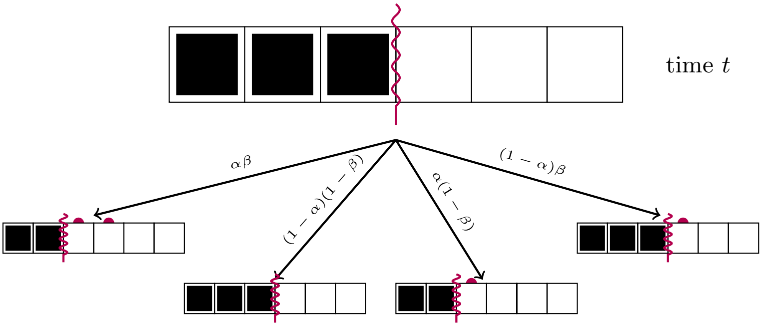


Fig. 2. Alternate dynamics according to the shift map. Remark that the alternate moves left or right according to the probabilities indicated on the arrows label.

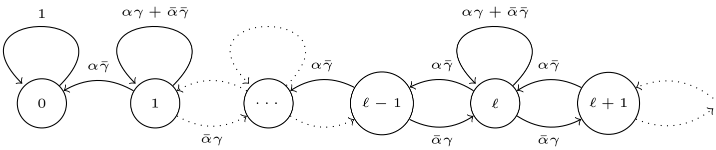


Fig. 3. The biased random walk characterizing segment length. Remark the absorbing state. The symbol $\bar{\xi}$ means $1 - \xi$.

a configuration with density of alternates equal to zero. We believe that similar formal tools and ideas can be applied successfully to all other doubly quiescent ECA extending the classification given in [21] to the the whole $\mathbb{Z}$.

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