Friendship and Stable Matching*

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Abstract. We study stable matching problems in networks where players are embedded in a social context, and may incorporate friendship relations or altruism into their decisions. Each player is a node in a social network and strives to form a good match with a neighboring player. We consider the existence, computation, and inefficiency of stable matchings from which no pair of players wants to deviate. When the benefits from a match are the same for both players, we show that incorporating the well-being of other players into their matching decisions significantly decreases the price of stability, while the price of anarchy remains unaffected. Furthermore, a good stable matching achieving the price of stability bound always exists and can be reached in polynomial time. We extend these results to more general matching rewards, when players matched to each other may receive different utilities from the match. For this more general case, we show that incorporating social context (i.e., "caring about your friends") can make an even larger difference, and greatly reduce the price of anarchy. We show a variety of existence results, and present upper and lower bounds on the prices of anarchy and stability for various matching utility structures.

1 Introduction

Stable matching problems capture the essence of many important assignment and allocation tasks in economics and computer science. The central approach to analyzing such scenarios is two-sided matching, which has been studied intensively since the 1970s in both the algorithms and economics literature. An important variant of stable matching is matching with cardinal utilities, when each match can be given numerical values expressing the *quality* or *reward* that the match yields for each of the incident players [5]. Cardinal utilities specify the quality of each match instead of just a preference ordering, and they allow the comparison of different matchings using measures such as social welfare. A particularly appealing special case of cardinal utilities is known as correlated stable matching, where both players who are matched together obtain the same reward. In addition to the wide-spread applications of correlated stable matching in, e.g., market sharing [16], social networks [17], and distributed computer

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networks [27], this model also has favorable theoretical properties such as the existence of a potential function. It guarantees existence of a stable matching even in the non-bipartite case, where every pair of players is allowed to match [2,27].

When matching individuals in a social environment, it is often unreasonable to assume that each player cares only about their own match quality. Instead, players incorporate the well-being of their friends/neighbors as well, or that of friends-of-friends. Players may even be altruistic to some degree, and consider the welfare of all players in the network. Caring about friends and altruistic behavior is commonly observed in practice and has been documented in laboratory experiments [14,25]. In addition, in economics there exist recent approaches towards modeling and analyzing *other-regarding preferences* [15]. Given that other-regarding preferences are widely observed in practice, it is a fundamental question to model and characterize their influence in classic game-theoretic environments. Recently, the impact of social influence on congestion and potential games has been characterized prominently in [8,10–12,18–20].

We consider a natural approach to incorporate social effects into partner selection and matching scenarios by studying how social context influences stability and efficiency in matching games. Our model of social context is similar to recent approaches in algorithmic game theory and uses dyadic influence values tied to the hop distance in the graph. In this way, every player may consider the well-being of every other player to some degree, with the degree of this regardfulness possibly decaying with hop distance. The perceived utility of a player is then composed of a weighted average of player utilities. Players who only care about their neighbors or fully altruistic players are special cases of this model.

For matching in social environments, the standard model of correlated stable matching may be too constraining compared to general cardinal utilities, because matched players receive exactly the same reward. Such an *equal sharing* property is intuitive and bears a simple beauty, but other reward sharing methods might be more natural in different contexts. For instance, in theoretical computer science it is common practice to list authors alphabetically, but in other disciplines the author sequence is carefully designed to ensure a proper allocation of credit to the authors of a joint paper. The credit is often supposed to be allocated in terms of input, i.e., the first author is the one that contributed most to the project. Such input-based or proportional sharing is then sometimes overruled with sharing based on intrinsic or acquired social status, e.g., when a distinguished expert in a field is easily recognized and subconsciously credited most with authorship of an article. We are interested in how such unequal reward sharing rules affect stable matching scenarios. We consider a large class of local reward sharing rules and characterize the impact of unequal sharing on existence and inefficiency of stable matchings, both in cases when players are embedded in a social context and when they are not.

