Chapter 15
Reduced Simplex Method

In this chapter and the following two chapters, some special forms of the LP problem, introduced in Sect. 25.1, will be employed to design new LP methods. In particular, this chapter will handle the so-called "reduced problem" (25.2), i.e.,

$$\begin{array}{ll} \min & x_{n+1}, \\ \text{s.t.} & (A \vdots a_{n+1}) \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = b, \quad x \geq 0, \end{array} \tag{15.1}$$

where $a_{n+1} = -e_{m+1}$. Note that the objective variable x_{n+1} is in the place of f (thereafter the two will be regarded equal), and hence there is no sign restriction on x_{n+1}.

In the conventional simplex context, x_{n+1} appears as a dependent variable. In each iteration, variation of the basic solution as well as the value of x_{n+1} comes from variation of a chosen nonbasic variable, x_q, corresponding to a negative cost. In contrast, the key of the "reduced simplex method", presented in this chapter, is to use x_{n+1} as a special nonbasic variable, an argument, which decreases in each iteration to push the associated basic solution toward optimality.

Effectiveness of algorithms derived in this chapter is to be investigated. There are no related numerical results available at this stage.

15.1 Derivation

Consider the reduced problem (15.1). As it plays a particular role, the objective variable x_{n+1} will be separated from the set of nonbasic variables.

Assume that the constraints of (15.1) are converted to the following equivalent canonical form by a series of elementary transformations:

$$x_B = \bar{b} - \bar{N} x_N - x_{n+1} \bar{a}_{n+1} \geq 0, \quad x_N \geq 0, \tag{15.2}$$

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where $\bar{a}_{n+1} = -B^{-1}e_{m+1} \neq 0$, and

$$B = \{j_1, \cdots, j_{m+1}\}, \quad N = A \backslash B, \quad n+1 \notin B. \tag{15.3}$$

Lemma 15.1.1. *If $\bar{a}_{n+1} \geq 0$, then problem (15.1) is infeasible or unbounded below.*

Proof. Assume that $(\bar{x}, \bar{x}_{n+1})$ is a feasible solution to (15.1), satisfying

$$\bar{x}_B = \bar{b} - \bar{N}\bar{x}_N - \bar{x}_{n+1}\bar{a}_{n+1} \geq 0. \tag{15.4}$$

Thus, for any $\alpha \geq 0$ and

$$\hat{x}_N = \bar{x}_N, \hat{x}_{n+1} = \bar{x}_{n+1} - \alpha,$$

it holds that

$$\hat{x}_B = \bar{b} - \bar{N}\hat{x}_N - \hat{x}_{n+1}\bar{a}_{n+1} = (\bar{b} - \bar{N}\bar{x}_N - \bar{x}_{n+1}\bar{a}_{n+1}) + \bar{a}_{n+1}\alpha \geq 0,$$

where the right-most inequality comes from (15.4), $\bar{a}_{n+1} \geq 0$ and $\alpha \geq 0$. This indicates that $(\hat{x}, \hat{x}_{n+1})$ is a feasible solution, and

$$\hat{x}_{n+1} \to -\infty, \quad \text{as } \alpha \to +\infty.$$

Therefore, the problem is unbounded below. □

Setting $x_N = 0$ in (15.2) leads to the following system of inequalities:

$$x_B = \bar{b} - x_{n+1}\bar{a}_{n+1} \geq 0. \tag{15.5}$$

Introduce the set of solutions to the system

$$\Phi(B) = \{x_{n+1} \mid \bar{b} - x_{n+1}\bar{a}_{n+1} \geq 0\}.$$

If this set is nonempty, then (15.5) is said *consistent*, and (15.2) is a *feasible canonical form*.

It is clear that any given $x_{n+1} = \bar{x}_{n+1}$ corresponds to a solution to (15.1), i.e.,

$$\begin{pmatrix} \bar{x}_B \\ \bar{x}_N \\ \bar{x}_{n+1} \end{pmatrix} = \begin{pmatrix} \bar{b} - \bar{x}_{n+1}\bar{a}_{n+1} \\ 0 \\ \bar{x}_{n+1} \end{pmatrix}.$$

Using the above notation, we have the following result.

Proposition 15.1.1. *$(\bar{x}, \bar{x}_{n+1})$ is a feasible solution to problem (15.1) if and only if $\bar{x}_{n+1} \in \Phi(B)$.*

Proof. Note that the constraints of (15.1) and (15.2) are equivalent. If $\bar{x}_{n+1} \in \Phi(B)$, then $(\bar{x}, \bar{x}_{n+1})$ clearly satisfies (15.2), hence is feasible. If, conversely, $(\bar{x}, \bar{x}_{n+1})$ is a feasible solution, then $\bar{x}_{n+1} \in \Phi(B)$ follows from (15.2). □

Definition 15.1.1. If, for some $p \in \{1, \cdots, m+1\}$, it holds that

$$\bar{x}_{j_p} = 0, \quad \bar{a}_{p,n+1} \neq 0, \tag{15.6}$$

then $(\bar{x}, \bar{x}_{n+1})$ is a basic solution; if, in addition,

$$\bar{x}_{j_i} \geq 0, \quad i = 1, \cdots, m+1, \tag{15.7}$$

it is a basic feasible solution. If

$$\bar{x}_{j_i} > 0, \quad \forall\, i = 1, \cdots, m+1, \quad \bar{a}_{i,n+1} < 0, \tag{15.8}$$

the basic feasible solution is said to be nondegenerate.

The preceding definitions of basic solution and basic feasible solution coincide with the same named items in the conventional simplex context. In fact, when (15.6) holds, $\bar{x}$ is just the basic solution, associated with the conventional simplex tableau, resulting from entering x_{n+1} to and dropping x_{j_p} from the basis; and if (15.7) holds, then components of the basic solution are all nonnegative, hence it is feasible. Therefore, the two will not be distinguished. It is noted however that the definition of nondegeneracy here is somewhat different from the conventional.

If $\Phi(B)$ is nonempty, it is logical to find the basic feasible solution, associated with its greatest lower bound. To this end, the following rule applies.

Rule 15.1.1 (Row rule) Assume $\bar{a}_{n+1} \not\geq 0$. Select pivot row index

$$p \in \arg\max\{\bar{b}_i/\bar{a}_{i,n+1} \mid \bar{a}_{i,n+1} < 0,\ i = 1, \cdots, m+1\}. \tag{15.9}$$

Let p be selected row index. Define

$$\begin{pmatrix} \hat{x}_B \\ \hat{x}_N \\ \hat{x}_{n+1} \end{pmatrix} = \begin{pmatrix} \bar{b} - (\bar{b}_p/\bar{a}_{p,n+1})\bar{a}_{n+1} \\ 0 \\ \bar{b}_p/\bar{a}_{p,n+1} \end{pmatrix}. \tag{15.10}$$

Using the preceding notation, we have the following result.

Lemma 15.1.2. *Assume $\Phi(B) \neq \emptyset$. If $\bar{a}_{n+1} \not\geq 0$, then $\hat{x}_{n+1}$ is its greatest lower bound, and $(\hat{x}, \hat{x}_{n+1})$ is a basic feasible solution.*

Proof. Note that condition $\bar{a}_{n+1} \not\geq 0$ ensures that (15.9) is well-defined.

Introduce notation

$$I = \{i = 1, \cdots, m+1 \mid \bar{a}_{i,n+1} < 0\}.$$

It is known from (15.9) and (15.10) that

$$\hat{x}_{n+1} \geq \bar{b}_i/\bar{a}_{i,n+1}, \quad i \in I,$$

from which it follows that

$$\bar{b}_i - \hat{x}_{n+1}\bar{a}_{i,n+1} \geq 0, \quad i \in I.$$

Note that (15.9) implies

$$\bar{a}_{p,n+1} < 0. \tag{15.11}$$

Now we show $\hat{x}_{n+1} \in \Phi(B)$. If, otherwise, it does not hold, then there is an $r \in \{1, \cdots, m+1\}$ such that

$$\bar{b}_r - \hat{x}_{n+1}\bar{a}_{r,n+1} < 0, \quad \bar{a}_{r,n+1} \geq 0.$$

There are following two cases arising:

Case (i) $\bar{a}_{r,n+1} = 0,\ \bar{b}_r < 0$. It clearly holds in this case that $\Phi(B) = \emptyset$.
Case (ii) $\bar{b}_r - \hat{x}_{n+1}\bar{a}_{r,n+1} < 0,\ \bar{a}_{r,n+1} > 0$. Then, it is known that

$$\bar{b}_p/\bar{a}_{p,n+1} = \hat{x}_{n+1} > \bar{b}_r/\bar{a}_{r,n+1}. \tag{15.12}$$

We show that

$$\bar{b}_p - x_{n+1}\bar{a}_{p,n+1} \geq 0 \tag{15.13}$$

and

$$\bar{b}_r - x_{n+1}\bar{a}_{r,n+1} \geq 0 \tag{15.14}$$

are inconsistent, as leads to $\Phi(B) = \emptyset$. In fact, for any x'_{n+1} satisfying (15.13), i.e.,

$$\bar{b}_p - x'_{n+1}\bar{a}_{p,n+1} \geq 0,$$

it follows from (15.11) and (15.12) that

$$x'_{n+1} \geq \bar{b}_p/\bar{a}_{p,n+1} > \bar{b}_r/\bar{a}_{r,n+1},$$

hence it is known by $\bar{a}_{r,n+1} > 0$ that

$$\bar{b}_r - x'_{n+1}\bar{a}_{r,n+1} < 0,$$

which indicates that x'_{n+1} does not satisfy (15.14).

