A Generic Convexification and Graph Cut Method for Multiphase Image Segmentation

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Abstract. We propose a unified graph cut based global minimization method for multiphase image segmentation by convexifying the non-convex image segmentation cost functionals. As examples, we shall apply this method to the non-convex multiphase Chan-Vese (CV) model and piecewise constant level set method (PCLSM). Both continuous and discretized formulations will be treated. For the discrete models, we propose a unified graph cut algorithm to implement the CV and PCLSM models, which extends the result of Bae and Tai [1] to any phases CV model. Moreover, in the continuous case, we further improve the model to be convex without any conditions using a number of techniques that are unique to the continuous segmentation models. With the convex relaxation and the dual method, the related continuous dual model is convex and we can mathematically show that the global minimization can be achieved. The corresponding continuous max-flow algorithm is easy and stable. Experimental results show that our model is very efficient.

1 Introduction

Many multiphase image segmentation models are non-convex and thus the corresponding numerical algorithms may sometimes get stuck at a local minimum close to the initial condition and produce undesirable segmentation results. For example, the multiphase Chan-Vese (CV) model [2] to partition an image into n parts by using $\log_2 n$ level set functions is non-convex. Its global minimization can not be guaranteed. Another multiphase segmentation method is to use a piecewise constant level set method PCLSM [3, 4] to represent different classes, the constraint of imposing the label function to be a piecewise constant function is non-convex, and thus the global minimization for such a model also can not be guaranteed.

Some efforts on the global minimization have been done in recent years. For the discrete methods, it is well-known that the global minimization can be attained by the graph cut approach. However, the graph cut method can only

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minimize some particular energies [5]. For a modified PCLSM [6], the global minimization can be obtained by Ishikawa graph cut method [7]. As to the multiphase CV model, generally speaking, the global minimization can not be achieved by the graph cut method since its associated discrete energy does not satisfy the graph representation condition in [5]. When a convex condition holds for the data term, Bae-Tai [8] have showed that the 4-phase CV model can be globally optimized by graph cut. If the convex condition fails, the authors also propose a truncation method to approximately minimize the energy. In [8], the weights assignment for the graph is derived from a nonnegative solution for some coupled linear equations, and the related convex condition makes sure that there is at least one nonnegative solution. To find such a nonnegative solution is not easy for any arbitrary phases segmentation, and thus this method is not convenient to be extended to any phases CV partition problems.

In the continuous case, convex relaxation method (e.g. [9–15]) is very popular in recent years. The main idea of the convex relaxation is to relax the binary characteristic function into a continuous interval $[0,1]$ such that the non-convex original problem becomes convex. Solving such a relaxed convex problem can enable one to find a global minimizer, and then the global binary solution of the original problem can be obtained by a threshold process. The functional lifting method [10] can be regarded as a convex relaxation of PCLSM, while the multi-dimensional generalization of the functional lifting [16] ensure that one can get a convex formulation of multiphase CV model. The continuous max-flow [15] approach shows that finding a max-flow on a discrete graph, namely graph cut method, corresponding to solving a continuous primal-dual problem. It gives the connections between the discrete approach and continuous method. More interestingly, it was found that the "cut" is just the Lagrangian multiplier for the flow conservation constraint when maximizing the total flow. Recently, some experimental comparisons between discrete and continuous segmentation methods has been given in [17] for some selected continuous multi-labelling approaches. It would be interesting to see a systematical comparison including these continuous max-flow with Augmented Lagrangian approaches.

The non-convex property of both PCLSM and CV models comes from the existence of some non-convex multiplication function terms. In this paper, we shall show that both of these two models can be written as some similar linear term plus multiplication function terms. Based on the graph cut minimization theory [5] and the property of *max* function which is associated to the convex envelope of the multiplication function, we propose a unified graph cut method to globally solve these two models. Since each term of the cost functional is convex after using the envelope functions, and thus the related relaxation continuous problem is also convex. Our method can easily handle any phases segmentation problems. For the discrete PCLSM, finding max-flows on the proposed graph is faster than Ishikawa's [7]. For the discrete 4-phase CV model, the proposed graph would be the same as [8], but it is easier to handle any phases segmentation. Moreover, to drop the convex condition in K-phase CV model in graph cut method, we propose a continuous relaxation max-flow method for multiphase

CV. Compared the earlier graph cut method, the proposed model is convex without any condition. Experimental results have shown that this can improve the quality of the segmentation results. A simultaneous work [18] appearing in this conference also derives a convex relaxation for the Chan-Vese model with any number of phases. However, their approach is based on directly computing the convex envelope of the data term.