1.1 Stable Matching within a Social Context

Correlated stable matching is a prominent subclass of general ordinary stable matching. We are given a (non-bipartite) graph $G = (V, E)$ with edge weights

r_e. In a matching M, if node u is matched to node v, the reward of node u is defined to be exactly r_e. This can be interpreted as both u and v getting an identical reward from being matched together. We will also consider unequal reward sharing, where u obtains reward r_e^u and v obtains reward r_e^v with $r_e^u + r_e^v = r_e$. Therefore, the preference ordering of each node over its possible matches is implied by the rewards that this node obtains from different edges. A pair of nodes (u, v) is called a *blocking pair* in matching M if u and v are not matched to each other in M, but can both strictly increase their rewards by being matched to each other instead. A matching with no blocking pairs is called a *stable matching.*

While the matching model above has been well-studied, we are interested in stable matchings that arise in the presence of social context. Denote the reward obtained by a node v in a matching M as $R_v(M)$. When it is clear which matching we are referring to, we will simply denote this reward by R_v. We now consider the case when node v not only cares about its own reward, but also about the rewards of its friends. Specifically, the *perceived* or *friendship utility* of node v in matching M is defined as

$$U_v = R_v + \sum_{d=1}^{diam(G)} \alpha_d \sum_{u \in N_d(v)} R_u,$$

where $N_d(v)$ is the set of nodes with shortest distance exactly d from v, and $1 \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq 0$ (we use $\boldsymbol{\alpha}$ to denote the vector of α_i values). In other words, for a node u that is distance d away from v, the utility of v increases by an α_d factor of the reward received by u. Thus, if $\alpha_d = 0$ for all $d \geq 2$, this means that nodes only care about their neighbors, while if all $\alpha_d > 0$, this means that nodes are altruistic and care about the rewards of everyone in the graph. The perceived utility is the quantity that the nodes are trying to maximize, and thus, in the presence of friendship, a blocking pair is a pair of nodes such that each node can increase its *perceived utility* by matching to each other. Given this definition of blocking pair, a stable matching is again defined as a matching without such a blocking pair. Note that while our definition includes α_d for all d, it is easy to see that only the values of α_1 and α_2 matter to the stability of a matching, since a deviation of a blocking pair only changes the R_v values of adjacent nodes.

Centralized Optimum and the Price of Anarchy. We study the social welfare of equilibrium solutions and compare them to an optimal centralized solution. The social welfare is the sum of rewards, i.e., a *social optimum* is a matching that maximizes $\sum_v R_v$. Notice that, while this is equivalent to maximizing the sum of player utilities when $\boldsymbol{\alpha} = 0$, this is no longer true with social context (i.e., when $\boldsymbol{\alpha} \neq 0$). Nevertheless, as in e.g. [11, 28], we believe this is a well-motivated and important measure of solution quality, as it captures the overall performance of the system, while ignoring the perceived "good-will" effects of friendship and altruism. For example, when considering projects done in pairs, the reward of an edge can represent actual productivity, while the perceived utility may not.

To compare stable solutions with a social optimum, we will often consider the price of anarchy and the price of stability. When considering stable matchings, by the price of anarchy (resp. stability) we will mean the ratio of social welfare of the social optimum and the social welfare of the worst (resp. best) stable matching.

1.2 New Results and Related Work

Our Results. In Section 2, we consider stable matching with friendship utilities and equal reward sharing. In this case, a stable matching exists and the price of anarchy (ratio of the maximum-weight matching with the worst stable matching) is at most 2, the same as in the case without friendship. The price of stability, on the other hand, improves significantly in the presence of friendship – we show a tight bound of $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$. Intuitively, the bound depends only on α_1, α_2 because a deviation by a blocking pair (u, v) only affects rewards R_w for nodes w neighboring u or v. Thus, the stability of a matching depends only on the graph and α_1, α_2; changing α_i with $i \geq 3$ does not change the stability of a matching. In addition to providing a tight bound on the price of stability, we present a dynamic process that converges to a stable matching of at least this quality in polynomial time, if initiated from the maximum-weight matching. Our results imply that for socially aware players, the price of stability can greatly improve: e.g., if $\alpha_1 = \alpha_2 = \frac{1}{2}$, then the price of stability is at most $\frac{6}{5}$, and a solution of this quality can be obtained efficiently.