Since either of the two cases leads to $\Phi(B) = \emptyset$, contradicting the assumption, it holds that $\hat{f} \in \Phi(B)$.

For any $x'_{n+1} \in \Phi(B)$, furthermore, it holds that

$$\bar{b} - x'_{n+1}\bar{a}_{n+1} \geq 0,$$

hence

$$\bar{b}_i/\bar{a}_{i,n+1} \leq x'_{n+1}, \quad i \in I,$$

which together with (15.9) and (15.10) gives

$$\hat{x}_{n+1} \leq x'_{n+1}.$$

Therefore, $\hat{x}_{n+1}$ is the greatest lower bound of $\Phi(B)$.

According the Lemma 15.1.1, on the other hand, $(\hat{x}, \hat{x}_{n+1})$ is a feasible solution. It is verified that

$$\hat{x}_{j_p} = 0. \tag{15.15}$$

Thus noting (15.11), it is known from Definition 15.1.1 that $(\hat{x}, \hat{x}_{n+1})$ is a basic feasible solution. □

After row index p determined, the following column rule is relevant.

Rule 15.1.2 (Column rule) Determine pivot column index

$$q \in \arg\min_{j \in N} \bar{a}_{pj}. \tag{15.16}$$

Theorem 15.1.1. *Assume $\Phi(B) \neq \emptyset$. If $\bar{a}_{pq} \geq 0$, then $(\hat{x}, \hat{x_{n+1}})$ is a basic feasible solution.*

Proof. From $\bar{a}_{p,q} \geq 0$ and (15.15), it is known that the pth row of $\bar{N}$ is nonnegative, i.e.,

$$e_p^{\mathrm{T}}\bar{N} \geq 0. \tag{15.17}$$

By Lemma 15.1.2, $(\hat{x}, \hat{f})$ is a basic feasible solution to (15.1). Assume that it is not optimal. Then there is a feasible solution, say $(\tilde{x}, \tilde{x_{n+1}})$, satisfies $\tilde{x}_{n+1} < \hat{x}_{n+1}$. Consequently, from (15.2) it follows that

$$\tilde{x}_{j_p} = \bar{b}_p - e_p^{\mathrm{T}}\bar{N}\tilde{x}_N - \bar{a}_{p,n+1}\tilde{x}_{n+1},$$

combining which, $\tilde{x}_N \geq 0$, (15.11), (15.15) and (15.17) leads to

$$\tilde{x}_{j_p} < \bar{b}_p - \bar{a}_{p,n+1}\hat{x}_{n+1} = \hat{x}_{j_p} = 0,$$

as contradicts that $\tilde{x}$ is a feasible solution. Therefore $\hat{x}$ is a basic feasible solution. □

Now assume $\bar{a}_{p,q} < 0$. Carry out the basis change by dropping j_p from and entering q to the basis. Assume that the new basis is

$$\hat{B} = \{j_1, \cdots, j_{p-1}, q, j_{p+1}, \cdots, j_{m+1}\}, \quad \hat{N} = A \backslash \hat{B},$$

where q is the pth index of $\hat{B}$. The according elementary transformations turn (15.2) to a new canonical form, setting $x_{\hat{N}} = 0$ in which gives the following system of inequalities:

$$x_{\hat{B}} = \hat{b} - x_{n+1}\hat{a}_{n+1} \geq 0. \tag{15.18}$$

Theorem 15.1.2. *Assume that the solution set $\Phi(\hat{B}) = \{x_{n+1} \mid \hat{b} - x_{n+1}\hat{a}_{n+1} \geq 0\}$ to (15.18) is nonempty, and that $\hat{x}_{n+1} \in \Phi(\hat{B})$. If $\Phi(\hat{B})$ is bounded below, then the largest lower bound of $\Phi(\hat{B})$ is less than or equal to $\hat{x}_{n+1}$.*

Proof. $Ax + x_{n+1}a_{n+1} = b$ and $x_{\hat{B}} \geq 0$ together are equivalent to

$$x_{\hat{B}} = \hat{b} - \hat{N}x_{\hat{N}} - x_{n+1}\hat{a}_{n+1} \geq 0.$$

By Lemma 15.1.2, $\hat{x}$ defined by (15.10) is a basic feasible solution, hence satisfying the preceding expression. Substituting it to the preceding and noting (15.15) gives

$$\hat{x}_{\hat{B}} = \hat{b} - \hat{x}_{n+1}\hat{a}_{n+1} \geq 0,$$

Therefore it holds that $\hat{x}_{n+1} \in \Phi(\hat{B})$. That $\Phi(\hat{B})$ is bounded below implies $\hat{a}_{n+1} \not\geq 0$, because it is unbounded below by Lemma 15.1.1, otherwise.

By Lemma 15.1.2, the greatest lower bound of $\Phi(\hat{B})$ is

$$\mu = \hat{b}_{p'}/\hat{a}_{p',n+1} = \max\{\hat{b}_i/\hat{a}_{i,n+1} \mid \hat{a}_{i,n+1} < 0,\ i = 1, \cdots, m+1\}. \tag{15.19}$$

Therefore, $\mu \leq \hat{x}_{n+1}$.

$\hat{a}_{n+1}$ can expressed in term of $\bar{a}_{n+1}$ as follows (see the first expression of (3.15)):

$$\hat{a}_{i,n+1} = \begin{cases} \bar{a}_{i,n+1} - (\bar{a}_{p,n+1}/\bar{a}_{pq})\bar{a}_{iq}, & i = 1, \cdots, m+1,\ i \neq p, \\ \bar{a}_{p,n+1}/\bar{a}_{pq}, & i = p, \end{cases}$$

Hence, from (15.15) and $\bar{a}_{pq} < 0$, it follows that

$$\hat{a}_{p,n+1} = \bar{a}_{p,n+1}/\bar{a}_{pq} > 0.$$

In addition, it is known from (15.19) that

$$\hat{a}_{p',n+1} < 0, \tag{15.20}$$

Therefore $p' \neq p$. From (15.19) and $\hat{x}$ satisfying the p'th expression of (15.18), it follows that

$$\hat{x}_{n+1} = (\hat{b}_{p'} - \hat{x}_{j_{p'}})/\hat{a}_{p',n+1} = \mu - \hat{x}_{j_{p'}}/\hat{a}_{p',n+1},$$

combining which, $p' \neq p$, (15.7) and (15.20) leads to $\mu \leq \hat{x}_{n+1}$. □

According to the preceding Theorem, such an iteration results in a new feasible canonical form, with objective value not increasing. It will be shown in Sect. 16.1 that under the nondegeneracy assumption, the objective value strictly monotonically decreases, and hence the solution process terminates in finitely many iterations, achieving optimality or detecting unboundedness of the problem.

15.2 Reduced Simplex Method

Based on the previous derivation, this section formulates the algorithm first, and then formulates its revised version.

Assume that via a series of elementary transformations, the initial tableau $(A \vdots -e_{m+1} \mid b)$ of the reduced problem (15.1) becomes

x_B^{T}	x_N^{T}	x_{n+1}	RHS
I	$\bar{N}$	$\bar{a}_{n+1}$	$\bar{b}$

(15.21)

which is termed reduced (simplex) tableau. If the system of inequalities

$$\bar{b} - x_{n+1}\bar{a}_{n+1} \geq 0$$

is consistent with respect to variable x_{n+1}, tableau (15.21) is said to be feasible.

Thereby, the overall steps described in Sect. 15.1 can be put in the following algorithm.

Algorithm 15.2.1 (Reduced simplex algorithm: tableau form). Initial: feasible reduced simplex tableau of form (15.21). This algorithm solves reduced problem (15.1).

1. Stop if $\bar{a}_{n+1} \geq 0$.

2. Determine row index p such that

$$\bar{x}_{n+1} = \bar{b}_p/\bar{a}_{p,n+1} = \max\{\bar{b}_i/\bar{a}_{i,n+1} \mid \bar{a}_{i,n+1} < 0,\ i = 1,\cdots,m+1\}.$$

3. Determine column index $q \in \arg\min_{j \in N} \bar{a}_{pj}$.
4. If $\bar{a}_{pq} \geq 0$, then compute $\bar{x}_B = \bar{b} - \bar{x}_{n+1}\bar{a}_{n+1}$, and stop.
5. Convert $\bar{a}_{pq}$ to 1, and eliminate the other nonzeros in the column by elementary transformations.
6. Go to step 1.

