Chapter 4
Duality Principle and Dual Simplex Method

The duality features a special relationship between a LP problem and another, both of which involve the same original data (A, b, c), located differently (except for the self-duality, see below). The former is referred to as *primal problem* while the latter as *dual problem*. It is important that there exists a close relationship between their feasible regions, optimal solutions and optimal values. The duality together with optimality conditions, yielding from it, constitute a basis for the LP theory. On the other hand, an economic interpretation of duality features its applications to practice. This chapter is devoted to these topics.

On the other hand, once any of primal and dual problems is solved, the problems are both solved due to duality. Thereby, a so-called *dual simplex method* will be derived by handling the dual problem in this chapter. Its tableau version will still proceed with the same simplex tableau.

From now on, "primal" will be added as a prefix, if necessary, to the simplex method and associated items to distinguish with their dual counterparts, introduced in this chapter.

4.1 Dual LP Problem

If the standard LP problem (1.8), i.e.,

$$(P) \qquad \begin{array}{ll} \min & f = c^T x, \\ \text{s.t.} & Ax = b, \qquad x \geq 0, \end{array} \tag{4.1}$$

is referred to as "primal", the following problem

$$(D) \qquad \begin{array}{ll} \max & g = b^T y, \\ \text{s.t.} & A^T y + z = c, \qquad z \geq 0, \end{array} \tag{4.2}$$

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constructed with the same data (A, b, c), is the "dual problem" of (4.1). There is 1–1 correspondence between variables of one of them and constraints of another.

Values of y, z, satisfying $A^T y + z = c$, are called *dual* solution. Set

$$D = \{(y, z) \in \mathcal{R}^m \times \mathcal{R}^n \mid A^T y + z = c,\ z \geq 0\}$$

is called *dual feasible region*, elements in which are *dual feasible solutions*.

It is clear that $\bar{y} = 0,\ \bar{z} = c$ is a dual solution; if, in addition, $c \geq 0$, it is a dual feasible solution. Given basis B, setting $z_B = 0$ in $B^T y + z_B = c_B$ gives

$$\bar{y} = B^{-T} c_B, \qquad \bar{z}_B = 0, \qquad \bar{z}_N = c_N - N^T \bar{y}, \tag{4.3}$$

called *dual basic solution*. $\bar{z}$ is just reduced costs; and $\bar{y}$ the simplex multiplier (see Note on Algorithm 3.5.1). If $\bar{z}_N \geq 0$, $(\bar{y}, \bar{z})$ is a dual basic feasible solution, corresponding to a vertex in D. For simplicity, thereafter $\bar{z}_N$ alone is often said to be dual basic solution.

The following alternative form of the dual problem (4.2):

$$\begin{array}{lll} (D) & \max & g = b^T y, \\ & \text{s.t.} & A^T y \leq c, \end{array} \tag{4.4}$$

is useful in some cases. Problems (4.2) and (4.4) will be regarded as the same.

As it can be converted to a standard one, any LP problem corresponds to a dual problem. By introducing slack variables $u \geq 0$, e.g., the problem

$$\begin{array}{ll} \max & c^T x, \\ \text{s.t.} & Ax \leq b, \quad x \geq 0, \end{array} \tag{4.5}$$

can be turned to the standard problem

$$\begin{array}{ll} \min & -c^T x, \\ \text{s.t.} & Ax + u = b, \quad x, u \geq 0, \end{array}$$

the dual problem of which is

$$\begin{array}{ll} \max & b^T y', \\ \text{s.t.} & \begin{pmatrix} A^T \\ I \end{pmatrix} y' \leq \begin{pmatrix} -c \\ 0 \end{pmatrix}. \end{array}$$

Setting $y = -y'$ turns the preceding to (4.5)'s dual problem below:

$$\begin{array}{ll} \min & b^T y, \\ \text{s.t.} & A^T y \geq c, \quad y \geq 0. \end{array}$$

Correspondence between primal and dual problems are summarized to the following table:

Primal problem		Dual problem	
Objective function	min	Objective function	max
Variables	Nonnegative nonpositive free	Constraints	$\leq$ $\geq$ $=$
Constraints	$\geq$ $\leq$ $=$	Variables	Nonnegative nonpositive free

Note: In applications of the preceding table, sign restriction is not handled as a constraint, but attributed to the associated variable

For example, the so-called "bounded-variable" LP problem

$$\begin{array}{ll} \min & c^T x, \\ \text{s.t.} & Ax = b, \quad l \leq x \leq u, \end{array} \tag{4.6}$$

can be transformed to

$$\begin{array}{llrcl} \min & c^T x, & & & \\ \text{s.t.} & Ax & & = & b, \\ & x + s & & = & u, \\ & -x & + t & = & -l, \\ & & s, t & \geq & 0. \end{array} \tag{4.7}$$

According the preceding table, the dual problem of (4.6) is

$$\begin{array}{ll} \max & b^T y - u^T v + l^T w, \\ \text{s.t.} & A^T y - v + w = c, \quad v, w \geq 0. \end{array} \tag{4.8}$$

The so-called *self-duality* referees to a special case when the dual problem of an LP problem is just itself. Combining the primal problem (4.1) and the dual problem (4.4), we construct the following problem:

$$\begin{array}{llll} \min & c^T x - b^T y, & & \\ \text{s.t.} & Ax & = b, & x \geq 0, \\ & A^T y & \leq c. & \end{array} \tag{4.9}$$

According the preceding table, the dual problem of it is

$$\begin{array}{lll} \max & b^T y + c^T v, & \\ \text{s.t.} & A^T y \qquad\quad \leq c, & \\ & \qquad\quad Av \;\; = -b, & v \leq 0. \end{array}$$

By setting $v = -x$ and handling the objective function properly, the preceding can be transformed to the original problem (4.9). Therefore, (4.9) is a self-dual problem.

4.2 Duality Theorems

This section only focuses on the duality of (P) and (D), as obtained results are valid for more general cases.