In Section 3 we instead study general reward sharing schemes. When two nodes matched together may receive different rewards, an integral stable matching may not exist. Thus, we focus on *fractional* stable matchings which we show to always exist, even with friendship utilities. Fractional matching is well-motivated in a social context, since the fractional amount of an edge in the matching corresponds to the strength of the link/relationship between this pair of nodes. The total relationships of any single node should add up to at most 1, modeling the fact that a single person cannot be involved in an unlimited amount of relationships. We show that for arbitrary reward sharing, prices of anarchy and stability depend on the level of inequality among reward shares. Specifically, if R is the maximum ratio over all edges $(u, v) \in E$ of the reward shares of node u and v, then the price of anarchy is at most $\frac{(1+R)(1+\alpha_1)}{1+\alpha_1 R}$. Thus, compared to the equal reward sharing case, if sharing is extremely unfair (R is unbounded), then friendship becomes even more important: changing α_1 from 0 to $\frac{1}{2}$ reduces the price of anarchy from unbounded to at most 3. In addition, for several particularly natural local reward sharing rules, we show that an integral stable matching exists, give improved price of anarchy guarantees, and show tight lower bounds.

Related Work. Stable matching problems have been studied intensively over the last few decades. On the algorithmic side, existence, efficient algorithms, and improvement dynamics for two-sided stable matchings have been of interest (for references, see standard textbooks, e.g., [29]). In this paper we address the more general stable roommates problem, in which every player can be matched to every other player. For general preference lists, there have been numerous works characterizing and algorithmically deciding existence of stable matchings [13, 30, 31]. In contrast, fractional stable matchings are always guaranteed to exist and exhibit various interesting polyhedral properties [1,31]. For the correlated stable roommates problem, existence of (integral) stable matchings is guaranteed by a potential function argument [2, 27], and convergence time of random improvement dynamics is polynomial [3]. In [6], price of anarchy and stability bounds for *approximate* correlated stable matchings were provided. In contrast, we study friendship, altruism, and unequal reward sharing in stable roommates problems with cardinal utilities.

Another line of research closely connected to some of our results involves game-theoretic models for contribution. In [7] we consider a contribution game tied closely to matching problems. Here players have a budget of effort and contribute parts of this effort towards specific projects and relationships. For more related work on the contribution game, see [7]. All previous results for this model concern equal sharing and do not address the impact of the player's social context. As we discuss in the full version of this paper in [4], most of our results for friendship utilities can also be extended to such contribution games.

Analytical aspects of reward sharing have been a central theme in game theory since its beginning, especially in cooperative games. Recently, there have been prominent algorithmic results also for network bargaining [21, 23] and credit allocation problems [22]. A recent line of work [32,33] treats extensions of cooperative games, where players invest into different coalitional projects. The main focus of this work is global design of reward sharing schemes to guarantee cooperative stability criteria. Our focus here is closer to, e.g., recent work on profit sharing games [9, 26]. We are interested in existence, computational complexity, and inefficiency of stable states under different reward sharing rules, with an aim to examine the impact of social context on stable matchings.

Our notion of a player's social context is based on numerical influence parameters that determine the impact of player rewards on the (perceived) utilities of other players. A recently popular model of altruism is inspired by Ledyard [24] and has generated much interest in algorithmic game theory [11, 12, 19]. In this model, each player optimizes a perceived utility that is a weighted linear combination of his own utility and the utilitarian welfare function. Similarly, for surplus collaboration [8] perceived utility of a player consists of the sum of players utilities in his neighborhood within a social network. Our model is similar to [10, 20] and smoothly interpolates between these global and local approaches.