Theorem 15.2.1. *Assume termination of Algorithm 15.2.1. It terminates at*

(i) Step 1, detecting unboundedness of the problem; or at
(ii) Step 4, providing a basic feasible solution $\bar{x}$.

Proof. The validity is shown by Theorems 15.1.2, 15.1.1 and 15.1.1, as well as related discussions in Sect. 15.1. □

Note It is possible to start solution process directly from a conventional feasible simplex tableau. To do so, assume availability of the following conventional simplex tableau, with f replaced by x_{n+1}:

x_B^{T}	x_N^{T}	x_{n+1}	RHS
I	$\bar{N}$		$\bar{b}$
	$\bar{z}_N^{\mathrm{T}}$	-1	

where $\bar{b} \geq 0$. Starting from it, the first iteration of Algorithm 15.2.1 needs to be replaced by the following steps:

1. $\bar{a}_{n+1} = -e_{m+1} \not\geq 0$.
2. $\bar{x}_{n+1} = 0/(-1) = 0$.
3. $q \in \arg\min_{j \in N} \bar{z}_j$.
4. Optimality is achieved if $\bar{c}_q \geq 0$.
5. Carry out elementary transformations to obtain a feasible reduced tableau by taking the entry in the bottom row and x_q column as the pivot.
6. Go to step 1.

Example 15.2.1. Solve the following problem by Algorithm 15.2.1:

$$\begin{aligned}
\min\quad & x_{10} = -2x_1 + 4x_2 + 3x_3 - 3x_4 - 4x_5,\\
\text{s.t.}\quad & -2x_1 - 6x_2 + 1x_3 - 3x_4 - x_5 + x_6 = 4,\\
& -x_1 - 9x_2 - 6x_3 + 2x_4 + 3x_5 + x_7 = 3,\\
& 8x_1 - 6x_2 + 3x_3 + 5x_4 + 7x_5 + x_8 = 2,\\
& 3x_1 - 2x_2 - 4x_3 - x_4 - 2x_5 + x_9 = 1,\\
& x_j \geq 0, \quad j = 1,\cdots,9.
\end{aligned}$$

Answer The problem has the following initial tableau:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
−2	−6	1	−3	−1	1					4
−1	−9	−6	2	3		1				3
8	−6	3	5	7			1			2
3	−2	−4	−1	−2				1		1
−2	4	3	−3	−4*					−1	

The right-hand side $(3, 7, 4, 5)^{\mathrm{T}}$ of which is nonnegative. Call Algorithm 15.2.1.
Iteration 1:

1. $\bar{a}_{10} \not\geq 0$.
2. $\bar{x}_{10} = \max\{0/(-1)\} = 0$, $p = 5$.
3. $\min\{-2, 4, 3, -3, -4\} = -4$, $q = 5$.
5. Multiply row 5 by $-1/4$, and then add $1, -3, -7, 2$ times of row 5 to rows 1,2,3,4, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
−3/2	−7	1/4	−9/4		1				1/4	4
−5/2	−6	−15/4	−1/4			1			−3/4	3
9/2	1	33/4	−1/4*				1		−7/4	2
4	−4	−11/2	1/2					1	1/2	1
1/2	−1	−3/4	3/4	1					1/4	

Iteration 2:

1. $\bar{a}_{10} \not\geq 0$.
2. $\bar{x}_{10} = \max\{3/(-3/4), 2/(-7/4)\} = 2/(-7/4) = -8/7,\ p = 3$.
3. $\min\{9/2, 1, 33/4, -1/4\} = -1/4,\ q = 4$.
5. Multiply row 3 by -4, and then add $9/4, 1/4, -1/2, -3/4$ times of row 3 to rows 1,2,4,5, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
−42	−16	−74			1		−9		16	−14
−7	−7	−12				1	−1		1	1
−18	−4	−33	1				−4		7	−8
13	−2	11					2	1	−3	5
14	2	24		1			3		−5	6

Table 15.1 Equivalence between the associated quantities

Quantity	Reduced	Relation	Revised reduced
Objective column	$\bar{a}_{n+1}$	$=$	$-B^{-1}e_{m+1}$
The righ-hand side	$\bar{b}$	$=$	$B^{-1}b$
Pivot row	$e_p^{\mathrm{T}}\bar{N}$	$=$	$e_p^{\mathrm{T}}B^{-1}N$
Pivot column	$\bar{a}_q$	$=$	$B^{-1}a_q$

Iteration 3:

1. $\bar{a}_{10} \not\geq 0$.
2. $\bar{x}_{10} = \max\{5/(-3), 6/(-5)\} = -6/5$, $p = 5$.
3. $\min\{14, 2, 24, 3\} \geq 0$.
4. $\bar{x}_{10} = -6/5$,

$$\begin{aligned} \bar{x}_B &= \bar{b} - \bar{x}_{10}\bar{a}_{n+1} = (-14, 1, -8, 5, 6)^{\mathrm{T}} - (-6/5)(16, 1, 7, -3, -5)^{\mathrm{T}} \\ &= (26/5, 11/5, 2/5, 7/5, 0)^{\mathrm{T}}. \\ B &= \{6, 7, 4, 9, 5\}. \end{aligned}$$

Basic optimal solution and according objective value are

$$\bar{x} = (0, 0, 0, 2/5, 0, 26/5, 11/5, 0, 7/5)^{\mathrm{T}}, \quad \bar{x}_{10} = -6/5.$$

Now let us derive the revised version of Algorithm 15.2.1.

Let (15.21) be the current reduced tableau, associated with basis and nonbasis matrices B, N. Premultiplying $(A\vdots - e_{m+1}|b)$ by B^{-1} gives a so-called "revised reduced tableau", as written

x_B^{T}	x_N^{T}	x_{n+1}	RHS
I	$B^{-1}N$	$B^{-1}\bar{a}_{n+1}$	$B^{-1}b$

(15.22)

Like in the conventional simplex context, reduced and revised reduced tableaus, associated with the same basis are equivalent; that is, their associated entries are equal. Based on such equivalence, it is easy to transform any reduced tableau to a revised version, and vice versa. As for the implementation of the reduced simplex method, the reduced tableau as a whole is not indispensable, and only a part of its entries are needed. Table 15.1 indicates equivalence relationship between quantities in reduce tableau (15.21) and revised reduced tableau (15.22).

Based on Table 15.1, Algorithm 15.2.1 can be revised as follows, in which $\bar{b}$ and $\bar{a}_{n+1}$ are generated recursively (see (17.13) and (17.15)).

Algorithm 15.2.2 (Reduced simplex algorithm). Initial: $(B, N), B^{-1}, \bar{b} = B^{-1}(b^{\mathrm{T}}, 0)^{\mathrm{T}}$, $\bar{a}_{n+1} = -B^{-1}e_{m+1}$, and consistent $\bar{b} - f\bar{a}_{n+1} \geq 0$. This algorithm solves the reduced problem (15.1).

1. Stop if $\bar{a}_{n+1} \geq 0$ (Unbounded).
2. Determine $\bar{x}_{n+1}$ and row index p such that

$$\bar{x}_{n+1} = \bar{b}_p/\bar{a}_{p,n+1} = \max\{\bar{b}_i/\bar{a}_{i,n+1} \mid \bar{a}_{i,n+1} < 0,\ i = 1, \cdots, m+1\}.$$

3. Compute $\sigma_N = N^{\mathrm{T}} B^{-\mathrm{T}} e_p$.
4. Determine column index $q \in \arg\min_{j \in N} \sigma_j$.
5. If $\sigma_q \geq 0$, then compute $\bar{x}_B = \bar{b} - \bar{x}_{n+1}\bar{a}_{n+1}$, and stop (optimality achieved).
6. Compute $\bar{a}_q = B^{-1}a_q$, $\nu = -\bar{a}_{p,n+1}/\sigma_q$, and $\tau = -\bar{b}_p/\sigma_q$.
7. If $\nu \neq 0$, update $\bar{a}_{n+1} = \bar{a}_{n+1} + \nu(\bar{a}_q - e_p)$.
8. If $\tau \neq 0$, update $\bar{b} = \bar{b} + \tau(\bar{a}_q - e_p)$.
9. Update B^{-1} by (3.23).
10. Update (B, N) by exchanging j_p and q.
11. Go to step 1.

It is noted that the preceding algorithm is practicable, compared with its tableau form, though the former is preferred in illustration in this book.

15.3 Reduced Phase-I: Single-Artificial-Variable

In general, an initial reduced simplex tableau is not feasible, from which the reduce simplex algorithm cannot get started. However, the algorithm can get started from a conventional feasible simplex tableau (see Note after Algorithm 15.2.1), and hence any conventional Phase-I method is applicable. In particular, the single-artificial-variable method, presented in Sect. 13.2, deserves attention, as the associated auxiliary program, involving a single artificial variable, is amenable to be solved by the reduced simplex method.