Theorem 4.2.1 (Symmetry). *The dual problem of a dual problem is the primal problem.*

Proof. Introduce slack variable $u \geq 0$ to dual problem (D), and make a variable transformation $y = y_1 - y_2$ to convert it to

$$\begin{array}{lll} \max & b^T(y_1 - y_2), & \\ \text{s.t.} & A^T(y_1 - y_2) + u = c, & y_1, y_2, u \geq 0, \end{array}$$

or equivalently,

$$\begin{array}{lll} \min & (-b^T, b^T, 0)(y_1^T, y_2^T, u^T)^T, & \\ \text{s.t.} & (A^T \vdots - A^T \vdots\, I)(y_1^T, y_2^T, u^T)^T = c, & y_1, y_2, u \geq 0. \end{array}$$

The dual problem of the preceding is

$$\begin{array}{ll} \max & c^T x', \\ \text{s.t.} & \begin{pmatrix} A \\ -A \\ I \end{pmatrix} x' \leq \begin{pmatrix} -b \\ b \\ 0 \end{pmatrix}, \end{array}$$

that is,

$$\begin{array}{lll} \max & c^T x', & \\ \text{s.t.} & Ax' = -b, & x' \leq 0, \end{array}$$

which becomes (P) by setting $x' = -x$. □

The preceding says that any of the primal and dual problems is the dual problem of the other. So, the two are symmetric in position. This is why any fact, holding for one of the primal and dual problems, has its counterpart for the other. It is of great importance that there is a close relationship between feasible or optimal solutions of the pair.

Theorem 4.2.2 (Weak duality). *If x and y are feasible solutions to primal and dual problems, respectively, then $c^T x \geq b^T y$.*

Proof. Premultiplying $c \geq A^T y$ by $x \geq 0$ gives $c^T x \geq y^T Ax$, substituting $b = Ax$ to which leads to $c^T x \geq b^T y$. □

According to the preceding, if there are feasible solutions to both primal and dual problems, any feasible value of the former is an upper bound of any feasible value of the latter, whereas any feasible value of the latter is a lower bound of any feasible value of the former.

Corollary 4.2.1. *If any of the primal and dual problems is unbounded, there exists no feasible solution to the other.*

Proof. By contradiction. Assume that there is a feasible solution to the dual problem. Then it follows from Theorem 4.2.2 that feasible values of the primal problem is bounded below. Analogously, if the primal problem is feasible, the dual problem is bounded above. □

Corollary 4.2.2. *Let $\bar{x}$ and $\bar{y}$ be feasible solutions to the primal and dual problems, respectively. If $c^T \bar{x} = b^T \bar{y}$, they are optimal solutions to the pair, respectively.*

Proof. According to Theorem 4.2.2, for any feasible solution x to the primal problem, it holds that $c^T x \geq b^T \bar{y} = c^T \bar{x}$, therefore $\bar{x}$ is an optimal solution to the primal problem. Analogously, $\bar{y}$ is an optimal solution to the dual problem. □

Theorem 4.2.3 (Strong duality). *If there exists an optimal solution to any of the primal and dual problems, then there exists an optimal one to the other, and the associated optimal values are equal.*

Proof. Assume that there is an optimal solution to the primal problem. According to Theorem 2.3.2, there is a basic optimal solution. Let B and N are optimal basis and nonbasis, respectively. Then

$$c_N^T - c_B^T B^{-1} N \geq 0, \qquad B^{-1} b \geq 0.$$

Thus, setting

$$\bar{y} = B^{-T} c_B, \tag{4.10}$$

leads to

$$A^T \bar{y} - c = \begin{pmatrix} B^T \\ N^T \end{pmatrix} \bar{y} - \begin{pmatrix} c_B \\ c_N \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore $\bar{y}$ is a feasible solution to the dual problem. By (4.10), on the other hand, the basic feasible solution ($\bar{x}_B = B^{-1}b$, $\bar{x}_N = 0$) satisfies

$$c^T \bar{x} = c_B^T \bar{x}_B = c_B^T B^{-1} b = b^T \bar{y}.$$

By Corollary 4.2.2, therefore, $\bar{x}$ and $\bar{y}$ are respective optimal solutions to the primal and dual problems with the same optimal value. Moreover, it is known by Theorem 4.2.1 that if there is an optimal solution to the dual problem, so is to the primal problem, with the same optimal value. □

Based on the strong duality, thereafter primal and dual optimal values will not be distinguished.

It is clear that if there is an optimal solution to one of the pair of (4.1) and (4.4), so is the self-dual problem (4.9). Moreover, the optimal value of the latter is equal to zero, and the optimal solution of the latter gives the primal and dual optimal solutions to the pair. A variation of it will be used to derive the so-called "homogeneous and self-dual interior-point method" (Sect. 9.4.4).

In case when any of the primal and dual problems is infeasible, it can be asserted that the other problem is infeasible or unbounded. The computation would be finished in this case. In some applications, however, it would be needed to determine which case the problem is. This can be resolved via the duality as follows.

Assume now that the primal problem (4.1) is infeasible. To determine whether the dual problem (4.2) is infeasible or unbounded, consider

$$\begin{array}{ll} \min & c^T x, \\ \text{s.t.} & Ax = 0, \qquad x \geq 0, \end{array} \tag{4.11}$$

which has a feasible solution $x = 0$. Solve the preceding program by the simplex algorithm. If (4.11) is unbounded, then the program

$$\begin{array}{ll} \min & 0, \\ \text{s.t.} & A^T y + z = c, \qquad z \geq 0, \end{array}$$

is infeasible (Corollary 4.2.1). Therefore, (4.2) is infeasible either. If an optimal solution to (4.11) is reached, then there exists an optimal solution to the preceding program (Theorem 4.2.3), as indicates that (4.2) is feasible; thereby, it can be further asserted that (4.2) is unbounded.

Using the duality, now we are able to prove Farkas Lemma 2.1 concisely.