The rest of the paper is organized as follows: section 2 is a brief introduction of the fundamental energy minimization theory with graph cut; in section 3, we propose the graph construction for the PCLSM and CV models; section 4 contains a continuous relaxation max-flow model for CV with super-level set representation; we show some experimental results and comparisons with other methods in section 5; finally, some conclusions and discussions are presented in section 6.

2 Energy Minimization with Graph Cut

In [5], the authors have given the condition of what energies can be minimized by graph cut. In this section, we shall briefly review the main results of [5].

Let $v_1, v_2, \cdots, v_n \in \{0,1\}$, $\mathcal{E}$ be a function of some binary variables. Then the following theorems hold:

Theorem 1 ([5]). *All the functions $\mathcal{E}(v_i)$ of one binary variable can be minimized by graph cut.*

Theorem 2 ([5]). *A function $\mathcal{E}(v_1, v_2)$ of 2 binary variables can be minimized by graph cut if and only if $\mathcal{E}$ is submodular, i.e. $\mathcal{E}(0,0)+\mathcal{E}(1,1) \leqslant \mathcal{E}(0,1)+\mathcal{E}(1,0)$. More generally, $\mathcal{E}(v_1, \cdots, v_n)$ of n binary variables can be minimized by graph cut if and only if $\mathcal{E}$ is submodular.*

Theorem 3 (additivity, [5]). *The sum of finite number of submodular functions is submodular.*

From the additivity theorem 3, one can conclude that if there are n functions which can be minimized by n different graphs, then the sum of the n functions also can be minimized by graph cut. This can be done by simply putting the vertices together and adding the n graphs' edge weights together (if any graphs have no edge between two vertices, one can add an edge with weight 0). For the proofs of these theorems, please refer to [5].

Suppose a s-t graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is constituted by a set of vertices $\mathbb{V}$ and a set of directed edges $\mathbb{E}$. Here there are two special distinguished vertices in $\mathbb{V}$, the source s and the sink t. A cut on the graph $\mathbb{G}$ is denoted by $(\mathbb{V}_t, \mathbb{V}_s)$, which is to partition the vertices $\mathbb{V}$ into two disjoint connected set $\mathbb{V}_s$ and $\mathbb{V}_t$ such that $s \in \mathbb{V}_s$ and $t \in \mathbb{V}_t$. For all binary variable $v_i \in \{0,1\}$, let $v_i = 0$ if the associated vertex belongs to $\mathbb{V}_s$ and $v_j = 1$ for vertex belongs to $\mathbb{V}_t$. Based on these theorems, we have the following conclusions:

Proposition 1 (linear function). *The minimization problem* $v_i^* = \arg\min_{v_i \in \{0,1\}} \{\mathcal{E}(v_i) = a_i v_i\}$, *where* a_i *is a known coefficient, corresponds to finding the min-cut on graph displayed in Fig.1(a).*

Proposition 2 (piecewise linear function). *The minimization problem* $(v_i^*, v_j^*) = \arg\min_{v_i, v_j \in \{0,1\}} \{\mathcal{E}(v_i, v_j) = b_{ij} \max\{v_j - v_i, 0\}\}$, *where* $b_{ij} \geqslant 0$ *is a known coefficient, corresponds to find the min-cut on graph displayed in Fig.1(b).*

These two properties are very important to construct a graph to minimize the energies of PCLSM and CV models. We shall show that both of these two models can be minimized by solving min cuts on graphs which constituted by sum of these two graphs. Since there are some multiplication functions terms in PCLSM and CV, for convenience, we firstly construct a graph to minimize such a term according to the propositions 1 and 2.

