Exponential Synchronization of a Class of RNNs with Discrete and Distributed Delays*

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Abstract. This paper studies the exponential synchronization of RNNs. The investigations are carried out by means of Lyapunov stability method and the Halanay inequality lemma. Finally, a numerical example with graphical illustrations is given to illuminate the presented synchronization scheme.

Keywords: Recurrent Neural Networks, Exponential synchronization, Stability.

1 Introduction

In the last decade, there has been increasing interest in exploring of recurrent neural networks (RNNs) since they have a wide range of applications, for instance, signal processing, pattern recognition, associative memory and combinatorial optimization. In particular, different types of recurrent neural networks (HNNs, CNNs) have been used and applied to study the qualitative properties such as existence and oscillations of solutions ([2], [4], [5]). Hence, there have been extensive results on the problem of the existence and synchronization of RNNs with constant time delays and time-varying delays in the literature. However, there exist few results on the dynamical behaviors of RNNs with continuously distributed delays. In particular, exponential synchronization of RNNs is of paramount importance in a variety of complex physical, chemical, and biological systems [13]. It is well known that such synchronization strategies have potential applications in several areas such as secure communication ([11], [14]) biological oscillators [3] and animal gaits [7]. It should be mentioned that there are different notions of synchronization, such as phase synchronization [16], generalized synchronization [17], lag synchronization [18], and identical synchronization [15]. In this paper, motivated by the above discussions, we are concerned with the exponential synchronization of a class of recurrent neural networks with varying-time

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coefficients and mixed delays. Thus, the goal in this paper is to design an appropriate controller such that the coupled neural networks remain synchronized. This paper is organized as follows. In Section 2, the synchronization problem to be considered is formulated. In Section 3, a new sufficient condition for the exponential synchronization is obtained. In Section 4, numerical simulations is given to show the validity of theoretical result.

2 Exponential Synchronization Problem

The model of the delayed recurrent neural network considered in this paper is described by the following state equations

$$\begin{aligned} \dot{x_i}(t) &= -a_i x_i(t) + \sum_{j=1}^{n} c_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} d_{ij}(t) f_j(x_j(t-\tau)) \\ &+ \sum_{j=1}^{n} p_{ij}(t) \int_{t-\sigma}^{t} f_j(x_j(s))ds + J_i(t), \\ x_i(t) &= \psi_i(t), \varrho \le t \le 0, 1 \le i \le n, \end{aligned} \tag{1}$$

where n is the number of the neurons in the neural network, $x_i(t)$ denotes the state of the ith neural neuron at time t, $f_j(x_j(t))$ is the activation function of jth neuron at time t. The functions $c_{ij}(\cdot)$, $d_{ij}(\cdot)$ and $p_{ij}(\cdot)$ denote, respectively, the connection weights, the discretely delayed connection weights, and the distributively delayed connection weights, of the jth neuron on the i neuron. $J_i(\cdot)$ is the external bias on the ith neuron, a_i denotes the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs. τ is the constant discrete time delay and $\varrho = \max(\tau, \sigma)$.

Now let us give the following notations and concepts used throughout this paper. For $x \in \mathbb{R}^n$, let $\|x\| = (x^T x)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} x_i^2\right)^{\frac{1}{2}}$ denote the Euclidean vector norm, and for a matrix $A \in \mathcal{M}_n(\mathbb{R})$, let $\|A\|$ indicate the norm of A induced by the Euclidean vector norm, i.e.,$\|A\| = (\lambda_{\max}(A^T A))^{\frac{1}{2}}$, where $\lambda_{\max}(A)$ represents the maximum eigenvalue of matrix A and T denotes the transpose of a matrix. We denote a vector solution of the above system as $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$. The neural network (1) can be rewritten in the following matrix-vector form

$$\begin{aligned} \dot{x}(t) &= -Dx(t) + Cf(x(t)) + Df(x(t-\tau)) + P\int_{t-\sigma}^{t} f(x(s))ds + J(t) \\ x(t) &= \psi(t), \varrho \le t \le 0. \end{aligned} \tag{2}$$

Throughout this paper, we make the following assumptions:

(H_1) For all $1 \le j \le n$, there exist positive constant numbers $L_j > 0$ such that for all $x, y \in \mathbb{R}$

$$|f_j(x) - f_j(y)| < L_j |x - y|,$$

(H_2) For all $1 \leq i \leq n$, $a_i > 0$ and $\tau, \sigma > 0$,

Let us introduce the following controlled slave (or response) system:

$$\begin{aligned} \dot{z}_i(t) &= -a_i z_i(t) + \sum_{j=1}^{n} c_{ij}(t) f_j(z_j(t)) + \sum_{j=1}^{n} d_{ij}(t) f_j(z_j(t-\tau)) \\ &+ \sum_{j=1}^{n} p_{ij}(t) \int_{t-\sigma}^{t} f_j(z_j(s))ds + J_i(t) + u_i \\ z_i(t) &= \varphi_i(t), \varrho \leq t \leq 0, 1 \leq i \leq n, \end{aligned} \tag{3}$$

in which $u_i(t)$ denotes the external control input that will be appropriately designed for an certain control objective.