Assume $\bar{b} = B^{-1}b \ngeq 0$, $\bar{N} = B^{-1}N$. Given some m-dimensional vector $\hat{x}_B \geq 0$, and set $\bar{a}_{n+1} = \bar{b} - \hat{x}_B$. Based on the canonical form of the constraint system, an auxiliary program of form (13.16) can be constructed, i.e.,

$$\begin{array}{ll} \min & x_{n+1}, \\ \text{s.t.} & x_B = \bar{b} - \bar{a}_{n+1}x_{n+1} - \bar{N}x_N, \\ & x, x_{n+1} \geq 0. \end{array}$$

The preceding program is lower bounded, associated with the feasible solution

$$\hat{x}_B = \hat{x}_B, \quad \hat{x}_N = 0, \quad \hat{x}_{n+1} = 1.$$

As the according auxiliary tableau of form (13.18) is itself a feasible reduce simplex tableau, it can be solved by the following slight variant of the reduced simplex algorithm.

Algorithm 15.3.1 (Tableau reduced Phase-I: single-artificial-variable). Initial: reduced simplex tableau of form (13.18). $\bar{a}_{n+1} = \bar{b} - \hat{x}_B$, $\hat{x}_B \geq 0$. This algorithm finds a feasible reduced simplex tableau.

1. Determine $\bar{x}_{n+1}$ and row index p such that

$$\bar{x}_{n+1} = \bar{b}_p/\bar{a}_{p,n+1} = \max\{\bar{b}_i/\bar{a}_{i,n+1} \mid \bar{a}_{i,n+1} < 0,\ i = 1, \cdots, m\}.$$

2. Select column index $q \in \arg\min_{j \in N} \bar{a}_{pj}$.
3. Stop if $\bar{a}_{pq} \geq 0$ (infeasible problem).
4. Convert $\bar{a}_{pq}$ to 1, and eliminate the other nonzeros in the column by elementary transformations.
5. If $\bar{b} \geq 0$, restore the original objective column, and stop (feasibility achieved).
6. Go to step 1.

Example 15.3.1. Solve that following problem, using Algorithm 15.3.1 as reduced Phase-I:

$$
\begin{array}{lrl}
\min & x_{10} = & -2x_1 + 4x_2 + 3x_3 - 3x_4 - 4x_5, \\
\text{s.t.} & & -2x_1 - 6x_2 + 1x_3 - 3x_4 - x_5 + x_6 = -3, \\
& & -x_1 - 9x_2 - 6x_3 + 2x_4 + 3x_5 + x_7 = -7, \\
& & 8x_1 - 6x_2 + 3x_3 + 5x_4 + 7x_5 + x_8 = 4, \\
& & 3x_1 - 2x_2 - 4x_3 - x_4 - 2x_5 + x_9 = -5, \\
& & x_j \geq 0, \quad j = 1, \cdots, 9.
\end{array}
$$

Answer Phase-I: To turn to Phase-II conveniently, it might be well still put the original objective row at the bottom of the tableau, but which will not take a part in pivoting in Phase-I. Set $\hat{x}_B = (1, 1, 0, 1)^{\mathrm{T}}$, $\bar{a}_{10} = (-4, -8, 0, -6)^{\mathrm{T}}$, and take x_{10} column as the auxiliary objective column. Then the initial auxiliary tableau is

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
−2	−6	1	−3	−1	1				−4	−3
−1	−9*	−6	2	3		1			−8	−7
8	−6	3	5	7			1			4
3	−2	−4	−1	−2				1	−6	−5
−2	4	3	−3	−4					−	

The auxiliary program has feasible solution $\bar{x}_B = (1, 1, 0, 1)^{\mathrm{T}}$, $\bar{x}_{10} = 1$.
Phase-I: Call Algorithm 15.3.1.

Iteration 1:

1. $\max\{-3/-4, -7/-8, -5/-6\} = 7/8$, $p = 2$.
2. $\min\{-1, -9, -6, 2, 3\} = -9 < 0$, $q = 2$.
4. Multiply row 2 by $-1/9$, and then add $6, 6, 2, -4$ times of row 2 to rows 1,3,4,5, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
−4/3		5	−13/3	−3	1	−2/3			4/3	5/3
1/9	1	2/3	−2/9	−1/3		−1/9			8/9	7/9
26/3		7	11/3	5		−2/3	1		16/3	26/3
29/9		−8/3	−13/9*	−8/3		−2/9		1	−38/9	−31/9
−22/9		1/3	−19/9	−8/3		4/9			−	−28/9

Iteration 2:

1. $\max\{(-31/9)/(-38/9)\} = 31/38,\ p = 4.$
2. $\min\{29/9, -8/3, -13/9, -8/3, -2/9\} = -8/3 < 0,\ q = 4.$
4. Multiply row 4 by $-9/13$, and then add $13/3, 2/9, -11/3, 19/94$ times of row 4 to rows 1,2,3,5, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
−11		13		5	1			−3	14	12
−5/13	1	14/13		1/13		−1/13		−2/13	20/13	17/13
219/13		3/13		−23/13*		−16/13	1	33/13	−70/13	−1/13
−29/13		24/13	1	24/13		2/13		−9/13	38/13	31/13
−93/13		55/13		16/13		10/13		−19/13	−	25/13

Iteration 3:

1. $\max\{(-1/13)/(-70/13)\} = 1/70,\ p = 3.$
2. $\min\{219/13, 3/13, -23/13, -16/13, 33/13\} = -23/13 < 0,\ q = 5.$
4. Multiply row 3 by $-13/23$, and then add $-5, -1/13, -24/13, -16/13$ times of row 3 to rows 1,2,4,5, respectively:
5. $\bar{b} \geq 0$, Take the original objective column to overwrite the current x_{10} column, resulting in a feasible reduced tableau below:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
842/23		314/23			1	−80/23	65/23	96/23		271/23
8/23	1	25/23				−3/23	1/23	−1/23		30/23
−219/23		−3/23		1		16/23	−13/23	−33/23		1/23
353/23		48/23	1			−26/23	24/23	45/23		53/23
105/23		101/23				−2/23*	16/23	7/23	−1	43/23

Phase-II: Call Algorithm 15.2.1.

Iteration 4:

1. $\bar{a}_{10} \not\geq 0.$
2. $\bar{x}_{10} = \max\{(43/23)/(-1)\} = -43/23,\ p = 5.$
3. $\min\{105/23, 101/23, -2/23, 16/23, 7/23\} = -2/23 < 0,\ q = 7.$
5. Multiply row 5 by $-23/2$, and then add $80/23, 3/23, -16/23, 26/23$ times of row 5 to rows 1,2,3,4, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
−146		−162			1		−25	−8	40	−63
−13/2	1	−11/2					−1	−1/2	3/2	−3/2
27		35		1			5	1	−8	15
−44		−55	1				−8	−2	13	−22
−105/2		−101/2				1	−8	−7/2	23/2	−43/2

Iteration 5:

2. $\bar{x}_{10} = \max\{15/(-8)\} = -15/8,\ p = 3.$
3. $\min\{27, 35, 5, 1, \} = 1 \geq 0.$
4. Basic optimal solution and according objective value:

$$\bar{x} = (0, 21/16, 0, 19/8, 0, 12, 1/16, 0, 0)^{\mathrm{T}}, \quad \bar{x}_{10} = -15/8.$$

15.4 Dual Reduced Simplex Method

This section describes a dual version of Algorithm 15.2.2, still using notations in the previous two sections. To this end, firstly established are optimality conditions and related properties in the reduced simplex context.

Theorem 15.4.1. (x, x_{n+1})*, where* $x_B = \bar{b} - x_{n+1}\bar{a}_{n+1},\ x_N = 0$*, is an optimal solution to (15.1) if the following conditions are satisfied:*

(i) $\bar{b} - x_{n+1}\bar{a}_{n+1} \geq 0, \quad \bar{b}_p - x_{n+1}\bar{a}_{p,n+1} = 0,$ *(primal feasibility)*
(ii) $e_p^{\mathrm{T}}\bar{N} \geq 0, \quad \bar{a}_{p,n+1} < 0.$ *(dual feasibility)*

Proof. The validity comes from Theorems 15.1.2, 15.1.1 and 15.1.1, as well as related discussions in Sect. 15.1. □

The pth row of the reduced simplex tableau, giving the according objective value, is called *objective row*.