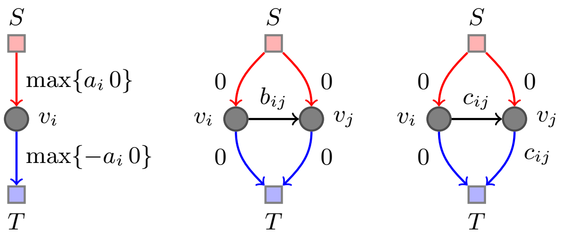
The multiplication function $\mathcal{E}(v_1, \cdots, v_n) = -c\prod_{i=1}^n v_i$, where $0 \leqslant v_i \leqslant 1$ and c is a known coefficient, is non-convex. From convex analysis theory (e.g. [19]), one can get its convex envelope

$$\mathcal{E}^{**}(v_1, \cdots, v_n) = \begin{cases} c\max\{-v_1, -v_2, \cdots, -v_n\}, & c \geqslant 0, \\ -c\max\{\sum_{i=1}^n v_i - n, 0\}, & c < 0, \end{cases}$$

which is the tightest convex function below $\mathcal{E}$. Moreover, in binary case, i.e. $v_i \in \{0, 1\}$, we have $\mathcal{E} = \mathcal{E}^{**}$. Thus, to minimize $\mathcal{E}$ with the binary constraint can be replaced by finding the minimizer of the convex function $\mathcal{E}^{**}$ with the same constraint. This is also the main idea of the convex relaxation method for product label spaces in [20]. When $c \geqslant 0$, in the two variables case, one can get

$$\mathcal{E}^{**}(v_1, v_2) = c\max\{v_2 - v_1, 0\} - cv_2.$$

Thus, based on the theorems 2, 3, and propositions 1, 2, we have the following conclusion:



(a) $a_i v_i$ (b) $b_{ij}\max\{v_j - v_i, 0\}$ (c) $-c_{ij} v_i v_j$

Fig. 1.

Corollary 1 (multiplication function). *The multiplication* $\mathcal{E}(\boldsymbol{v}) = -c_n \prod_{i=1}^{n} v_i, v_i \in \{0,1\}$ *is submodular if and only if the coefficient* $c_n \geqslant 0$. *In particular, the 2 variables multiplication* $\mathcal{E}(v_i, v_j) = -c_{ij} v_i v_j, v_i, v_j \in \{0,1\}$ *is submodular if and only if the coefficient* $c_{ij} \geqslant 0$. *Moreover, if* $c_{ij} \geqslant 0$, *the related binary minimization problem corresponds to find the min-cut on the graph displayed in Fig.1(c).*

3 Graph Construction for PCLSM and CV Models

In this section, we shall use the results in section 2 to construct graphs to solve PCLSM and CV models.

3.1 PCLSM

Now, we use the former method to minimize PCLSM with graph cut.

With the label function representation, the energy of multi-phase segmentation problem can be written as the following modified PCLSM [3]

$$\mathcal{E}^{PCLSM}(l) = \sum_{k=0}^{K-1} \int_{\Omega} \delta_{l,k} d^k \mathrm{d}x + \mu \int_{\Omega} |\nabla l| \mathrm{d}x, \tag{1}$$

Here $l : \Omega \to \{0, 1, \cdots, K-1\}$ is an unknown integer function whose K different values are used to represent the K number of phases, and

$$\delta_{l,k} = \begin{cases} 1, \, l(x) = k, \\ 0, \, l(x) \neq k. \end{cases}$$

The first term is the data term which gives the classification criterion, and each d^k should depend on the input image I. For example, $d^k(x) = |I(x) - c^k|^\lambda$, $\lambda = 1, 2$ represents that the pixels are classified in terms of the intensity means $\{c^k\}_{k=1}^K$. In this paper, we suppose $d^k \geqslant 0$ are known. While the second term is the regularization term and μ is a parameter which controls the balance of these two terms. This functional is non-convex because of the existence of composite function of delta function and l. However, with the help of the γ-super-level set function

$$\phi^\gamma(x) = \begin{cases} 1, \text{ when } \; l(x) \geqslant \gamma, \\ 0, \text{ when } \; l(x) < \gamma, \end{cases} \tag{2}$$

we have

$$\phi^k(x) - \phi^{k+1}(x) = \delta_{l(x),k}, \tag{3}$$

for $k = 0, 1, \cdots, K-1$. Together with the generalized co-area formula [21]

$$\int_{\Omega} |\nabla l| \mathrm{d}x = \int_{\Omega} \left(\int_{-\infty}^{+\infty} |\nabla \phi^\gamma(x)| \mathrm{d}\gamma \right) \mathrm{d}x,$$

the functional can be formulated as

$$\mathcal{E}^{PCLSM}(\boldsymbol{\phi}) = \sum_{k=0}^{K-1} \int_{\Omega} (\phi^k - \phi^{k+1}) d^k \mathrm{d}x + \mu \sum_{k=1}^{K-1} \int_{\Omega} |\nabla \phi^k| \mathrm{d}x, \tag{4}$$

where $\boldsymbol{\phi} = (\phi^1, \phi^2, \cdots, \phi^{K-1}), \phi^1 \geqslant \phi^2 \geqslant \cdots \geqslant \phi^{K-1}$, and $\phi^0 = 1, \phi^K = 0$.