2 Matching with Equal Reward Sharing

We begin by considering correlated stable matching in the presence of friendship utilities. In this section, the reward received by both nodes of an edge in a matching is the same, i.e., we use equal reward sharing, where every edge e has an inherent value r_e and both endpoints receive this value if edge e is in the

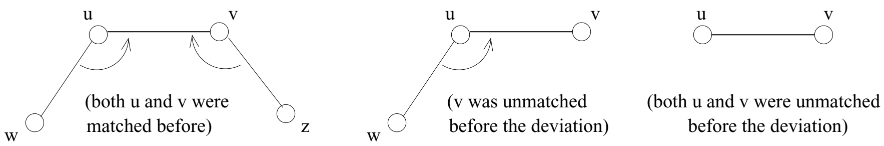

Fig. 1. (Left) Biswivel deviation (Right,Middle) Swivel deviation

matching. We consider more general reward sharing schemes in Section 3. Recall that the friendship utility of a node v increases by $\alpha_d R_u$ for every node u, where d is the shortest distance between v and u. We abuse notation slightly, and let α_{uv} denote α_d, so if u and v are neighbors, then $\alpha_{uv} = \alpha_1$.

Given a matching M, let us classify the following types of improving deviations that a blocking pair can undergo.

Definition 1. *We call an improving deviation a* **biswivel** *whenever two neighbors u and v switch to match to each other, such that both u and v were matched to some other nodes before the deviation in M.*

See Figure 1 for explanation. For such a biswivel to exist in a matching, the following necessary and sufficient conditions must hold.

$$(1+\alpha_1)r_{uv} > (1+\alpha_1)r_{uw} + (\alpha_1 + \alpha_{uz})\, r_{vz} \tag{1}$$

$$(1+\alpha_1)r_{uv} > (1+\alpha_1)r_{vz} + (\alpha_1 + \alpha_{vw})\, r_{uw} \tag{2}$$

Intuitively, the left side of Inequality (1) quantifies the utility gained by u because of getting matched to v and the right side quantifies the utility lost by u because of u and v breaking their present matchings with w and z respectively. Hence, Inequality (1) implies that u gains more utility by getting matched with v than it loses because of u and v breaking their matchings with v and z. Inequality (2) can similarly be explained in the context of node v.

Definition 2. *We call an improving deviation a* **swivel** *whenever two neighbors get matched such that at least one node among the two neighbors was not matched before the deviation.*

See Figure 1 for explanation. For a swivel to occur, it is easy to see that the reward r_{uv} of the new edge added to the matching must be strictly larger than the rewards of edges that u or v were matched to before (if any).

2.1 Existence and Social Welfare

Theorem 1. *A stable matching exists in stable matching games with friendship utilities. Moreover, the set of stable matchings without friendship (i.e., when $\boldsymbol{\alpha} = \mathbf{0}$) is a subset of the set of stable matchings with friendship utilities on the same graph.*

Theorem 2. *The price of anarchy in stable matching games with friendship utilities is at most 2, and this bound is tight.*

2.2 Price of Stability and Convergence

The main result in this section bounds the price of stability in stable matching games with friendship utilities to $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$, and this bound is tight (see Theorem 4 below). This bound has some interesting implications. It is decreasing in each α_1 and α_2, hence having friendship utilities always yields a lower price of stability than without friendship utilities. Also, note that values of $\alpha_3, \alpha_4, ..., \alpha_{diam(G)}$ have no influence. This is not surprising: after a deviation by a blocking pair (u, v), the rewards R_w remain the same for all w except those neighboring u or v. Thus, caring about players more than distance 2 away does not improve the price of stability in any way. Also, if $\alpha_1 = \alpha_2 = 1$, then PoS = 1, i.e., there will exist a stable matching which will also be a social optimum. Thus *loving thy neighbor and thy neighbor's neighbor but nobody beyond* is sufficient to guarantee that there exists at least one socially optimal stable matching. In fact, due to the shape of the curve, even small values of friendship quickly decrease the price of stability; e.g., setting $\alpha_1 = \alpha_2 = 0.1$ already decreases the price of stability from 2 to ~ 1.7.