3 Exponential Synchronization of the RNNs

Definition 1. *The systems* (1) *and the uncontrolled system* (2) *(i.e.* $u_i = 0, \forall 1 \leq i \leq n$ *in* (3)*) are said to be exponentially synchronized if there exist constants* $\eta \geq 1$ *and* $\alpha > 0$ *such that*

$$|x_i(t) - z_i(t)| \leq \eta |x_i(0) - z_i(0)| e^{-\alpha t}$$

for any $t \geq 0$. *Moreover, the constant* α *is defined as the exponential synchronization rate.*

From (1) and (3), the following error dynamics equation can be obtained:

$$\begin{aligned} \dot{e}_i(t) &= -a_i e_i(t) + \sum_{j=1}^{n} c_{ij}(t) F_j(e_j(t)) + \sum_{j=1}^{n} d_{ij}(t) F_j(e_j(t-\tau)) \\ &+ \sum_{j=1}^{n} p_{ij}(t) \int_{t-\sigma}^{t} F_j(e_j(s))ds + u_i, 1 \leq i \leq n, \end{aligned} \tag{4}$$

where $e(t) = x(t) - z(t)$ is the error term, and $F(e(t)) = f(x(t)) - f(z(t))$; $F(e(t-\tau)) = f(x(t-\tau)) - f(z(t-\tau))$.

As long as the control input stabilize the system, the error vector $e(t)$ converges to zero as time t goes to infinity i.e. $\lim_{t\to+\infty} e(t) = \lim_{t\to+\infty} x(t) - z(t) = 0$. If the state variables of the drive system are used to drive the response system, then the control input vector with state feedback is designed as follows:

$$\begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix} = M \begin{pmatrix} x_1(t) - z_1(t) \\ \vdots \\ x_n(t) - z_n(t) \end{pmatrix} = M \begin{pmatrix} e_1(t) \\ \vdots \\ e_n(t) \end{pmatrix} \tag{5}$$

where $M = (m_{ij})_{n\times n}$ is the controller gain matrix and will be appropriately chosen for exponentially synchronizing both drive system and response system. It follows that the error dynamics can be expressed by the following compact form:

$$\dot{e}(t) = -Ae(t) + CF(e(t)) + DF(e(t-\tau)) + P \int_{t-\sigma}^{t} F(e(s))ds + u$$

Lemma 1. *([1]) For all $(n \times n)$ real symmetric matrix M, one has M is positive definite if and only if all its eigenvalues are positive. Furthermore, for all $x \in \mathbb{R}^n$*

$$\lambda_{\min}(M)\|x\|^2 \leq x^T M x \leq \lambda_{\max}(M)\|x\|^2$$

where $\lambda_{\min}(M)$ $(\lambda_{\max}(M))$ represents the minimum (resp. the maximum) eigenvalue of the matrix M.

Lemma 2. *(Halanay inequality lemma [9]). Let $\rho \geq 0$ be a constant, and $V(\cdot)$ be a non-negative continuous function defined for $[-\rho, +\infty[$ which satisfies*

$$\dot{V}(t) \leq -pV(t) + q\left(\sup_{t-\rho \leq s \leq t} V(s)\right)$$

for $t \geq 0$, where p and q are constants. If $p > q > 0$, then

$$V(t) \leq \left(\sup_{-\rho \leq s \leq 0} V(s)\right) e^{-\delta t}$$

for $t > 0$, where δ is a unique positive root of the equation $\delta = p - qe^{\delta\tau}$.

Theorem 1. *Suppose that the conditions $(H_1)-(H_2)$ hold. If the controller gain matrix M in (5) is real symmetric and positive definite satisfying*

$$\frac{\max\limits_{1 \leq i \leq n} L_i\left(2\|C\| + \|D\| + \sigma\|P\|\right)}{2\min\limits_{1 \leq i \leq n} a_i + 2\lambda_{\min}(M)} < 1, (H_3)$$

then the exponential error system (4) converges exponentially.