The functional (4) is now convex, compared with the original formulation (1). The above process is the functional lifting method (FLM) that has been developed in [10], but the data term is slightly different from [10]. The data term here is linear but L^1 in [10].

In discrete case, (4) can be globally minimized by graph cut. In the next, we shall construct such a graph.

Let $\mathbb{P}$ be the set of mesh grid points in Ω, and $\mathbb{N}_p^4$ be the set of 4 nearest neighbors of $p \in \mathbb{P}$. For $\Omega \subset \mathbb{R}^2$, $\mathbb{P} = \{(i,j) \subset \mathbb{Z}^2\}$ and for each $p = (i,j) \in \mathbb{P}$

$$\mathbb{N}_p^4 = \{(i \pm 1, j), (i, j \pm 1)\},$$

Let ϕ_p^k, d_p^k be the function values of ϕ^k and d^k at $p \in \mathbb{P}$. If we choose the anisotropic TV

$$TV_1(\phi^k) = \int_{\Omega} |\nabla \phi^k|_1 \mathrm{d}x = \int_{\Omega} (|\partial_{x_1} \phi^k| + |\partial_{x_2} \phi^k|) \mathrm{d}x_1 \mathrm{d}x_2$$

for the regularization term in (4), and employ the difference schemes

$$|\partial_{x_1} \phi^k| = \frac{|\partial_{x_1} \phi^k|}{2} + \frac{|\partial_{x_1} \phi^k|}{2} = \frac{|\phi^k_{i+1,j} - \phi^k_{i,j}|}{2} + \frac{|\phi^k_{i-1,j} - \phi^k_{i,j}|}{2},$$
$$|\partial_{x_2} \phi^k| = \frac{|\partial_{x_2} \phi^k|}{2} + \frac{|\partial_{x_2} \phi^k|}{2} = \frac{|\phi^k_{i,j+1} - \phi^k_{i,j}|}{2} + \frac{|\phi^k_{i,j-1} - \phi^k_{i,j}|}{2},$$

then the discrete formulation of (4) should be

$$\mathcal{E}^{PCLSM-D}(\boldsymbol{\phi}) = \sum_{k=0}^{K-1} \sum_{p \in \mathbb{P}} \left(\phi_p^k - \phi_p^{k+1}\right) d_p^k + \frac{\mu}{2} \sum_{k=1}^{K-1} \sum_{p \in \mathbb{P}} \sum_{q \in \mathbb{N}_p^4} |\phi_q^k - \phi_p^k|.$$

From propositon 1, $\phi_p^k d_p^k, -\phi_p^{k+1} d_p^k$ can be minimized by solving the min-cuts on graphs displayed in Fig.2(a) and 2(b), respectively. According to the additivity theorem 3 and combining the constraint condition $\phi^1 \geqslant \cdots \geqslant \phi^{K-1}$, the data term at each pixel $p \in \mathbb{P}$ in the above energy can be minimized through searching the min-cut on graph defined in Fig.2(c). For a cut $(\mathbb{V}_s, \mathbb{V}_t)$, we say $(\mathbb{V}_s, \mathbb{V}_t)$ is a feasible cut when the cost of the cut $C(\mathbb{V}_s, \mathbb{V}_t) < +\infty$. Besides, let us denote

$$\mathbb{B} = \{\boldsymbol{\phi} = (\phi^1, \phi^2, \cdots, \phi^{K-1}) : \phi^k \in \{0,1\}, 1 = \phi^0 \geqslant \phi^1 \geqslant \phi^2 \geqslant \cdots \geqslant \phi^{K-1} \geqslant \phi^K = 0\}. \tag{5}$$

Then we have the following result:

Proposition 3. *There is a one-to-one correspondence between the feasible cuts of graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ defined on Fig.2(c) and the binary super-level set function $\boldsymbol{\phi}$, and*