Lemma 15.4.1. *Assume that*

$$e_p^{\mathrm{T}}\bar{N} \geq 0, \qquad p \in \{1, \cdots, m+1\}. \tag{15.23}$$

If $\bar{a}_{p,n+1} = 0$ *and* $\bar{b}_p < 0$*, then there is no feasible solution to (15.1).*

If $\bar{a}_{p,n+1} \neq 0$ *and* $\bar{x}_{n+1}$ *satisfies*

$$\bar{b}_p - \bar{x}_{n+1}\bar{a}_{p,n+1} = (<)\ 0,$$

then for any feasible value x'_{n+1} *(if any), the following hold:*

(i) $x'_{n+1} \leq (<) \bar{x}_{n+1}$ *when* $\bar{a}_{p,n+1} > 0.$
(ii) $x'_{n+1} \geq (>) \bar{x}_{n+1}$ *when* $\bar{a}_{p,n+1} < 0.$

Proof. The pth equality constraint of (15.2) is

$$x_p = \bar{b}_p - e_p^{\mathrm{T}}\bar{N}x_N - \bar{a}_{p,n+1}x_{n+1}.$$

Let $\tilde{x} \geq 0$ be a feasible solution, associated with objective value x'_{n+1}. Substituting it to the preceding gives

$$\tilde{x}_p = \bar{b}_p - e_p^{\mathrm{T}}\bar{N}\tilde{x}_N - \bar{a}_{p,n+1}x'_{n+1}. \tag{15.24}$$

In addition, from (15.21) and $\tilde{x} \geq 0$, it follows that

$$-e_p^{\mathrm{T}}\bar{N}\tilde{x}_N \leq 0. \tag{15.25}$$

Condition $\bar{a}_{p,n+1} = 0$ and $\bar{b}_p < 0$ together with (15.25) leads to negativity of the right-hand side of (15.24), as contradicts the nonnegative lett-hand side. Therefore, there is no feasible solution to (15.1).

(i) When $\bar{a}_{p,n+1} > 0$ and $x'_{n+1} > (\geq)\bar{x}_{n+1}$, it holds that

$$\bar{b}_p - \bar{a}_{p,n+1}x'_{n+1} < (\leq)\ \bar{b}_p - \bar{a}_{p,n+1}\bar{x}_{n+1} = (<)0, \tag{15.26}$$

combining which and (15.25) leads to negativeness of the right-hand side of (15.24), as contradicts the nonnegative lett-hand side. Therefore, $x'_{n+1} \leq (<)\bar{x}_{n+1}$.

(ii) When $\bar{a}_{p,n+1} < 0$ and $x'_{n+1} < (\leq)\bar{x}_{n+1}$, (15.26) still holds, as also leads to negativity of the right-hand side of (15.24), leading to a contradiction. Therefore, $x'_{n+1} \geq (>)\bar{x}_{n+1}$. □

As was shown, the reduced simplex method pursues dual feasibility while maintaining primal feasibility. Conversely, pursuing primal feasibility while maintaining dual feasibility will lead to its dual version.

Assume now that the dual feasibility condition (ii) holds. Define $\bar{x}$ as follows:

$$\bar{x}_{n+1} = \bar{b}_p/\bar{a}_{p,n+1}, \quad \bar{x}_B = \bar{b}_B - \bar{x}_{n+1}\bar{a}_{n+1}, \quad \bar{x}_N = 0. \tag{15.27}$$

It is clear that $\bar{x}_{j_p} = 0$.

If $\bar{x}_B \geq 0$ holds, then the primal feasibility condition (i) is satisfied. Thus, $\bar{x}$ is an optimal solution to (15.1). Assume $\bar{x}_B \not\geq 0$. Then the following rule is applicable.

Rule 15.4.1 (Dual row rule) Select row index r such that

$$\bar{x}_{j_r} = \min\{\bar{x}_{j_i} \mid i = 1, \cdots, m+1\} < 0.$$

If $e_r^{\mathrm{T}}\bar{N} \not\geq 0$, in addition, the following rule is well-defined:

Rule 15.4.2 (Dual column rule) Select column index q such that

$$\beta = -\bar{a}_{pq}/\bar{a}_{rq} = \min\{-\bar{a}_{pj}/\bar{a}_{rj} \mid \bar{a}_{rj} < 0,\ j \in N\} \geq 0.$$

If $\beta > 0$, the reduced simplex tableau is said to be *dual nondegenerate*.

Once a pivot is determined, a basis change is executed to drop x_r from and enter x_q to the basis. It is not difficult to show that the resulting pth row still satisfies $e_p^{\mathrm{T}}\bar{N} \geq 0$. If $\bar{a}_{p,n+1} < 0$, then go to the next iteration.

The solution steps are summarized to the following algorithm.

Algorithm 15.4.1 (Dual reduced simplex algorithm: tableau form). Initial: Reduced simplex tableau of form (15.21), where $e_p^{\mathrm{T}}\bar{N} \geq 0$, $\bar{a}_{p,n+1} < 0$. This algorithm solves the reduced problem (15.1).

1. Compute $\bar{x}_{n+1} = \bar{b}_p/\bar{a}_{p,n+1}$.
2. Compute $\bar{x}_B = \bar{b} - \bar{x}_{n+1}\bar{a}_{n+1}$.
3. Select row index $r \in \arg\min\{\bar{x}_{j_i} \mid i = 1, \cdots, m+1, i \neq p\}$.
4. Stop if $\bar{x}_{j_r} \geq 0$.
5. If $J = \{j \in N \mid \bar{a}_{rj} < 0\} = \emptyset$, set $p = r$ and go to step 8.
6. Determine β and column index q such that $\beta = -\bar{a}_{pq}/\bar{a}_{rq} = \min_{j \in J} -\bar{a}_{pj}/\bar{a}_{rj}$.
7. Convert $\bar{a}_{rq}$ to 1, and eliminate the other nonzeros in the pivot column by elementary transformations.
8. Go to step 1 if $\bar{a}_{p,n+1} < 0$.
9. Stop.

Theorem 15.4.2. *Assuming dual nondegeneracy, Algorithm 15.4.1 terminates either at*

(i) Step 4, achieving a basic optimal solution $\bar{x}$; or at
(ii) Step 9, detecting infeasibility of the problem.

Proof. Termination is shown first. Assume that it does not terminate at the current iteration. It is known from steps 1 and 2 that the basic solution $\bar{x}$ is associated with objective value

$$\bar{x}_{n+1} = \bar{b}_p/\bar{a}_{p,n+1}.$$

And $\bar{x}_{j_r} < 0$ implies that

$$\bar{b}_r - \bar{x}_{n+1}\bar{a}_{r,n+1} < 0. \tag{15.28}$$

From the preceding two expressions and $\bar{a}_{p,n+1} < 0$, it follows that

$$\bar{b}_r\bar{a}_{p,n+1} - \bar{b}_p\bar{a}_{r,n+1} > 0. \tag{15.29}$$

There are the following two cases only:

(i) Passing from step 7 to step 8 to go to the next iteration. The new entry in the pth row and $n+1$ column yielded from the basis change in step 7 satisfies

$$\hat{a}_{p,n+1} = \bar{a}_{p,n+1} + \beta\bar{a}_{r,n+1} < 0. \tag{15.30}$$

and the pth component of the new right-hand side is equal to

$$\hat{b}_p \bar{b}_p + \beta \bar{b}_r. \tag{15.31}$$

In step 1 of the next iteration, the objective value calculated from the preceding two impressions is then

$$\hat{x}_{n+1} = \frac{\hat{b}_p}{\hat{a}_{p,n+1}} = \frac{\bar{b}_p + \beta \bar{b}_r}{\bar{a}_{p,n+1} + \beta \bar{a}_{r,n+1}},$$

The difference between the new and old objective values is

$$\begin{aligned}\hat{x}_{n+1} - \bar{x}_{n+1} &= \frac{\bar{b}_p \bar{a}_{p,n+1} + \beta \bar{b}_r \bar{a}_{p,n+1} - \bar{b}_p \bar{a}_{p,n+1} - \beta \bar{b}_p \bar{a}_{r,n+1}}{\bar{a}_{p,n+1}(\bar{a}_{p,n+1} + \beta \bar{a}_{r,n+1})} \\ &= \frac{\beta(\bar{b}_r \bar{a}_{p,n+1} - \bar{b}_p \bar{a}_{r,n+1})}{\bar{a}_{p,n+1}(\bar{a}_{p,n+1} + \beta \bar{a}_{r,n+1})}.\end{aligned}$$

It is known from $\bar{a}_{p,n+1} < 0$ and (15.30) that the denominator in the preceding expression is positive, whereas it is known from $\beta \geq 0$ and (15.29) that the numerator is nonnegative, therefore the objective value never decreases. Under the dual nondegeneracy assumption, the objective value strictly increases.

(ii) Passing from step 5 to step 8 to go to the next iteration. It is noted that $\bar{a}_{r,n+1} < 0$ holds in this case. The difference between the new and old objective values is

$$\frac{\bar{b}_r}{\bar{a}_{r,n+1}} - \frac{\bar{b}_p}{\bar{a}_{p,n+1}} = \frac{\bar{b}_r \bar{a}_{p,n+1} - \bar{b}_p \bar{a}_{r,n+1}}{\bar{a}_{r,n+1} \bar{a}_{p,n+1}},$$

where the denominator is clearly positive whereas, by (15.29), the numerator is also positive. Therefore, the objective value strictly increases.

If the algorithm does not terminate, then under the dual nondegeneracy assumption, the objective value strictly increases monotonically, hence no cycling occurs. This means that there are infinitely many basic solutions, as is a contradiction. Therefore, the algorithm terminates.