Proof. First, it is clear that in view of (H_1)

$$\begin{aligned}\|F(e(t-\tau))\|^2 &= \sum_{i=1}^{n} F_i^2(e(t-\tau)) \leq \sum_{i=1}^{n} L_i^2(e_i^2(t-\tau)) \\ &\leq \max_{1 \leq i \leq n} L_i^2 \|e(t-\tau))\|^2 .\end{aligned}$$

Similarly, $\|F(e(t))\| \leq \max\limits_{1 \leq i \leq n} L_i^2 \|e(t))\|^2$. In order to confirm that the origin of (4) is globally exponential synchronization, let us consider the continuous function, V defined as follows: $V(t) = \frac{1}{2} e(t)^T e(t) = \frac{1}{2}\|e\|^2$. Calculating the time derivative of V along the trajectory by using the vector norm, matrix norm and from the inequalities above, we obtain immediately

$$\begin{aligned}\dot{V}(t) = &-e(t)^T Ae(t) + e(t)^T CF(e(t)) + e(t)^T DF(e(t-\tau)) \\ &+e(t)^T P\int\limits_{t-\sigma}^{t} K(t-s)F(e(s))ds - e(t)^T Me(t)\end{aligned}$$

$$\leq -\sum_{i=1}^{n} a_i e_i^2 + \|e\| \|C\| \|F(e(t))\| + \|e\| \|D\| \|F(e(t-\tau))\|$$
$$+ \|e\| \|P\| \int_{t-\sigma}^{t} \|F(e(s))\| \, ds - \lambda_{\min}(M) \|e(t)\|^2$$

By Cauchy Shwartz inequality one can obtain

$$\begin{aligned}
\dot{V}(t) \leq & -\min_{1\leq i\leq n} a_i \|e\|^2 + \|e(t)\| \|C\| \max_{1\leq i\leq n} L_i \|e(t))\| + \max_{1\leq i\leq n} L_i \|e(t)\| \|D\| \|e(t-\tau))\| \\
& + \max_{1\leq i\leq n} L_i \|e(t)\| \|P\| \left(\int_{t-\sigma}^{t} ds \right)^{\frac{1}{2}} \left(\int_{t-\sigma}^{t} \|e(s)\|^2 \, ds \right)^{\frac{1}{2}} - \lambda_{\min}(M) \|e(t)\|^2 \\
\leq & \left(-\min_{1\leq i\leq n} a_i + \|C\| \max_{1\leq i\leq n} L_i - \lambda_{\min}(M) \right) \frac{1}{2} \|e(t)\|^2 + \max_{1\leq i\leq n} L_i \|e(t)\| \times \\
& \times \|D\| \|e(t-\tau))\| + \max_{1\leq i\leq n} L_i \|e(t)\| \|P\| \sqrt{\sigma} \left(\int_{t-\sigma}^{t} \|e(s)\|^2 \, ds \right)^{\frac{1}{2}} \\
\leq & \left(-\min_{1\leq i\leq n} a_i + \|C\| \max_{1\leq i\leq n} L_i - \lambda_{\min}(M) \right) \|e(t)\|^2 + \frac{1}{2} \|D\| \times \\
\times & \max_{1\leq i\leq n} L_i \left(\|e(t)\|^2 + \|e(t-\tau)\|^2 \right) + \sqrt{\sigma} \max_{1\leq i\leq n} L_i \|e(t)\| \|P\| \sqrt{\sigma} \left(\max_{t-\sigma\leq s\leq t} \|e(s)\|^2 \right)^{\frac{1}{2}} \\
\leq & \left(-\min_{1\leq i\leq n} a_i + \|C\| \max_{1\leq i\leq n} L_i - \lambda_{\min}(M) \right) \|e(t)\|^2 + \frac{1}{2} \|D\| \max_{1\leq i\leq n} L_i \times \\
\times & \left(\|e(t)\|^2 + \|e(t-\tau)\|^2 \right) + \frac{\sigma}{2} \max_{1\leq i\leq n} L_i \|P\| \left(\|e(t)\|^2 + \max_{t-\sigma\leq s\leq t} \|e(s)\|^2 \right) \\
\leq & \left(-2 \min_{1\leq i\leq n} a_i + 2 \|C\| \max_{1\leq i\leq n} L_i + \|D\| \max_{1\leq i\leq n} L_i - 2\lambda_{\min}(M) + \sigma \max_{1\leq i\leq n} L_i \|P\| \right) V(t) \\
& + \frac{1}{2} \|D\| \max_{1\leq i\leq n} L_i \|e(t-\tau))\|^2 + \frac{\sigma}{2} \max_{1\leq i\leq n} L_i \|P\| \max_{t-\sigma\leq s\leq t} \|e(s)\|^2 \\
\leq & -\left(2 \min_{1\leq i\leq n} a_i - 2 \|C\| \max_{1\leq i\leq n} L_i - \|D\| \max_{1\leq i\leq n} L_i + 2\lambda_{\min}(M) - \sigma \max_{1\leq i\leq n} L_i \|P\| \right) V(t) \\
& + \max_{1\leq i\leq n} L_i \left(\|D\| + \frac{\sigma}{2} \|P\| \right) \max_{t-\rho\leq s\leq t} V(s)
\end{aligned}$$