We will establish the price of stability bound by defining an algorithm that creates a good stable matching in polynomial time. One possible idea to create a stable matching that is close to optimum is to use a Best-Blocking-Pair algorithm: start with the best possible matching, i.e., a social optimum, which may or may not be stable. Now choose the "best" blocking pair (u, v): the one with maximum edge reward r_{uv}. Allow this blocking pair to get matched to each other instead of their current partners. Check if the resulting matching is stable. If it is not stable then allow the best blocking pair for this matching to get matched. Repeat the procedure until there are no more blocking pairs, thereby obtaining a stable matching.

This algorithm gives the desired price of stability and running time bounds for the case of "altruism" when all α_i are the same, see Corollary 1 below. To provide the desired bound with general friendship utilities, we must alter this algorithm slightly using the concept of *relaxed* blocking pair.

Definition 3. *Given a matching M, we call a pair of nodes (u, v) a relaxed blocking pair if either (u, v) form an improving swivel, or u and v are matched to w and z respectively, with the following inequalities being true:*

$$(1 + \alpha_1)r_{uv} > (1 + \alpha_1)r_{uw} + (\alpha_1 + \alpha_2)\, r_{vz} \tag{3}$$

$$(1 + \alpha_1)r_{uv} > (1 + \alpha_1)r_{vz} + (\alpha_1 + \alpha_2)\, r_{uw} \tag{4}$$

In other words, a relaxed blocking pair ignores the possible edges between nodes u and z, and has α_2 in the place of α_{uz} (similarly, α_2 in the place of α_{vw}). It is clear from this definition that a blocking pair is also a relaxed blocking pair, since the conditions above are less constraining than Inequalities (1) and (2). Thus a matching with no relaxed blocking pairs is also a stable matching. We will call a relaxed blocking pair satisfying Inequalities (3) and (4) a *relaxed biswivel,* which may or may not correspond to an improving deviation, since a relaxed blocking pair is not necessarily a blocking pair. We define the Best-Relaxed-Blocking-Pair Algorithm to be the same as the Best-Blocking-Pair algorithm, except at each step it chooses the best *relaxed* blocking pair.

Dynamics: To establish the efficient running time of Best-Relaxed-Blocking-Pair and the price of stability bound of the resulting stable matching, we first analyze the dynamics of this algorithm and prove some helpful lemmas. We can interpret the algorithm as a sequence of swivel and relaxed biswivel deviations, each inserting one edge into M, and removing up to two edges. Note that it is not guaranteed that the inserted edge will stay forever in M, as a subsequent deviation can remove this edge from M. Let $O_1, O_2, O_3, \cdots$ denote this sequence of deviations, and $e(i)$ denote the edge which got inserted into M because of O_i. We analyze the dynamics of the algorithm by using the following key lemma.

Lemma 1. *Let O_j be a relaxed biswivel that takes place during the execution of the best relaxed blocking pair algorithm. Suppose a deviation O_k takes place before O_j. Then we have $r_{e(k)} \geq r_{e(j)}$. Furthermore, if O_k is a relaxed biswivel then $e(k) \neq e(j)$ (thus at most $|E(G)|$ relaxed biswivels can take place during the execution of the algorithm).*

It is important to note that this lemma does *not* say that $r_{e(i)} \geq r_{e(j)}$ for $i < j$. We are only guaranteed that $r_{e(i)} \geq r_{e(j)}$ for $i < j$ if O_j is a *relaxed biswivel*. Between two successive relaxed biswivels O_k and O_j, the sequence of $r_{e(i)}$ for consecutive swivels can and does increase as well as decrease, and the same edge may be added to the matching multiple times. All that is guaranteed is that $r_{e(j)}$ for a biswivel O_j will have a lower value than all the preceding $r_{e(i)}$'s. Thus, this lemma suggests a nice representation of Best-Relaxed-Blocking-Pair in terms of phases, where we define a *phase* as a subsequence of deviations that begins with a relaxed biswivel and ends with the next relaxed biswivel. Lemma 1 guarantees that at the start of each phase, the $r_{e(j)}$ value is smaller than the values in all previous phases, and that there is only a polynomial number of phases.