Lemma 4.2.1 (Farkas). *Assume $A \in \mathcal{R}^{m\times n}$ and $b \in \mathcal{R}^m$. The feasible region P is nonempty if and only if*

$$b^T y \geq 0, \qquad \forall\ y \in \{y \in \mathcal{R}^m \mid A^T y \geq 0\}.$$

Proof. Consider the following LP problem

$$\begin{aligned} &\min \quad 0, \\ &\text{s.t.} \quad Ax = b, \qquad x \geq 0, \end{aligned} \tag{4.12}$$

the dual problem of which is

$$\begin{aligned} &\max \quad b^T y', \\ &\text{s.t.} \quad A^T y' \leq 0. \end{aligned} \tag{4.13}$$

Note that $y' = 0$ is a feasible solution to it, with feasible value 0.

Necessity. Assume that the feasible region P of (4.12) is nonempty, hence all feasible solutions correspond to the same objective value 0. According to Theorem 4.2.2, for any $y' \in \{y' \in \mathcal{R}^m \mid A^T y' \leq 0\}$ it holds that $b^T y' \leq 0$. By setting $y = -y'$, it is known that $b^T y \geq 0$ holds for $y \in \{y \in \mathcal{R}^m \mid A^T y \geq 0\}$.

Sufficiency. Assume that for any $y \in \{y \in \mathcal{R}^m \mid A^T y \geq 0\}$ it holds that $b^T y \geq 0$. Then, for $y' \in \{y' \in \mathcal{R}^m \mid A^T y' \leq 0\}$ we have $b^T y' \leq 0$, hence there is an optimal solution to (4.13). According to Theorem 4.2.3, therefore, there is an optimal solution to (4.12), as implies nonempty of P. □

4.3 Optimality Condition

From duality theorems presented in the previous section, it is possible to derive a set of conditions for primal and dual solutions to be optimal, as stands as a theoretical basis for LP. We consider the standard LP problem first, and then more general LP problems.

Assume that x and (y, z) are primal and dual (not necessarily feasible) solutions.

Definition 4.3.1. Difference $c^T x - b^T y$ between the primal and dual objective values is *duality gap* between x and (y, z).

Definition 4.3.2. If $x^T z = 0$, x and (y, z) are *complementary*; if $x + z > 0$, in addition, the two are strictly complementary.

Quantity $x^T z$ is termed *complementarity residual.*

Lemma 4.3.1. *The duality gap and complementarity residual of x and (y, z) are equal; x and (y, z) are complementary if and only if their duality gap equals zero.*

Proof. Since x and (y, z) satisfy the equality constrains, it is easy to derive that

$$c^T x - b^T y = x^T c - (Ax)^T y = x^T (c - A^T y) = x^T z.$$

□

If it holds as an equality, an "≥" or "≤" type of inequality is said to be tightly satisfied, and if it does as a strict inequality, it is said to be slackly satisfied.

In case when the nonnegative constraints are satisfied, the complementarity of x and (y, z) is equivalent to satisfaction of

$$x_j z_j = 0, \qquad \forall \quad j = 1, \ldots, n. \tag{4.14}$$

It is clear that for $j = 1, \ldots, n$, it holds that $x_j = 0$ (or $z_j = 0$) if $z_j > 0$ (or $x_j > 0$). If a component of x (or z) slackly satisfies the associated nonnegativity constraint, therefore, the corresponding component of z (or x) must satisfy the associated nonnegativity constraint tightly.

Theorem 4.3.1 (Optimality conditions for the standard LP problem). *x is an optimal solution of the standard LP problem if and only if there exist y, z such that*

$$\begin{array}{llll} (i) & Ax = b, & x \geq 0, & \text{(primal feasibility)} \\ (ii) & A^T y + z = c, & z \geq 0, & \text{(dual feasibility)} \\ (iii) & x^T z = 0. & & \text{((slackness) complementarity)} \end{array} \tag{4.15}$$

Proof. Note that for x and (y, z), zero duality gap is equivalent to complementarity (Lemma 4.3.1).

Sufficiency. By Corollary 4.2.2 and the equivalence of zero duality gap and complementarity, it follows from (4.15) that x and (y, z) are primal and dual optimal solutions, respectively.

Necessity. If x is a primal optimal solution, then it satisfies condition (i). By Theorem 4.2.3, in addition, there is a dual optimal solution (y, z) such that the duality gap is zero, hence conditions (ii) and (iii) are satisfied. □

The following result is stated without proof (Goldman and Tucker 1956b).

Theorem 4.3.2 (Strict complementarity). *If there exists a pair of primal and dual optimal solutions, then there exists a strictly complementary pair of such solutions.*

The significance of the optimal conditions speaks for itself. As for algorithm research, any type of optimality criterions in various contexts must coincide with these conditions. From them, it is understandable that LP algorithms always solve the pair of problems at the same time. For instance, once the simplex algorithm reaches primal optimal solution

$$\bar{x}_B = B^{-1} b, \qquad \bar{x}_N = 0,$$

it also gives a dual optimal solution

$$\bar{y} = B^{-T} c_B, \qquad \bar{z}_B = 0, \quad \bar{z}_N = c_N - N^T \bar{y}.$$

In view of the symmetry between primal and dual problems, moreover, it is not surprising why LP algorithms often present in pair: if there is an algorithm for solving the primal problem, there is an according algorithm for solving the dual problem, and vise versa. As an example, the dual algorithm, presented in Sect. 4.5, matches the simplex algorithm.

Interior-point methods often judge the degree of approaching optimality through the complementarity residual: the smaller the residual is, the closer to optimality; when it vanishes, primal and dual optimal solutions are attained respectively. It is noticeable, moreover, that direct dealing with the optimal conditions as a system of equalities and inequalities can lead to some interior-point algorithms (Sect. 9.4). Such algorithms usually generate a strictly complementary pair of optimal solutions in the limit, as is of importance for asymptotic analysis (Güler and Ye 1993).