Now, in virtue of lemma 1 and (H_3) it follows that $V(t) \leq \left(\sup_{-\rho\leq s\leq t0} V(s) \right) e^{-\delta t}$ where

$$\delta = \left(2 \min_{1\leq i\leq n} a_i - \max_{1\leq i\leq n} L_i \left(2 \|C\| + \|D\| + \sigma \|P\| \right) + 2\lambda_{\min}(M) \right)$$
$$- \max_{1\leq i\leq n} L_i \left(\|D\| + \frac{\sigma}{2} \|P\| \right) e^{\delta\rho}.$$

Therefore, $V(e(t))$ converges to zero exponentially, which in turn implies that $e(t)$ also converges globally and exponentially to zero with a convergence rate of $\frac{\delta}{2}$, i.e.$\|e(t)\| \leq \left(\sup_{-\rho\leq s\leq s} \|\phi(s) - \psi(s)\| \right) e^{-\delta \frac{t}{2}}$.

In other words, every trajectory $z_i(t)$ of (3) must synchronize exponentially toward the $x_i(t)$ with a convergence rate of $\frac{\delta}{2}$. This completes the proof.

Remark 1. Clearly and from the above study, the sufficient condition for exponential synchronization of systems (1) and (3) depends only on the continuous delay but relies on the connection weights and the controller gain. Besides, when for all $1 \leq i, j \leq n$, $p_{ij} = 0$ and $J(\cdot)$ is constant, model (1) and (2) in this paper become the models in [12]. On the other hand, in [6] under similar hypothesis, authors derive exponential synchronization criteria for two chaotic neural networks under the configuration of the master slave mode by applying the Lyapunov stability approach and the Halanay inequality. However, the conditions of Theorem 1 in this paper is easy to test in practice. So, the results in [6] is a special case of the results in this paper. It should be mentioned that the method in this paper is not as same as the method in [8] and [10].

4 An Illustrative Example

In order to illustrate some feature of our main results, in this section, we will apply our main results to some special three-dimensional systems and demonstrate the efficiencies of our criteria.

Example 1. Let us consider the following delayed recurrent neural network

$$\dot{x_i}(t) = -a_i x_i(t) + \sum_{j=1}^{3} c_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{3} d_{ij}(t) f_j(x_j(t-\tau)) + \sum_{j=1}^{3} p_{ij}(t) \int_{t-\sigma}^{t} f_j(x_j(s))ds + J_i(t),$$

and the response recurrent neural network is designed as follows:

$$\dot{z_i}(t) = -a_i z_i(t) + \sum_{j=1}^{3} c_{ij}(t) f_j(z_j(t)) + \sum_{j=1}^{3} d_{ij}(t) f_j(z_j(t-\tau)) + \sum_{j=1}^{3} p_{ij}(t) \int_{t-\sigma}^{t} f_j(z_j(s))ds + J_i(t) + u_i(t)$$

$$\dot{e}(t) = -Ae(t) + CF(e(t)) + DF(e(t-\tau)) + P \int_{t-\sigma}^{t} F(e(s))ds + u$$

Pose: $a_1 = 11, a_2 = 17, a_3 = 13, f_j(x) = x, \tau = 1, \sigma = 2$ and

$$C = \begin{pmatrix} 1 & -3 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, D = \begin{pmatrix} 2 & -3 & -1 \\ 0 & -1 & 4 \\ -1 & 0 & 2 \end{pmatrix}, P = \begin{pmatrix} 0.5 & -1.5 & 1 \\ 1 & 0 & 2 \\ 2 & -0.5 & 1 \end{pmatrix}, M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

So the condition (H_3) is satisfied since

$$\frac{\max\limits_{1\leq i\leq 3} L_i\left(2\times 3.8408+4.6862+2\times 3.3126\right)}{2\times 9+2\times\lambda_{\min}\left(M\right)}=0.86332<1$$