Note that $e_p^{\mathrm{T}} \bar{N} \geq 0$ always holds for the algorithm. By Lemma 15.4.1, optimality is achieved while termination occurs at step 4. Now assume that it occurs at step 9, hence the new entry in the pth row and $n + 1$ column satisfies

$$\hat{a}_{p,n+1} = \bar{a}_{p,n+1} + \beta \bar{a}_{r,n+1} \geq 0. \tag{15.32}$$

Since $\bar{a}_{p,n+1} < 0$, in this case the β, determined in step 6, is positive. Assume there is a feasible solution, associated with objective value x'_{n+1}. It is clear that the $\bar{x}_{n+1}$, determined in step 1, satisfies

$$\bar{b}_p - \bar{x}_{n+1}\bar{a}_{p,n+1} = 0, \tag{15.33}$$

and $\bar{a}_{p,n+1} < 0$. Thus it holds by Lemma 15.4.1 that

$$x'_{n+1} \geq \bar{x}_{n+1}. \tag{15.34}$$

In case when passing through steps $7 \rightarrow 8 \rightarrow 9$, on the other hand, it follows from (15.31), (15.32), (15.33), (15.28) and $\beta > 0$ that

$$\hat{b}_p - \bar{x}_{n+1}\hat{a}_{p,n+1} = (\bar{b}_p - \bar{x}_{n+1}\bar{a}_{p,n+1}) + \beta(\bar{b}_r - \bar{x}_{n+1}\bar{a}_{r,n+1}) = \beta(\bar{b}_r - \bar{x}_{n+1}\bar{a}_{r,n+1}) < 0. \tag{15.35}$$

If $\hat{a}_{p,n+1} > 0$, then it is know by Lemma 15.4.1 that

$$x'_{n+1} < \bar{x}_{n+1},$$

which contradicts (15.34), therefore there is no feasible solution; if $\hat{a}_{p,n+1} = 0$, then it is known by (15.35) that $\hat{b}_p < 0$; consequently, there is still no feasible solution, by Lemma 15.4.1. In case when passing through steps $5 \rightarrow 8 \rightarrow 9$,

$$e_r^{\mathrm{T}}\bar{N} \geq 0, \quad \bar{a}_{r,n+1} \geq 0$$

and (15.28) hold. Then it can be similarly shown that there is no feasible solution. □

Example 15.4.1. Solve the following problem by Algorithm 15.4.1:

$$\begin{array}{lrl}
\min & x_{10} = x_1 + 4x_2 + 3x_3 + 2x_4 + 9x_5, & \\
\text{s.t.} & -x_1 + 5x_2 - 4x_4 - 2x_5 + x_6 & = -1, \\
 & -3x_1 - 2x_2 - 6x_3 + x_4 - x_5 + x_7 & = -7, \\
 & -x_2 + 4x_3 - 6x_4 + 4x_5 + x_8 & = 4, \\
 & 5x_1 + 3x_2 - 3x_3 + 3x_4 + 5x_5 + x_9 & = 0, \\
 & x_j \geq 0, \quad j = 1, \cdots, 9. &
\end{array}$$

Answer The initial tableau is

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
−1	5		−4	−2	1					−1
−3	−2	−6	1	−1		1				−7
	−1	4	−6	4			1			4
5	3	−3	3	5				1		
1*	4	3	2	9					−1	

Iteration 1:
To convert it to a reduced simplex tableau, take the smallest, 1, among the first 5 entries in the bottom row (in x_1 column) as pivot. Add $1, 3, -5$ times of row 5 to rows 1,2,4, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
	9	3	−2	7	1				−1	−1
	10	3	7	26		1			−3	−7
	−1	4	−6	4			1			4
	−17	−18*	−7	−40				1	5	
1	4	3	2	9					−1	

which is a dual feasible reduced tableau with $p = 5$.
Call Algorithm 15.4.1.

Iteration 2:

1. $\bar{x}_{10} = 0/(-1) = 0$.
2. $\bar{x}_B = (-1, -7, 4, 0, 0)^{\mathrm{T}}$.
3. $\min\{-1, -7, 4, 0, 0\} = -7 < 0,\ r = 2$.
5. $J = \emptyset,\ p = 2$.
8. $\bar{a}_{2,10} = -3 < 0$.

Iteration 3:

1. $\bar{x}_{10} = (-7)/(-3) = 7/3$.
2. $\bar{x}_B = (-1, -7, 4, 0, 0)^{\mathrm{T}} - (7/3)(-1, -3, 0, 5, -1)^{\mathrm{T}} = (4/3, 0, 4, -35/3, 7/3)^{\mathrm{T}}$.
3. $\min\{4/3, 4, -35/3, 7/3\} = -35/3 < 0,\ r = 4$.
5. $J = \{2, 3, 4, 5\} \neq \emptyset$.
6. $\beta = \min\{-10/(-17), -3/(-18), -7/(-7), -26/(-40)\} = 3/18,\ q = 3$.
7. Multiply row 4 by $-1/18$, and then add $-3, -3, -4, 3$ times of row 4 to rows $1, 2, 3, 5$, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
	37/6		−19/6*	1/3	1			1/6	−1/6	−1
	43/6		35/6	58/3		1		1/6	−13/6	−7
	−43/9		−68/9	−44/9			1	2/9	10/9	4
	17/18	1	7/18	20/9				−1/18	−5/18	
1	7/6		5/6	7/3				1/6	−1/6	

8. $\bar{a}_{2,10} = -13/6 < 0$.

Iteration 4:

1. $\bar{x}_{10} = (-7)/(-13/6) = 42/13$.
2. $\bar{x}_B = (-1, -7, 4, 0, 0)^{\mathrm{T}} - (42/13)(-1/6, -13/6, 10/9, -5/18, -1/6)^{\mathrm{T}}$
   $= (-6/13, 0, 16/39, 35/39, 7/13)^{\mathrm{T}}$.
3. $\min\{-6/13, 16/39, 35/39, 7/13\} = -6/13 < 0,\ r = 1$.
5. $J = \{4\} \neq \emptyset$.
6. $\beta = \min\{-(35/6)/(-19/6)\} = 35/19,\ q = 4$.
7. Multiply row 1 by $-6/19$, and add $-35/6, 68/9, -7/18, -5/6$ times of row 1 to rows 2, 3, 4, 5, respectively :

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	RHS
	−37/19		1	−2/19	−6/19			−1/19	1/19	6/19
	352/19			379/19	35/19	1		9/19	−47/19	−168/19
	−1,111/57			−108/19	−136/57		1	−10/57	86/57	364/57
	97/57	1		43/19	7/57			−2/57	−17/57	−7/57
1	53/19			46/19	5/19			4/19	−4/19	−5/19

8. $\bar{a}_{2,10} = -47/19 < 0$.

Iteration 5:

1. $\bar{x}_{10} = (-168/19)/(-47/19) = 168/47$.
2. $\bar{x}_B = (6/19, -168/19, 364/57, -7/57, -5/19)^{\mathrm{T}}$
   $-(168/47)(1/19, -47/19, 86/57, -17/57, -4/19)^{\mathrm{T}}$
   $= (6/47, 0, 140/141, 133/141, 23/47)^{\mathrm{T}} \geq 0$.
4. Basic optimal solution and according objective value:

$$\bar{x} = (23/47, 0, 133/141, 6/47, 0, 0, 0, 140/141, 0)^{\mathrm{T}}, \quad \bar{x}_{10} = 168/47.$$

Based on the equivalence between the reduced tableau (15.21) and the revised tableau (15.22), it is not difficult to transfer Algorithm 15.4.1 to its revision.

Algorithm 15.4.2 (Dual reduced simplex algorithm). Initial: $(B, N), B^{-1}, \bar{b} = B^{-1}b, \bar{a}_{n+1} = -B^{-1}e_{m+1}$. $\sigma_N = e_p^{\mathrm{T}} B^{-1} N \geq 0,\ \bar{a}_{p,n+1} < 0$. This algorithm solves the reduced problem (15.1).

1. Compute $\bar{x}_{n+1} = \bar{b}_p / \bar{a}_{p,n+1}$.
2. Compute $\bar{x}_B = \bar{b} - \bar{x}_{n+1}\bar{a}_{n+1}$.
3. Determine row index $r \in \arg\min\{\bar{x}_{j_i} \mid i = 1, \cdots, m+1, i \neq p\}$.
4. Stop if $\bar{x}_{j_r} \geq 0$ (optimality achieved).
5. Compute $\omega_N = N^{\mathrm{T}} B^{-\mathrm{T}} e_r$.
6. If $J = \{j \in N \mid \omega_j < 0\} = \emptyset$, set $p = r,\ \sigma_N = \omega_N$, and go to step 14.