Theorem 3. Best-Relaxed-Blocking-Pair *outputs a stable matching after $O(m^2)$ iterations, where m is the number of edges in the graph.*

Notice that in each phase, the value of the matching only increases, since swivels only remove an edge if they add a better one. Below, we use the fact that only relaxed biswivel operations reduce the cost of the matching to bound the cost of the stable matching this algorithm produces. We do this by tracing what an edge of M^* "gets mapped to" as swivel and biswivel operations "change" this edge into another one, and showing that the image of an edge can experience at most one relaxed biswivel. The proof appears in the full version [4] of this paper.

Theorem 4. *The price of stability in stable matching games with friendship utilities is at most $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$, and this bound is tight.*

From Theorems 3 and 4, we immediately get the following corollary about the behavior of best blocking pair dynamics. This corollary applies in particular to the model of altruism when $\alpha_i = \alpha$ for all $i = 1, \ldots, diam(G)$.

Corollary 1. *If $\alpha_1 = \alpha_2$ and we start from the social optimum matching,* Best-Blocking-Pair *converges in $O(m^2)$ time to a stable matching that is at most a factor of $\frac{2+2\alpha_1}{1+3\alpha_1}$ worse than the optimum.*

3 Matching with Friendship and General Reward Sharing

In the previous section we assumed that for $(u, v) \in M$ both u and v get the *same reward* r_{uv}. Let us now treat the more general case where u and v receive different rewards for $(u, v) \in M$. We define r^x_{xy} as the reward of x from edge $(x, y) \in M$. We interpret our model in a reward sharing context, where x and y share a total reward of $r_{xy} = r^x_{xy} + r^y_{xy}$. The correlated matching model of Section 2 can equivalently be formulated as equal sharing with nodes u and v receiving a reward of $r_{uv}/2$.

Without friendship utilities, our stable matching game reduces to the stable roommates problem (i.e., non-bipartite stable matching), since reward shares can be arbitrary and thus induce arbitrary preference lists for each node. It is well known that a stable matching may not exist in instances of the stable roommates problem. While we are able to prove existence of integral stable matching for several interesting special cases (see Section 3.1 below), the addition of friendship further complicates matters. In Section 2.1 we showed that for equal sharing, a stable matching without friendship utilities (i.e., $\boldsymbol{\alpha} = \mathbf{0}$) is also a stable matching when we have friendship utilities. This is no longer true for unequal reward sharing: adding a social context can completely change the set of stable matchings. In the full version [4] of this paper we give such examples, including an example where adding a social context (i.e., increasing $\boldsymbol{\alpha}$ above zero) destroys all stable matchings that exist when $\boldsymbol{\alpha} = \mathbf{0}$.

Although stable matchings may not exist in general non-bipartite graphs, *fractional* stable matchings are guaranteed to exist [1]. Fortunately, as we prove below, this holds even in the presence of friendship utilities with general reward sharing: A fractional stable matching always exists. By a "fractional stable matching" we simply mean a fractional matching (where the total fractional matches for a node v add up to at most 1) with no blocking pairs.