By using the matlab Toobox, one can obtain the graphical illustration Fig. 1

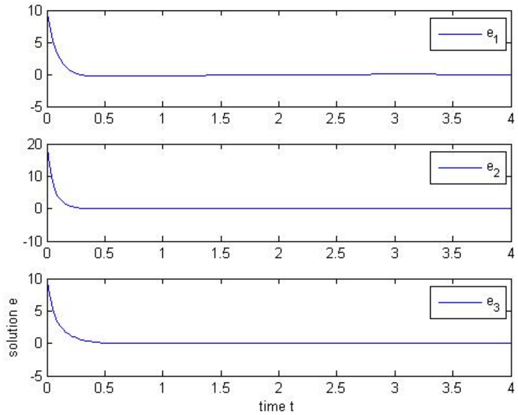

Fig. 1. The exponential synchronization error

5 Conclusion

By Constructing an appropriate linear feedback controller, this paper addresses the problem of exponential synchronization of a class of recurrent neural networks with mixed delays. Based on the properties of a recurrent attractor, we gave a new synchronization criterion for the considered system by using Lyapunov method and the well known Halanay lemma. To demonstrate the effectiveness of the proposed method, a numerical example is used.

References

1. Allen, L., Bridges, T.J.: Numerical exterior algebra and the compound matrix method. Numerische Mathematik 92, 197–232 (2002)
2. Ammar, B., Chérif, F., Alimi, M.A.: Existence and Uniqueness of Pseudo Almost Periodic Solutions of Recurrent Neural Networks with Time-Varying Coefficients and Mixed Delays. IEEE Transactions on Neural Networks and Learning Systems 23(1), 109–118 (2012)
3. Bazhenov, M., Huerta, R., Rabinovich, M.I., Sejnowski, T.: Cooperative behavior of a chain of synaptically coupled chaotic neurons. Phys. D 116, 392–400 (1998)
4. Cao, J.D., Wang, J.: Global exponential stability and periodicity of recurrent neural networks with time delay. IEEE Trans. Circ. Syst. I 52(5), 920–931 (2005)

5. Chérif, F.: Existence and global exponential stability of pseudo almost periodic solution for SICNNs with mixed delays. JAMC 39(1) (2011)
6. Cheng, C.-J., Liao, T.-L., Hwang, C.-C.: Exponential synchronization of a class of chaotic neural networks. Chaos, Solitons and Fractals 24, 197–206 (2005)
7. Collins, J.J., Stewart, I.N.: Coupled nonlinear oscillators and the symmetries of animal gaits. J. Nonlinear Sci. 3, 349–392 (1993)
8. Cui, B., Lou, X.: Synchronization of chaotic recurrent neural networks with time-varying delays using nonlinear feedback control. Chaos, Solitons and Fractals 39, 288–294 (2009)
9. Gopalsamy, K.: Stability and oscillations in delay differential equations of population dynamics. Kluwer Academic Publishers, The Netherlands (1992)
10. Huang, T., Chen, G., Kurths, J.: Synchronization of chaotic systems with time-varying coupling delays. Discrete and Continuous Dynamical Systems, Series B 16(4) (2011)
11. Itoh, M., Wu, C.W., Chua, L.O.: Communication systems via chaotic signals from a reconstruction viewpoint. Int. J. Bifur. Chaos Appl. Sci. Eng. 7, 275–286 (1997)
12. Lu, H., van Leeuwen, C.: Synchronization of chaotic neural networks via output or state coupling. Chaos, Solitons and Fractals 30, 166–176 (2006)
13. Ott, E., Grebogi, C., Yorke, J.A.: Controlling chaos. Rev. Lett. 64, 1196–1199 (1990)
14. Parlitz, U., et al.: Transmission of digital signals by chaotic synchronization. Int. J. Bifur. Chaos Appl. Sci. Eng. 2, 973–977 (1992)
15. Pecora, L.M., Carroll, T.L.: Synchronization in chaotic systems. Phys. Rev. Lett. 64, 821–824 (1990)
16. Rosenblum, M.G., Pikovsky, A.S., Kurths, J.: Phase synchronization of chaotic oscillators. Phys. Rev. Lett. 76, 1804–1807 (1996)
17. Rulkov, N.F., Sushchik, M.M., Tsimring, L.S., Abarbanel, H.D.I.: Generalized synchronization of chaos in directionally coupled chaotic systems. Phys. Rev. E 51, 980–994 (1995)
18. Rosenblum, M.G., Pikovsky, A.S., Kurths, J.: From phase to lag synchronization in coupled chaotic oscillators. Phys. Rev. Lett. 78, 4193–4196 (1997)