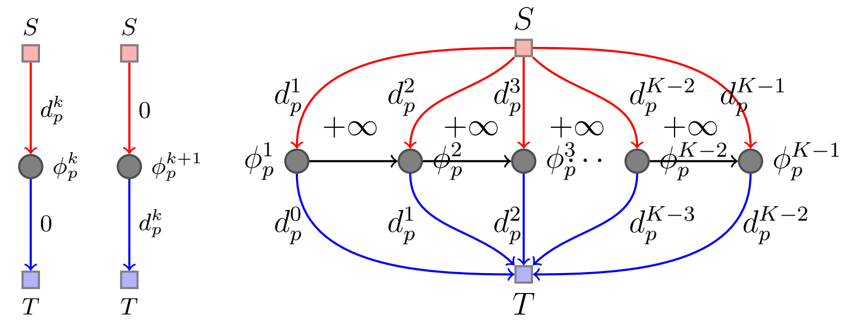
$$\min_{(\mathbb{V}_s, \mathbb{V}_t)} C(\mathbb{V}_s, \mathbb{V}_t) = \min_{\boldsymbol{\phi} \in \mathbb{B}} \sum_{k=0}^{K-1} \sum_{p \in \mathbb{P}} \left(\phi_p^k - \phi_p^{k+1}\right) d_p^k.$$

As to the regularization term, since ϕ_p^k, ϕ_q^k are binary and we have

$$\frac{\mu}{2}\sum_{k=1}^{K-1}\sum_{p\in\mathbb{P}}\sum_{q\in\mathbb{N}_p^4}|\phi_q^k-\phi_p^k| = \frac{\mu}{2}\sum_{k=1}^{K-1}\sum_{p\in\mathbb{P}}\sum_{q\in\mathbb{N}_p^4}|\phi_q^k-\phi_p^k|^2 = \frac{\mu}{2}\sum_{k=1}^{K-1}\sum_{p\in\mathbb{P}}\sum_{q\in\mathbb{N}_p^4}(\phi_q^k+\phi_p^k-2\phi_q^k\phi_p^k).$$

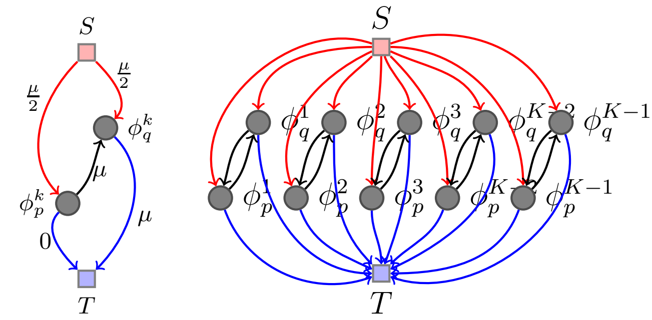
By applying the additivity theorem 3 and corollary 1, the regularization term can be minimized by solving the min-cut on the graph displayed in Fig.3.

Finally, we can get the graph for minimizing the energy $\mathcal{E}^{PCLSM-D}$ by simply adding the edge weights of graphs defined in Fig.2(c) and Fig.3(b) together according to the additivity of the graph.



(a) $\phi_p^k d_p^k$ (b) $-\phi_p^{k+1} d_p^k$ (c) $\sum_{k=0}^{K-1}\left(\phi_p^k-\phi_p^{k+1}\right)d_p^k$

Fig. 2. Graph construction for data term at each pixel $p \in \mathbb{P}$ in PCLSM



(a) $\frac{\mu}{2}(\phi_q^k+\phi_p^k-2\phi_q^k\phi_p^k)$ (b) $\frac{\mu}{2}\sum_{k=1}^{K-1}\sum_{q\in\mathbb{N}_p^4}(\phi_q^k+\phi_p^k-2\phi_q^k\phi_p^k)$

Fig. 3. Graph construction for regularization term at each pixel $p \in \mathbb{P}$ in PCLSM. In Fig.3(b), all the weights of red and blue edges (t-link) are 4μ, and all the weights of black edges (n-link) are μ.