For the bounded-variable LP problem (4.6), we have the following result.

Theorem 4.3.3 (Optimal conditions for the bounded-variable problem). *x is an optimal solution of the bounded-variable LP problem if and only if there exist y, v, w such that*

$$\begin{array}{llll} (i) & Ax = b, & l \le x \le u, & \text{(primal feasibility)} \\ (ii) & A^T y - v + w = c, & v, w, \ge 0, & \text{(dual feasibility)} \\ (iii) & (x-l)^T w = 0, & (u-x)v = 0. & \text{((slackness) complementarity)} \end{array} \tag{4.16}$$

Proof. It is derived from (4.6) to (4.8) and Theorem 4.3.1. □

Finally, consider the general problem of form

$$\begin{array}{ll} \min & c^T x, \\ \text{s.t.} & x \in \Omega, \end{array} \tag{4.17}$$

where $\Omega \subset \mathcal{R}^n$ is a convex set.

Lemma 4.3.2. *It is an optimal solution to (4.17) if and only if x^* satisfies*

$$c^T(x - x^*) \ge 0, \qquad \forall \quad x \in \Omega \tag{4.18}$$

Proof. Assume that x^* is an optimal solution. If (4.18) does not hold, i.e., there is a point $\bar{x} \in \Omega$ such that

$$c^T(\bar{x} - x^*) < 0, \tag{4.19}$$

then it holds that

$$c^T \bar{x} < c^T x^*, \tag{4.20}$$

which contradicts optimality of x^*. Conversely, assume that (4.18) holds. If x^* is not optimal to (4.17), then there exists $\bar{x} \in \Omega$ satisfying (4.20), which implies (4.19), as contradicts satisfaction of (4.18). □

It is noted from the proof that the preceding Lemma is actually valid for an arbitrary set Ω. Geometrically, it says that a sufficient and necessary condition for x^* to be an optimal solution to (4.17) is that the angle between the vector from x^* to any point $x \in \Omega$ and the gradient of the objective function is no more than $\pi/2$.

Vector $x^* \in \Omega$ is termed *local optimal solution* if it is an optimal solution over some spherical neighborhood of it; or more precisely, there exists $\gamma > 0$ such that

$$c^T x^* = \min \{c^T x \mid x \in (\Omega \cap \Sigma)\}, \qquad \Sigma = \{x \in \mathcal{R}^n \mid \|x - x^*\| \leq \gamma\}. \quad (4.21)$$

Theorem 4.3.4. *A vector is an optimal solution to (4.17) if and only if it is a local optimal solution to it.*

Proof. The necessity is clear. The sufficiency. Assume that x^* is a local optimal solution. So, there is $\gamma > 0$ so that (4.21) holds. For any $\bar{x} \in \Omega$ with $\bar{x} \neq x^*$, define

$$x' = x^* + \gamma/\|\bar{x} - x^*\|(\bar{x} - x^*). \quad (4.22)$$

Then, $\|x' - x^*\| = \gamma$, hence $x' \in \Omega \cap \Sigma$. Thereby, it follows from (4.22) that $c^T x' - c^T x^* \geq 0$ together with

$$(\bar{x} - x^*) = \|\bar{x} - x^*\|/\gamma c^T (x' - x^*),$$

gives

$$c^T (\bar{x} - x^*) = \|\bar{x} - x^*\|/\gamma c^T (x' - x^*) \geq 0.$$

According to Lemma 4.3.2, x^* is an optimal to (4.17). □

4.4 Dual Simplex Method: Tableau Form

The dual simplex method is of great importance, as it can be even more efficient than the simplex method, and serve as a basic tool to solve integer or mixed LP problems. This section will derive its tableau version.

As was well-known, simplex tableaus created by a series of elementary transformations are equivalent in the sense that they represent problems equivalent to the original one. No matter how pivots are selected, a resulting simplex tableau offers a pair of complementary primal and dual solutions, which are optimal whenever entries in the right-hand side and the objective row are all nonnegative, or in other words, both primal and dual feasibility achieved. Starting from a feasible simplex tableau, e.g., the tableau simplex algorithm generates a sequence of feasible simplex tableaus, until dual feasibility achieved.

Using an alternative pivot rule, the so-called "dual simplex algorithm" presented in this section generates a sequence of dual feasible simplex tableaus, until primal feasibility achieved. To do so, of course, it has to start from a dual feasible simplex tableau.

Consider the standard LP problem (4.1). Let (3.18) be a current dual feasible simplex tableau, satisfying $\bar{z}_N \geq 0$ but $\bar{b} \not\geq 0$.

In contrast to the simplex method, we first determine pivot row rather than column.

Rule 4.4.1 (Dual row rule) Select row index by

$$p \in \arg\min\{\bar{b}_i \mid i = 1, \ldots, m\}.$$

It is clear that the preceding will drop the basic variable x_{j_p} from the basis, turning it to primal feasible.

Lemma 4.4.1. *Assume that $\bar{z}_N \geq 0$ and $\bar{b}_p < 0$. If column index set*

$$J = \{j \in N \mid \bar{a}_{pj} < 0\} \tag{4.23}$$

is empty, then the LP problem is infeasible.

Proof. $\bar{z}_N \geq 0$ indicates dual feasibility. Assume that the dual problem is bounded, hence there is an optimal dual solution. According to the strong duality Theorem, there is a optimal primal solution. Assume that $\hat{x} \geq 0$ is such an optimal primal solution, which satisfies the equality, corresponding to the pth row of the simplex tableau, i.e.,

$$\hat{x}_{j_p} + \sum_{j \in N} \bar{a}_{pj} \hat{x}_j = \bar{b}_p.$$

From (4.23) and $\hat{x} \geq 0$, it follows that the left-side of the preceding is nonnegative, as contradicts $\bar{b}_p < 0$. Therefore, the problem is dual unbounded, and hence infeasible. □

Assume that p has been determined and (4.23) does not hold. Then the following is well-defined.