7. Determine β and column index q such that $\beta = -\sigma_q/\omega_q = \min_{j\in J} -\sigma_j/\omega_j$.
8. Solve $B\bar{a}_q = a_q$ for $\bar{a}_q$, and compute $\nu = -\bar{a}_{r,n+1}/\omega_q$ and $\tau = -\bar{b}_r/\omega_q$
9. Update: $\bar{a}_{n+1} = \bar{a}_{n+1} + \nu(\bar{a}_q - e_r)$, where.
10. Update: $\bar{b} = \bar{b} + \tau(\bar{a}_q - e_r)$, where.
11. Update B^{-1} by (3.23) $(p = r)$.
12. Update (B, N) by exchanging j_p and q.
13. Solve $B^{\mathrm{T}}h = e_p$ and compute $\sigma_N = N^{\mathrm{T}}h$.
14. Go to step 1 if $\bar{a}_{p,n+1} < 0$.
15. Stop (infeasible problem).

It is possible to improve the dual reduced method by replacing the row Rule 15.4.1. Analogous to the dual largest-distance rule (Sect. 12.3), some rule based on how much the point $(\bar{x}_N, \bar{f})$ violates the constraints seems to be attractive as derived as follows.

Introduce a set of row vectors

$$(w^i)^{\mathrm{T}} = e_i^{\mathrm{T}} B^{-1}(N \mid a_{n+1}), \quad i = 1, \cdots, m+1.$$

For any $i = 1, \ldots, m+1$, the signed distance from point $(\bar{x}_N, \bar{f})$ to the boundary (associated with the ith row of the canonical form)

$$(B^{-1}b)_i - (w^i)^{\mathrm{T}}(x_N^T, f)^{\mathrm{T}} = 0$$

is defined by (see Sect. 2.1)

$$\bar{x}_{j_i}/\|w^i\|.$$

Rule 15.4.3 (Dual row rule: largest-distance) Select pivot row index r such that

$$\bar{x}_{j_r} = \min\{\bar{x}_{j_i}/\|w^i\| \mid i = 1, \cdots, m+1\}.$$

The recurrence formulas of $\|w^i\|^2, i = 1, \cdots, m$ are the same as (12.13) and (12.14).

Alternatively, the following approximate formulas may be used to simplify computations.

Rule 15.4.4 (Dual row rule: approximate largest-distance) Select pivot row index r such that

$$\bar{x}_{j_r} = \min\{\bar{x}_{j_i}/|\bar{a}_{i,n+1}| \mid i = 1, \cdots, m+1\}.$$

It is promising if the other rules, described in Chaps. 11 and 12, are adapted within the reduce simplex framework.

15.5 Dual Reduced Phase-I: The Most-Obtuse-Angle

Algorithm 15.4.1 requires availability of a dual feasible reduced tableau. Dual Phase-I methods presented in Chap. 14 may be applied to provide a conventional dual feasible simplex tableau of (15.1). Then, letting the objective variable x_{n+1} leave the basis gives a dual feasible reduced tableau. However, it would be more direct and effective to achieve the goal in the reduced simplex context based on the most-obtuse-angle heuristics.

The procedure can be written as follows.

Algorithm 15.5.1 (Tableau dual reduced Phase-I: the most-obtuse-angle). Initial: Reduced simplex tableau of form (15.21). This algorithm finds a dual feasible reduced simplex tableau.

1. Select pivot row index $p \in \arg\min\{\bar{a}_{i,n+1} \mid i = 1, \cdots, m+1\}$.
2. Stop if $\bar{a}_{p,n+1} \geq 0$.
3. Select pivot column index $q \in \arg\min_{j \in N} \ \bar{a}_{pj}$.
4. Stop if $\bar{a}_{pq} \geq 0$.
5. Convert $\bar{a}_{pq}$ to 1, and eliminate the other nonzeros in the pivot column by elementary transformations.
6. Go to step 1.

Theorem 15.5.1. *Assume finiteness of Algorithm 15.5.1. It terminates either at*

(i) Step 2, detecting infeasibility or lower unboundedness of the problem; or at
(ii) Step 4, obtaining a dual feasible reduced simplex tableau.

Proof. When it terminates at step 4, dual feasibility condition is satisfied clearly. Assume that termination occurs at step 2. If $\bar{x}$ is a feasible solution to the problem, then it satisfies

$$x_B = \bar{b} - \bar{N}x_N - x_{n+1}\bar{a}_{n+1} \geq 0, \quad x_N \geq 0.$$

Since $-\bar{a}_{n+1} \leq 0$, the preceding expression holds for all x_{n+1} satisfying $x_{n+1} \leq \bar{x}_{n+1}$, therefore the problem is unbound below. □

The preceding Algorithm is used as a dual Phase-1 procedure in the following three examples. In the first example, the infeasibility of the problem will be detected after dual Phase-1. In the second, an optimal solution will be found at the end of the dual Phase-I. In the third, an optimal solution will be achieved after the first iteration of Phase-II.

Example 15.5.1. Solve the following problem by two-phase dual reduced simplex method:

$$\begin{array}{llrcrcrcrcrcrcl}
\min & x_7 = x_1 - x_2 - 2x_3, \\
\text{s.t.} & -x_1 + x_2 + x_3 + x_4 & = & 0, \\
& x_1 - 2x_2 + x_3 + x_5 & = & 1, \\
& x_1 + 2x_2 - 2x_3 + x_6 & = & -8, \\
& x_j \geq 0, \quad j = 1, \cdots, 6.
\end{array}$$

Answer Dual Phase-I: Initial tableau is

x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
−1	1	1	1				
1	−2	1		1			1
1	2	−2			1		−8
1	−1	−2*				−1	

Iteration 1: To drop x_7 from the basis, take $p = 4$. $\min\{0, -1, -2\} = -2, q = 3$.

Multiply row 4 by $-1/2$, and add $-1, -1, 2$ times of row 4 to rows 1,2,3, respectively, obtaining the following reduced simplex tableau:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
−1/2*	1/2		1			−1/2	
3/2	−5/2			1		−1/2	1
	3				1	1	−8
−1/2	1/2	1				1/2	

Phase-I: Call Algorithm 15.5.1.

Iteration 2:

1. $\min\{-1/2, -1/2, 1, 1/2\} = -1/2 < 0, \ p = 1$.
3. $\min\{-1/2, 1/2\} = -1/2, \ q = 1$.
5. Multiply row 1 by -2, and add $-3/2, 1/2$ times of row 1 to rows 2,4, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
1	−1		−2			1	
	−1*		3	1		−2	1
	3				1	1	−8
		1	−1			1	

Iteration 3:

1. $\min\{1, -2, 1, 1\} = -2,\ p = 2$.
3. $\min\{-1, 3\} = -1,\ q = 2$.
5. Multiply row 2 by −1, and add 1, −3 times of row 2 to rows 1,2, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
1			−5	−1		3	−1
	1		−3	−1		2	−1
			9	3	1	−5	−5
		1	−1			1	

Iteration 4:

1. $\min\{3, 2, -5, 1\} = -5,\ p = 3$.
3. $\min\{9, 3\} = 3 > 0$.
4. Dual feasibility achieved.

Dual Phase-II: Call Algorithm 15.4.1.

Iteration 5:

1. $\bar{x}_7 \ = (-5)/(-5) = 1$.
2. $\bar{x}_B \ = (-1, -1, -5, 0)^{\mathrm{T}} - 1 \times (3, 2, -5, 1)^{\mathrm{T}} = (-4, -3, 0, -1)^{\mathrm{T}}, B = \{1, 2, 6, 3\}$.
3. $\min\{-4, -3, 0, -1\} = -4, r = 1$.
6. $\beta \ = \min\{-9/(-5), -3/(-1)\} = 9/5,\ q = 4$.
7. Multiply row 1 by $-1/5$, and add $3, -9, 1$ times of row 1 to rows 2,3,4, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
−1/5			1	1/5		−3/5	1/5
−3/5	1			−2/5		1/5	−2/5
9/5				6/5	1	2/5	−34/5
−1/5		1		1/5		2/5	1/5

8. $\bar{a}_{3,7} = 2/5 > 0$.
9. Stop, detecting infeasibility of the problem.

Example 15.5.2. Solve the following problem by two-phase dual reduced simplex method:

$$\begin{array}{lrrrrrrcr} \min & x_7 = x_1 & - 2x_2 & - 5x_3, & & & & & \\ \text{s.t.} & -2x_1 & + x_2 & + x_3 & +x_4 & & & = & 1, \\ & 2x_1 & - 3x_2 & + x_3 & & +x_5 & & = & -1, \\ & x_1 & + 2x_2 & - x_3 & & & +x_6 & = & 2, \\ & & & x_j \geq 0, & j = 1, \cdots, 6. & & & & \end{array}$$

Answer Initial tableau is

x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
−2	1	1	1				1
2	−3	1		1			−1
1	2	−1			1		2
1	−2	−5				−1	

Iteration 1: $p = 4, \min\{1, -2, -5\} = -5, q = 3$.

Multiply row 4 by $-1/5$, and add $-1, -1, 1$ times of row 4 to rows 1,2,3, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
−9/5*	3/5		1			−1/5	1
11/5	−17/5			1		−1/5	−1
4/5	12/5				1	1/5	2
−1/5	2/5	1				1/5	

Dual Phase-I: Call Algorithm 15.5.1.