3.2 CV Model

The Chan-Vese model [22] is a popular image segmentation model, a generalization of Chan-Vese model has been proposed in [2] to partition an image into K parts by using $\log_2 K$ level set functions. In [4], vector binary functions were used to represent the different phases and this has a very close relationship to the things we are going to discuss here. For the general CV model with any number of phases, we firstly derive a general formulation for multi-phase CV model from the binary representation of integers. It is well known that for any integer $k \in \{0, 1, \cdots, K-1\}$, there is a mapping $\Lambda : \mathbb{Z}^+ \cup \{0\} \to \mathbb{B}$ such that $\Lambda(k) = b_k^{M-1} \cdots b_k^1 b_k^0$, where $\mathbb{B}$ is the set of the binary representation of integers. Here $b_k^m \in \{0, 1\}, m = 0, \cdots, M-1$, $M = \lceil \log_2 K \rceil$, and $\lceil \cdot \rceil$ is a ceiling operator. For example, the binary representation of integer $\Lambda(5) = 101$. In essence, the CV model is closely related to this binary representation. For simplification, we first consider the $K = 2^M, M = 1, 2 \cdots$ case. Let us write the binary representation of the label function $l : \Omega \to \{0, 1, \cdots, 2^M - 1\}$ as $\Lambda(l) = \psi^{M-1} \cdots \psi^0$, and in this case it is easy to check Λ is a 1-1 mapping, then

$$\delta_{l,k} = \delta_{\Lambda(l),\Lambda(k)} = \delta_{\psi^{M-1}\cdots\psi^0, b_k^{M-1}\cdots b_k^0} = \prod_{m=0}^{M-1} \delta_{\psi^m, b_k^m} = \begin{cases} 1, \psi^m = b_k^m, \\ 0, \mathit{else}. \end{cases}$$

Using the γ-super-level set function representation for ψ^m and let us denote

$$\phi^{m,b_k^m}(x) = \begin{cases} 1, \text{when } \ \psi^m(x) \geqslant b_k^m, \\ 0, \text{when } \ \psi^m(x) < b_k^m, \end{cases}$$

for all $m = 0, 1, \cdots, M-1$. Now, we have

$$\phi^{m,b_k^m} - \phi^{m,b_k^m+1} = \delta_{\psi^m, b_k^m},$$

and thus

$$\delta_{l,k} = \prod_{m=0}^{M-1} (\phi^{m,b_k^m} - \phi^{m,b_k^m+1}).$$

In the above equation,

$$\phi^{m,0} = 1, \phi^{m,1} = \psi^m, \phi^{m,2} = 0,$$

since $b_k^m, \psi^m \in \{0, 1\}$, so the unknown variables are only $\boldsymbol{\phi}^{\cdot,1} = (\phi^{0,1}, \phi^{1,1}, \cdots, \phi^{M-1,1})$.

If the segmentation phase $K \neq 2^M$, i.e. $2^{M-1} < K < 2^M$, then Λ may not be a 1-1 mapping, which means that we may use serval labels to indicate one class. This can be achieved by adding several d^{K-1} in the energy.

Replacing the δ function in (4) with the above expression and modifying the regularization term, we get the multi-phase CV model for any K phases:

$$\mathcal{E}^{CV-K}(\boldsymbol{\phi}^{\cdot,1}) = \sum_{k=0}^{2^M-1} \int_\Omega d^k \prod_{m=0}^{M-1} (\phi^{m,b_k^m} - \phi^{m,b_k^m+1}) \mathrm{d}x + \mu \sum_{m=0}^{M-1} \int_\Omega |\nabla \phi^{m,1}| \mathrm{d}x, \tag{6}$$

where $d^k = d^{K-1}$ when $K-1 \leqslant k \leqslant 2^M - 1$. Let point out that the choice of d^k when $K-1 \leqslant k \leqslant 2^M - 1$ is not unique, for example, one can use $d^k = \frac{\sum_{k=0}^{K-1} d^k}{K}$ for $k > K-1$.

The discrete formulation of the regularization term in CV model (6) is

$$\mathcal{R}^{CV-K}(\boldsymbol{\phi}) = \frac{1}{2}\sum_{m=0}^{M-1}\sum_{p\in\mathbb{P}}\sum_{q\in\mathbb{N}_p^4}|\phi_q^m - \phi_p^m|,$$

which can be minimized by graph cut in terms of our earlier discussion.