Rule 4.4.2 (Dual column rule) Determine β and column index q such that

$$\beta = -\bar{z}_q / \bar{a}_{pq} = \min_{j \in J} -\bar{z}_j / \bar{a}_{pj} \geq 0. \tag{4.24}$$

β is referred to as *dual stepsize*.

Once a pivot, say $\bar{a}_{pq}$, is determined, perform elementary transformations to turn it to 1 and eliminate all other nonzeros in the q-indexed column. Then adding β times of the pth row to the objective row results in a new simplex tableau (see (3.13)). It is not difficult to show that the new tableau remains dual feasible, that is, its objective row remains nonnegative.

The objective value of the resulting tableau is then

$$-\hat{f} = -\bar{f} + \beta\bar{b}_p \leq -\bar{f}, \tag{4.25}$$

Therefore, $\hat{f} \geq \bar{f}$, indicating that the objective value does not decrease. When $\bar{z}_N > 0$, the objective value strictly increases, as is a case in which the simplex tableau (or the dual feasible solution) is said to be *dual nondegenerate*.

The overall steps can be put in the following algorithm (Beale 1954; Lemke 1954).

Algorithm 4.4.1 (Dual simplex algorithm: tableau form). Initial: a dual feasible simplex tableau of form (3.18). This algorithm solves the standard LP problem (1.7).

1. Select pivot row index $p \in \arg\min\{\bar{b}_i \mid i = 1, \ldots, m\}$.
2. Stop if $\bar{b}_p \geq 0$.
3. Stop if $J = \{j \in N \mid \bar{a}_{p\,j} < 0\} = \emptyset$.
4. Determined pivot column $q \in \arg\min_{j\in J} -\bar{z}_j/\bar{a}_{p\,j}$.
5. Convert $\bar{a}_{p\,q}$ to 1, and eliminate the other nonzeros in the column by elementary transformations.
6. Go to step 1.

Theorem 4.4.1. *Under the dual nondegenerate assumption, Algorithm 4.4.1 terminates either at*

(i) Step 2, achieving a pair of primal and dual optimal solutions; or at
(ii) Step 3, detecting infeasibility of the problem.

Proof. The proof on the termination is similar to the simplex algorithm. The meanings of its exits come from Lemmas 3.2.1 and 4.4.1 and discussions preceding the algorithm. □

If dual degeneracy presents, the dual simplex method would stall in solution process, even fail to solve a problem due to cycling (Beale 1955). Despite the dual degeneracy almost always occurs, the dual simplex method perform successfully in practice, as is just in the primal simplex context.

Algorithm 4.4.1 starts from a dual feasible simplex tableau. In general, there is a need for a dual Phase-I procedure to serve for this purpose. This topic will be delayed to Chap. 14.

Example 4.4.1. Solve the following LP problem by Algorithm 4.4.1:

$$\begin{array}{ll} \min & f = x_1 + 2x_2 + x_3, \\ \text{s.t.} & 2x_1 + \ \ x_2 + x_3 - x_4 = 1, \\ & -x_1 + 4x_2 + x_3 \qquad \geq 2, \\ & \ \ x_1 + 3x_2 \qquad\qquad \leq 4, \\ & x_j \geq 0, \ j = 1, \ldots, 4. \end{array}$$

Answer Initial: turn the problem to the standard form by introducing slack variables $x_5, x_6 \geq 0$ in constrains. Then, premultiply the first two constraints by -1 respectively:

$$\begin{array}{ll} \min & f = x_1 + 2x_2 + x_3, \\ \text{s.t.} & -2x_1 - \ \ x_2 - x_3 + x_4 \qquad\qquad = -1, \\ & \ \ x_1 - 4x_2 - x_3 \qquad + x_5 \qquad = -2, \\ & \ \ x_1 + 3x_2 \qquad\qquad\qquad + x_6 = \ \ 4, \\ & \qquad x_j \geq 0, \quad j = 1, \ldots, 6, \end{array}$$

which corresponds to an available dual feasible simplex tableau, i.e.,

x_1	x_2	x_3	x_4	x_5	x_6	RHS
−2	−1	−1	1			−1
1	−4*	−1		1		−2
1	3				1	4
1	2	1				

Iteration 1:

1. $\min\{-1, -2, 4\} = -2 < 0, \ p = 2$.
3. $J = \{2, 3\}$.
4. $\min\{-2/(-4), -1/(-1)\} = 1/2, \ q = 2$.
5. Multiply row 2 by $-1/4$, and then add $1, -3, -2$ times of row 2 to rows 1,3,4, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	RHS
−9/4*		−3/4	1	−1/4		−1/2
−1/4	1	1/4		−1/4		1/2
7/4		−3/4		3/4	1	5/2
3/2		1/2		1/2		−1

Iteration 2:

1. $\min\{-1/2, 1/2, 5/2\} = -1/2 < 0,\ p = 1.$
3. $J = \{1, 3, 5\}.$
4. $\min\{-(3/2)/(-9/4), -(1/2)/(-3/4), -(1/2)/(-1/4)\} = 2/3,\ q = 1.$
5. Multiply row 1 by $-4/9$, and then add $1/4, -7/4, -3/2$ times of row 1 to rows 2,3,4, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	RHS
1		1/3	−4/9	1/9		2/9
	1	1/3	−1/9	−2/9		5/9
		−4/3	7/9	5/9	1	19/9
			2/3	1/3		−4/3

The right-hand side of the preceding is now nonnegative, hence obtained is an optimal simplex tableau. The optimal solution and associated objective value are

$$\bar{x} = (2/9, 5/9, 0, 0)^T, \qquad \bar{f} = 4/3.$$

4.5 Dual Simplex Method

In the preceding section, the tableau dual simplex algorithm was formulated. In this section, we first derive its revised version based on equivalence between the simplex tableau and the revised simplex tableau, just as what we have done for deriving the simplex method from its tableau version. Then, we derive it alternatively to reveal the fact that it essentially solves the dual problem.