Iteration 2:

1. $\min\{-1/5, -1/5, 1/5, 1/5\} = -1/5, \ p = 1$.
3. $\min\{-9/5, 3/5\} = -9/2, q = 1$.
5. Multiply row 1 by $-5/9$, and add $-11/5, -4/5, 1/5$ times of row 1 to rows 2,3,4, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
1	−1/3		−5/9			1/9	−5/9
	−8/3*		11/9	1		−4/9	2/9
	8/3		4/9		1	1/9	22/9
	1/3	1	−1/9			2/9	−1/9

Iteration 3:

1. $\min\{1/9, -4/9, 1/9, 2/9\} = -4/9,\ p = 2.$
3. $\min\{-8/3, 11/9\} = -8/3,\ q = 2.$
5. Multiply row 2 by $-3/8$, and add $1/3, -8/3, -1/3$ times of row 2 to rows 1,3,4, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
1			−17/24	−1/8		1/6	−7/12
	1		−11/24	−3/8		1/6	−1/12
			5/3	1	1	−1/3	8/3
		1	1/24	1/8		1/6	−1/12

$\min\{1/6, 1/6, -1/3, 1/6\} = -1/3, p = 3$; $\min\{5/3, 1\} > 0$, dual feasibility achieved.

Dual Phase-II: Call Algorithm 15.4.1.

Iteration 4:

1. $\bar{x}_7 = (8/3)/(-1/3) = -8.$
2. $\bar{x}_B = (-7/12, -1/12, 8/3, -1/12)^{\mathrm{T}} - (-8)(1/6, 1/6, -1/3, 1/6)^{\mathrm{T}}$
   $= (3/4, 5/4, 0, 5/4)^{\mathrm{T}} \geq 0.$
4. Basic optimal solution and according objective value:

$$\bar{x} = (3/4, 5/4, 5/4, 0, 0, 0)^{\mathrm{T}}, \quad \bar{x}_7 = -8.$$

Example 15.5.3. Solve the following problem by two-phase dual reduced simplex method:

$$\begin{array}{ll} \min & x_9 = -2x_1 - x_2 + 2x_3 + 4x_4, \\ \text{s.t.} & x_1 - 2x_2 + 4x_3 - x_4 + x_5 = 4, \\ & 2x_1 - 3x_2 - x_3 + x_4 + x_6 = -6, \\ & x_1 + x_3 + x_4 + x_7 = 2, \\ & 2x_1 + x_2 - x_3 - 4x_4 + x_8 = -1, \\ & x_j \geq 0, \quad j = 1, \cdots, 8. \end{array}$$

Answer Initial tableau is

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	RHS
1	−2	4	−1	1					4
2	−3	−1	1		1				−6
1		1	1			1			2
2	1	−1	−4				1		−1
−2*	−1	2	4					−1	

Iteration 1:

To drop objective variable x_9 from the basis, take $p = 5$; $\min\{-2, -1, 2, 4\} = -2, q = 1$.

Multiply row 5 by $-1/2$, and add $-1, -2, -1, -2$ times of row 5 to rows 1,2,3,4, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	RHS
	$-5/2$	5	1	1				$-1/2$	4
	-4^*	1	5		1			-1	-6
	$-1/2$	2	3			1		$-1/2$	2
		1					1	-1	-1
1	$1/2$	-1	-2					$1/2$	

Dual Phase-I: Call Algorithm 15.5.1.

Iteration 2:

1. $\min\{-1/2, -1, -1/2, -1, 1/2\} = -1, \ p = 2$.
3. $\min\{-4, 1, 5\} = -4, q = 2$.
5. Multiply row 2 by $-1/4$, and add $5/2, 1/2, -1/2$ times of row 2 to rows 1,3,5, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	RHS
		$35/8$	$-17/8$	1	$-5/8$			$1/8$	$31/4$
	1	$-1/4$	$-5/4$		$-1/4$			$1/4$	$3/2$
		$15/8$	$19/8$		$-1/8$	1		$-3/8$	$11/4$
		1					1	-1	-1
1		$-7/8$	$-11/8^*$		$1/8$			$3/8$	$-3/4$

Iteration 3:

1. $\min\{1/8, 1/4, -3/8, -1, 3/8\} = -1, p = 4$.
3. $\min\{1, 0, 0\} \geq 0$. Dual feasibility achieved.

Dual Phase-II: Call Algorithm 15.4.1.

Iteration 3:

1. $\bar{x}_9 \ = (-1)/(-1) = 1$.
2. $\bar{x}_B \ = (31/4, 3/2, 11/4, -1, -3/4)^{\mathrm{T}} - (1/8, 1/4, -3/8, -1, 3/8)^{\mathrm{T}}$
   $= (61/8, 5/4, 25/8, 0, -9/8)^{\mathrm{T}}$.
3. $\min\{61/8, 5/4, 25/8, 0, -9/8\} = -9/8 < 0, r = 5$.
5. $J \ = \{3, 4\} \neq \emptyset$.
6. $\min\{-1/(-7/8), 0/(-11/8)\} = 0, 0 \leq (-1)/(-3/8), q = 4$.
7. Multiply row 5 by $-8/11$, and add $17/8, 5/4, -19/8$ times of row 5 to rows 1,2,3, respectively:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	RHS
−17/11		63/11		1	−9/11			−5/11	98/11
−10/11	1	6/11			−4/11			−1/11	24/11
19/11		4/11			1/11	1		3/11	16/11
		1					1	−1	−1
−8/11		7/11	1		−1/11			−3/11	6/11

Iteration 4:

1. $\bar{x}_9 = (-1)/(-1) = 1.$
2. $\bar{x}_B = (98/11, 24/11, 16/11, -1, 6/11)^{\mathrm{T}} - (1)(-5/11, -1/11, 3/11, -1, -3/11)^{\mathrm{T}}$
   $= (103/11, 25/11, 13/11, 0, 9/11)^{\mathrm{T}} \geq 0.$
4. Basic optimal solution and according objective value are

$$\bar{x} = (0, 25/11, 0, 9/11, 103/11, 0, 13/11, 0)^{\mathrm{T}}, \quad \bar{x}_9 = 1.$$

15.6 Notes

The reduced simplex method can be traced back to the publication of the "bisection simplex method" (Pan 1991, 1996a), which bisections an interval, including the optimal value, iteration by iteration until achieving optimality. Pan wrote (1991, p. 724).

> Finally, we indicate that justifications of …in fact describes an approach to improving feasible solutions with dual type of canonical form,[1] in a manner similar to that in the conventional method. We are not interested in on this line though, and will develop another method…

At that time, the reduced simplex method seemed ready to come out at one's call. But unfortunately it had been overlooked by not regarding its prospects favorably until recently drawing attention again from the author.

Although there are no numerical results available at present, the method is promising for the following reasons, at least:

Firstly, while its computational effort per iteration is about the same as the conventional simplex method, a novel pivot rule is employed. Consequently, the resulting search direction corresponds to the negative reduced objective gradient as a whole (since the objective function involves a single variable), as seems to be advantageous to the conventional search direction which corresponds to a negative component only. In each iteration, as a result, the decrement in the objective variable's value is just equal to decrement in the original objective value (see also Vemuganti 2004).

[1]It is noting but the reduced simplex tableau.

Secondly, as was mentioned in Sect. 3.9, the numerical stability of the conventional simplex method is actual not good enough, since it could select a too small pivot in magnitude. Occasionally, it must restart from scratch to handle the troublesome case when the basis matrix is close to singularity (Sect. 5.1). In contrast, the reduced simplex method is numerically stable, since it tends to select a large pivot in magnitude. Consequently, the risk of using the restarting remedy is significantly reduced, if not avoided completely. In the stability point of view, therefore, Harris practicable row rule becomes unnecessary, although it would remain useful in the sense of the most-obtuse-angle heuristics (see Sect. 5.6).

Thirdly, a variant of the reduced simplex method shows a bright application outlook, as it seems to be a desirable framework for the implementation of the so-called "controlled-branch method" for solving ILP problems; favorably the associated LP relaxation subprograms can be handled without increasing their sizes at all (see Sect. 25.7).

Finally, the reduced simplex method would become more powerful if the basis is generalized to allow the so-called "deficient-basis" (Sect. 20.6). Moreover, a method for generating an initial deficient-basis and an associated Phase-I method can be derived using the reduced simplex framework (Sect. 20.7). These methods seems to be simple as well as efficient.

As for sparsity, on the other hand, a large amount of fill-ins could yield from transforming the conventional simplex tableau to a reduced one if nonzero costs in the original problem occupy a high proportion (see Sect. 15.2). In this case, the column, firstly selected to enter the basis, should be as sparse as possible. It would be a good idea to obtain a reduced simplex tableau directly from some crash procedure (Sect. 5.5) (excluding the objective variable from the set of basic variables).

On the other hand, some particular scaling should be applied with respect to the objective function, as the reduced simplex method would be sensitive to it.