Theorem 5. *A fractional stable matching always exists, even in the case of friendship utilities and general reward sharing.*

Since an integral stable matching may not exist, we instead consider fractional matching; by price of anarchy here we mean the ratio of the total reward in a socially optimum *fractional* matching with the worst *fractional* stable matching. The corresponding ratio between the integral versions is trivially upper bounded by this amount as well. We define $R = \max_{(u,v)\in E(G)} \frac{r^u_{uv}}{r^v_{uv}}$. Note that $R \geq 1$. With this notation, we have the following theorem:

Theorem 6. *The (fractional) price of anarchy for general reward sharing with friendship utilities is at most* $1 + Q$*, where* $Q = \frac{R+\alpha_1}{1+\alpha_1 R}$*, and this bound is tight.*

Let us quickly consider the implications of the bound in Theorem 6. If $R = 1$, the bound is 2. This result implies Theorem 2, since when we have $R = 1$, then both u and v get the same reward from an edge $(u, v) \in M$. If $\alpha_1 = 0$, the bound is $1+R$. The tightness of this bound implies that as sharing becomes more unfair, i.e., as $R \to \infty$, we can find instances where the price of anarchy is unbounded. Unequal sharing can make things much worse for the stable matching game.

Notice, however, that $\frac{R+\alpha_1}{1+\alpha_1 R}$ is a decreasing function of α_1. As α_1 goes from 0 to 1, the bound goes from $1 + R$ to 2. Without friendship utilities ($\boldsymbol{\alpha} = \mathbf{0}$), we have a tight upper bound of $1 + R$, which is extremely bad for large R. As α_1 tends to 1, however, the price of anarchy drops to 2, independent of R. For example, for $\alpha_1 = 1/2$ it is only 3. Thus, social context can drastically improve the outcome for the society, especially in the case of unfair and unequal reward sharing.

For price of stability of general reward sharing with friendship utilities, we have a lower bound within an additive factor of 1 of optimum. Specifically, define $Q' = \frac{(1+\alpha_1)(1+R)}{1+\alpha_1(R+1)}$, then we have the following theorem for the price of stability:

Theorem 7. *The price of stability of stable matching games with friendship and general reward sharing is in* $[Q', Q + 1]$*, with* $Q < Q' \le Q + 1$*.*

3.1 Specific Reward Sharing Rules

In this section we consider some particularly natural reward sharing rules and show that games with such rules have nice properties. Specifically, while for general reward sharing an (integral) stable matching may not exist, for the reward sharing rules below we show they always exist (although only if there is no social context involved) and how to compute them efficiently. We also give improved bounds on prices of anarchy for these special cases. Specifically, we consider the following sharing rules:

- *Matthew Effect sharing:* In sociology, "Matthew Effect" is a term coined by Robert Merton to describe the phenomenon which says that, when doing similar work, the more famous person tends to get more credit than other less-known collaborators. We model such phenomena for our network by associating brand values λ_u with each node u, and defining the reward that node u gets by getting matched with node v as $r^u_{uv} = \frac{\lambda_u}{\lambda_u+\lambda_v} \cdot r_{uv}$. Thus nodes u and v split the edge reward in the ratio of $\lambda_u : \lambda_v$, and a node with high λ_u value gets a disproportionate amount of reward.
- *Trust sharing:* Often people collaborate based on not only the quality of a project but also how much they trust each other. We model such a situation by associating a value β_u with each node u, which represents the *trust value* of player u, or how pleasant they are to work with. Each edge (u, v) also has an inherent quality h_{uv}. Then, the reward obtained by node u from partnering with node v is $r^u_{uv} = h_{uv} + \beta_v$.

With friendship utilities, even these intuitive special cases of reward sharing do not guarantee the existence of an integral stable matching [4]. Without friendship, however, integral stable matching exists and can be efficiently computed for Matthew Effect sharing and Trust sharing, unlike in the case of general reward sharing.

Theorem 8. *An integral stable matching always exists in stable matching games with Matthew Effect sharing and Trust sharing if* $\boldsymbol{\alpha} = 0$ *(i.e., if there is no friendship). Furthermore, this matching can be found in* $O(|V||E|)$ *time.*

The price of anarchy of Matthew effect sharing can be as high as the guarantee of Theorem 6, with $R = \max_{(uv)} \frac{\lambda_u}{\lambda_v}$. For Trust sharing, however:

Theorem 9. *The price of anarchy for (fractional) stable matching games with Trust sharing and friendship utilities is at most* $\max\{2 + 2\alpha_1, 3\}$.

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