Like the simplex Algorithm 3.5.2, in each iteration the dual simplex algorithm computes the objective row, the right-hand side, pivot column and row. The objective row and/or the right-hand side can be computed in a recurrence manner (see (3.13) and (3.14)). The pivot column and row can be computed through B^{-1} and the original data, just as in the simplex method.

If nonbasic entries in the pivot row are all nonnegative, i.e.,

$$\sigma_N^T \triangleq e_p^T \bar{N} = e_p^T B^{-1} N \geq 0,$$

the dual problem is unbounded, hence the original problem is infeasible. B^{-1} will be updated in the same way as in the simplex method.

Based on Table 3.2, therefore, Algorithm 4.4.1 can be revised to the following algorithm.

Algorithm 4.5.1 (Dual simplex algorithm). Initial: $(B, N), B^{-1}$. $\bar{z}_N \geq 0$, $\bar{x}_B = B^{-1}b$ and $\bar{f} = c_B^T \bar{x}_B$. This algorithm solves the standard LP problem (1.8).

1. Select row index $p \in \arg\min\{\bar{x}_{j_i} \mid i = 1, \ldots, m\}$.
2. Stop if $\bar{x}_{j_p} \geq 0$ (optimality achieved).
3. Compute $\sigma_N = N^T B^{-T} e_p$.
4. Stop if $J = \{j \mid \sigma_j < 0, \ j \in N\} = \emptyset$ (infeasible problem).
5. Determine β and column index q such that
   $\beta = -\bar{z}_q/\sigma_q = \min_{j \in J} -\bar{z}_j/\sigma_j$.
6. Set $\bar{z}_{j_p} = \beta$, and update $\bar{z}_N = \bar{z}_N + \beta\sigma_N, \ \bar{f} = \bar{f} - \beta\bar{x}_{j_p}$ if $\beta \neq 0$.
7. Compute $\bar{a}_q = B^{-1}a_q$.
8. Update by $\bar{x}_B = \bar{x}_B - \alpha\bar{a}_q$, $\bar{x}_q = \alpha$, where $\alpha = \bar{x}_{j_p}/\sigma_q$.
9. Update B^{-1} by (3.23).
10. Update (B, N) by exchanging j_p and q.
11. Go to step 1.

Alternatively, Algorithm 4.5.1 can be derived by solving the dual problem itself as follows.

Consider the dual problem

$$(D) \qquad \begin{array}{ll} \max & g = b^T y, \\ \text{s.t.} & A^T y \leq c. \end{array}$$

Given (B, N), B^{-1}. It is easy to verify that $\bar{y} = B^{-T} c_B$ satisfies the dual constraints, i.e.,

$$\bar{z}_B = c_B - B^T \bar{y} = 0, \qquad \bar{z}_N = c_N - N^T \bar{y} \geq 0. \tag{4.26}$$

$\bar{y}$ is a dual basic feasible solution, or geometrically a vertex in the dual feasible region

$$D = \{y \mid A^T y \leq c\}.$$

In the primal simplex context, $\bar{y}$ is usually called "simplex multipliers".

Consider the associated primal basic solution

$$\bar{x}_B = B^{-1}b, \qquad \bar{x}_N = 0.$$

If $\bar{x}_B \geq 0$, then $\bar{x}$ and $(\bar{y}, \bar{z})$ satisfy the optimality condition, and are therefore a pair of primal and dual basic optimal solutions.

Now assume that $\bar{x}_B = B^{-1}b \ngeq 0$. Determine row index p such that

$$\bar{x}_{j_p} = \min\{\bar{x}_{j_i} \mid i = 1, \ldots, m\} < 0. \tag{4.27}$$

Introduce vector

$$h = B^{-T} e_p, \tag{4.28}$$

which with (4.27) gives

$$-b^T h = -b^T B^{-T} e_p = -e_p^T (B^{-1} b) = \bar{x}_{j_p} > 0, \tag{4.29}$$

implying that $-h$ is an uphill direction, with respect to the dual objective function g.

Now consider the line search scheme below:

$$\hat{y} = \bar{y} - \beta h, \tag{4.30}$$

where β is a dual stepsize to be determined. From (4.30), (4.28) and (4.26), it follows that

$$\hat{z}_B = c_B - B^{\mathrm{T}} \hat{y} = c_B - B^{\mathrm{T}} (\bar{y} - \beta h) = \beta e_p \geq 0, \tag{4.31}$$

$$\hat{z}_N = c_N - N^{\mathrm{T}} \hat{y} = c_N - N^{\mathrm{T}} (\bar{y} - \beta h) = \bar{z}_N + \beta N^{\mathrm{T}} h. \tag{4.32}$$

If $\sigma_N = N^T h \not\geq 0$, then it is seen from (4.32) that a too large $\beta > 0$ will lead to $\hat{z}_N \not\geq 0$, as violates the dual feasibility. It is easy to determine the largest possible β and according column index q, subject to $\hat{z}_N \geq 0$ (see step 5 of Algorithm 4.4.1). Then, drop j_p from and enter q to the basis. It is easy to verify that $\hat{y}$ is just the dual basic feasible solution, corresponding to the resulting basis.

The following is valid in the other case.

Proposition 4.5.1. *If $\sigma_N = N^T h \geq 0$, the dual problem is unbounded, and $-h$ is an uphill extreme direction in the dual feasible region D.*

Proof. If $\sigma_N \geq 0$, it is seen from (4.26) and (4.31), (4.32) that

$$\hat{z} = c - A^{\mathrm{T}} \hat{y} \geq 0, \qquad \forall \beta \geq 0,$$

implying feasibility of the new solution $\hat{y}$ given by (4.30). On the other hand, it is known from (4.30) and (4.29) that the associated new objective value is

$$b^{\mathrm{T}} \hat{y} = b^{\mathrm{T}} \bar{y} - \beta \bar{x}_{j_p}, \tag{4.33}$$

which goes to ∞, as β tends to ∞. Thus the dual problem is unbounded. This means that $-h$ is a uphill unbounded direction of D. In fact, it is seen that $-h$ is the direction of 1-dimensional face or edge

$$\{y \in \mathcal{R}^m \mid A^{\mathrm{T}} y \leq c;\ a_{j_i}^{\mathrm{T}} y = c_{j_i},\ i = 1, \cdots, m,\ i \neq p\},$$

and therefore a uphill extreme direction. $\square$

The preceding analysis is actually valid for any given negative $\bar{x}_{j_p}$. Though far from the best, Rule (4.27) is simple and easy to use, and the corresponding objective increment is the largest possible for a unit stepsize. More efficient dual pivot rules will be presented in Chap. 12.

Since the beginning of 1990s of the last century, successful applications of the dual steepest-edge rule (Forrest and Goldfarb 1992), some of its approximations and bound-flipping (Kirillova et al. 1979) have injected fresh vigoure to the dual simplex method, so that it becomes one of the most powerful methods for solving LP problems (Bixby 2002; Koberstein 2008).

Example 4.5.1. Solve the following problem by Algorithm 4.5.1:

$$\begin{array}{lrcrcrcrcrcrcr} \min & f = x_1 + 2x_2 + x_3, \\ \text{s.t.} & -2x_1 & - & x_2 & - & x_3 & + & x_4 & & & & & = & -1, \\ & x_1 & - & 4x_2 & - & x_3 & & & + & x_5 & & & = & -2, \\ & x_1 & + & 3x_2 & & & & & & & + & x_6 & = & 4, \\ & & & & & x_j \geq 0, & & j = 1, \ldots, 6. \end{array}$$

Answer Initial: $B = \{4, 5, 6\},\ N = \{1, 2, 3\},\ B^{-1} = I,$
$\bar{z}_N = (1, 2, 1)^T,\ \bar{x}_B = (-1, -2, 4)^T,\ f = 0.$

Iteration 1:

1. $\min\{-1, -2, 4\} = -2 < 0,\ p = 2,\ x_5$ leaves the basis.
3. $\sigma_N = (1, -4, -1)^T.$
5. $\beta = \min\{2/4, 1/1\} = 1/2,\ q = 2,\ x_2$enters the basis.
6. $\bar{z}_N = (3/2, 0, 1/2)^T,\ \bar{z}_{j_p} = 1/2,\ \bar{f} = 0 - (1/2)(-2) = 1.$
7. $\bar{a}_q = (-1, -4, 3)^T.$
8. $\alpha = -2/-4 = 1/2,\ \bar{x}_B = (-1, -2, 4)^T - (1/2)(-1, -4, 3)^T$ $= (-1/2, 0, 5/2)^T,\ \bar{x}_2 = \alpha = 1/2.$
9. $B^{-1} = \begin{pmatrix} 1 & 1/-4 & \\ & 1/-4 & \\ & 3/4 & 1 \end{pmatrix}.$
10. $B = \{4, 2, 6\},\ N = \{1, 5, 3\},\ \bar{z}_N = (3/2, 1/2, 1/2)^T \geq 0,\ \bar{x}_B$ $= (-1/2, 1/2, 5/2)^T.$

Iteration 2:

1. $\min\{-1/2, 1/2, 5/2\} = -1/2 < 0,\ p = 1, x_4$ leaves the basis.
3. $\sigma_N = (-9/4, -1/4, -3/4)^T.$
5. $\beta = \min\{(3/2)/(9/4), (1/2)/(1/4), (1/2)/(3/4)\} = 2/3,\ q = 1,\ x_1$ enters the basis.
6. $\bar{z}_N = (3/2, 1/2, 1/2)^T + (2/3)(-9/4, -1/4, -3/4)^T = (0, 1/3, 0)^T.$
   $\bar{z}_{j_p} = 2/3,\ \bar{f} = 1 - (2/3)(-1/2) = 4/3.$
7. $\bar{a}_q = (-9/4, -1/4, 7/4)^T.$

8. $\alpha = (-1/2)/(-9/4) = 2/9,\ \bar{x}_B = (-1/2, 1/2, 5/2)^T - (2/9)(-9/4, -1/4, 7/4)^T = (0, 5/9, 19/9)^T,\ \bar{x}_1 = 2/9.$

9. $$B^{-1} = \begin{pmatrix} -4/9 & & \\ -1/9 & 1 & \\ 7/9 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/-4 & \\ & 1/-4 & \\ & 3/4 & 1 \end{pmatrix} = \begin{pmatrix} -4/9 & 1/9 & \\ -1/9 & -2/9 & \\ 7/9 & 5/9 & 1 \end{pmatrix}.$$

10. $B = \{1, 2, 6\},\ N = \{4, 5, 3\},\ \bar{z}_N = (2/3, 1/3, 0)^T \geq 0,$
$\bar{x}_B = (2/9, 5/9, 19/9)^T \geq 0.$
The optimal solution and objective value:

$$\bar{x} = (2/9, 5/9, 0, 0, 0, 19/9)^T, \qquad \bar{f} = 4/3.$$

4.6 Economic Interpretation of Duality: Shadow Price

The dual problem is of an interesting economic interpretation. Assume that the primal problem

$$\begin{array}{ll} \max & f = c^T x, \\ \text{s.t.} & Ax \leq b, \quad x \geq 0, \end{array}$$

is a plan model for a manufacturer to produce n products using m resources. The available amount of resource i is $b_i,\ i = 1, \ldots, m$, units; producing an unit of product j consumes a_{ij} units of resource i. The profit of an unit of product j is $c_j,\ j = 1, \ldots, n$. The goal is to achieve the highest profit with the limited resources.

The dual problems is

$$\begin{array}{ll} \min & b^T y, \\ \text{s.t.} & A^T y \geq c, \quad y \geq 0. \end{array}$$

Let $\bar{x}$ and $\bar{y}$ be primal and dual optimal solutions, respectively. According to the strong duality Theorem, associated primal and dual optimal values are equal, i.e.,

$$v = c^T \bar{x} = b^T \bar{y}.$$

Optimal value's partial derivative with respect to b_i is

$$\frac{\partial v}{\partial b_i} = \bar{y}_i.$$

Therefore, $\bar{y}_i$ is equal to the increment of the highest profit, created by adding one unit of resource i, and can be taken as manufacturer's assessment for resource i, as is named *shadow price* by Paul Samuelson.[1] Shadow price $\bar{y}_i$ is the upper price limit that the manufacturer can afford to buy resource i. When market price of resource i is lower than shadow price $\bar{y}_i$, the manufacturer should consider to buy it to expand the production scale, whereas he should consider to sell it to reduce the production scale in the other case. The manufacturer will not buy resource i any more, no matter how low its price is, whenever the optimal solution $\bar{x}$ satisfies the ith primal inequality constraint slackly, as implies that resource i is not fully used. In fact, the shadow price $\bar{y}_i$ vanishes in this case.

Let x and y be any primal and dual feasible solutions respectively, hence $c^T x \leq b^T y$ holds according to the weak duality. Inequality

$$c^T x < b^T y,$$

implies that the total profit (output) of the plan is less than the available value (input) of the resources. In economic terms, the input-output system is said "instable (non-optimal)" in this case. It is a stable (optimal) system only when output is equal to input.

Consider economic implication of the dual constraints. The manufacturer negotiates with the supplier at price y_i for resource i, as is calculated to purchase resources $b_i, i = 1, \ldots, m$ by overall payment $b^T y$. For $j = 1, \ldots, n$, on the other hand, the supplier asks for resource prices to produce an unit product j being no less than the profit of an unit of product j, as satisfies the jth dual constraint

$$\sum_{i=1}^{m} a_{i,j} y_i \geq c_j.$$

If the suppler asks for too high prices, that is, the dual optimal solution $\bar{y}$ satisfies the jth dual constraint slackly, then $\bar{x}_j$ vanishes, as implies that the manufacturer should not arrange for producing product j at all, no matter how high the profit of an unit of the product is.

4.7 Notes

The concept and theorems of duality were first proposed by famous mathematician von Neumann. In October 1947, he made foundational discussions on the topic in a talk with George B. Dantzig and in a working paper, finished a few weeks later. In 1948, Dantzig provided a rigorous proof on the duality theorems in a report. Subsequently, Gale et al. (1951) formulated the duality theorems and proved them

[1]Paul Samuelson (1915–2009), American economist, the winner of The Nobel Economics Prize (1970), the first American winning this prize.

using Farkas Lemma, independently. Goldman and Tucker (1956b) and Balinski and Tucker (1969) discussed theoretical properties of the dual problem systematically.

As was stressed, the simplex tableau is just a concise expression of a LP problem itself, and all such tableaus created by the primal or dual simplex algorithm are equivalent in the sense of their representation of the LP problem. Then the following question arises:

Are dual problems corresponding to the simplex tableaus equivalent?

Consider the dual problem, corresponding to tableau (3.18), i.e.,

$$\begin{array}{ll} \max & \bar{f} + \bar{b}^{\mathrm{T}} y', \\ \text{s.t.} & \begin{pmatrix} I \\ \bar{N}^{\mathrm{T}} \end{pmatrix} y' + \begin{pmatrix} z'_B \\ z'_N \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{z}_N \end{pmatrix}, \quad z'_B, z'_N \geq 0. \end{array}$$

As their $\bar{b}, \bar{N}, \bar{z}_N$ are not the same, the dual problems corresponding to different simplex tableaus are also different. However, such differences are not essential. In fact, Making variable transformations $y' = B^{\mathrm{T}} y - c_B,\ z' = z$ and noting

$$\bar{b} = B^{-1} b, \quad \bar{N} = B^{-1} N, \quad \bar{z}_N = c_N - N^{\mathrm{T}} B^{-\mathrm{T}} c_B, \quad \bar{f} = c_B^{\mathrm{T}} B^{-1} b,$$

the dual problem can be converted to

$$\begin{array}{ll} \max & b^{\mathrm{T}} y, \\ \text{s.t.} & \begin{pmatrix} B^{\mathrm{T}} \\ N^{\mathrm{T}} \end{pmatrix} y + \begin{pmatrix} z_B \\ z_N \end{pmatrix} = \begin{pmatrix} c_B \\ c_N \end{pmatrix}, \quad z_B, z_N \geq 0, \end{array}$$

which is the original dual problem. Therefore, all the generated simplex tableaus can be regarded as equivalent with respect to represented dual problems.

In summary, elementary transformations generate equivalent simplex tableaus. On the primal side, the right-hand sides give primal basic solutions and the bottom rows give primal reduced objective functions. On the dual side, the right-hand sides render dual reduced objective functions and the bottom rows dual basic solutions.

Based on duality, the following is also valid.

Proposition 4.7.1. *If it has a dual solution* $(\bar{y},\ \bar{z})$*, the standard problem (4.1) is equivalent to*

$$\begin{array}{lll} \max & f = \bar{y}^T b + \bar{z}^T x, & \\ \text{s.t.} & Ax = b, & x \geq 0. \end{array} \tag{4.34}$$

Proof. $(\bar{y},\ \bar{z})$ satisfies $A^{\mathrm{T}} \bar{y} + \bar{z} = c$ or equivalently,

$$c^T = \bar{y}^T A + \bar{z}^T.$$

Substituting the preceding to the objective of the standard problem and noting the constraint system gives the objective of (4.34), i.e.,

$$f = c^T x = \bar{y}^T A x + \bar{z}^T x = \bar{y}^T b + \bar{z}^T x,$$

and vice versa. □

The preceding says that the cost vector c in the standard problem can be replaced by any dual solution $\bar{z}$, with only a constant difference in objective value.