Let $i_0 < i_1 < \cdots < i_{m-1}$ be any m-combination of the set $\{0,1,\cdots,M-1\}$ and $\mathbb{S}^m$ be a set which contains the C_M^m different m-combinations. For simplicity, we rewrite $\phi^{m,1}$ as ϕ^m. In the discrete case, the data term in CV model (6) can be expressed as

$$\mathcal{D}^{CV-K}(\boldsymbol{\phi}) = \sum_{p\in\mathbb{P}}\left(d_p^0 + \sum_{m=1}^{M}\sum_{i_0,i_1,\cdots,i_{m-1}\in\mathbb{S}^m}\sum_{t_0=0}^{1}\sum_{t_1=0}^{1}\cdots\sum_{t_{m-1}=0}^{1} c_{t_{m-1},\cdots,t_0}^{i_{m-1},\cdots,i_0}\phi_p^{i_0}\phi_p^{i_1}\cdots\phi_p^{i_{m-1}}\right),$$

where the coefficients

$$c_{t_{m-1},\cdots,t_0}^{i_{m-1},\cdots,i_0} = (-1)^{(m-\sum_{j=0}^{m-1}t_j)}\, d_p^{\sum_{j=0}^{m-1}t_j 2^{i_j}}.$$

Now, the data term $\mathcal{D}^{CV-K}$ only contains some multiplying functions, we can use the former theorems. According to proposition 1, corollary 1 and theorem 3, we can get the following result

Proposition 4. *The K-phases discrete CV model can be exactly minimized by graph cut if the coefficients $\sum_{t_0=0}^{1}\cdots\sum_{t_{m-1}=0}^{1} c_{t_{m-1},\cdots,t_0}^{i_{m-1},\cdots,i_0} \leqslant 0$ for all $2 \leqslant m \leqslant M$.*

In particular, when $K=3$, $M = \lceil\log_2 3\rceil = 2$, then

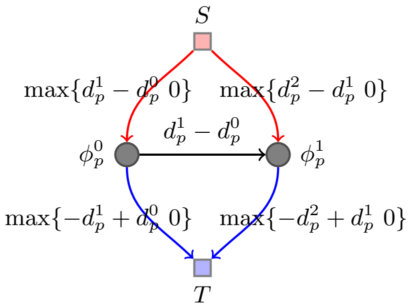
$$\begin{aligned}\mathcal{D}^{CV-3}(\boldsymbol{\phi}) &= \textstyle\sum_{p\in\mathbb{P}}\left(d_p^0 + (c_0^0 + c_1^0)\phi_p^0 + (c_0^1 + c_1^1)\phi_p^1 + (c_{0,0}^{1,0} + c_{0,1}^{1,0} + c_{1,0}^{1,0} + c_{1,1}^{1,0})\phi_p^0\phi_p^1\right)\\ &= \textstyle\sum_{p\in\mathbb{P}}\left(d_p^0 + (-d_p^0 + d_p^1)\phi_p^0 + (-d_p^0 + d_p^2)\phi_p^1 + (d_p^0 - d_p^1)\phi_p^0\phi_p^1\right).\end{aligned}$$

Similarly, for $K=4$,

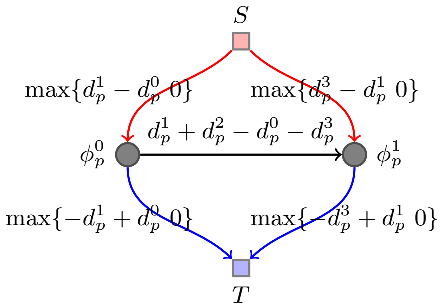
$$\begin{aligned}\mathcal{D}^{CV-4}(\boldsymbol{\phi}) &= \textstyle\sum_{p\in\mathbb{P}}\left(d_p^0 + (c_0^0 + c_1^0)\phi_p^0 + (c_0^1 + c_1^1)\phi_p^1 + (c_{0,0}^{1,0} + c_{0,1}^{1,0} + c_{1,0}^{1,0} + c_{1,1}^{1,0})\phi_p^0\phi_p^1\right)\\ &= \textstyle\sum_{p\in\mathbb{P}}\left(d_p^0 + (-d_p^0 + d_p^1)\phi_p^0 + (-d_p^0 + d_p^2)\phi_p^1 + (d_p^0 - d_p^1 - d_p^2 + d_p^3)\phi_p^0\phi_p^1\right).\end{aligned}$$

Thus we have the following conclusion:

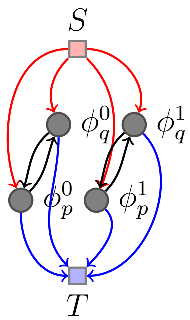
Corollary 2. *The 3-phase discrete CV model can be exactly minimized by graph cut if and only if $d^0 - d^1 \leqslant 0$. Similarly, the condition of the 4-phase discrete CV model is $d^0 - d^1 - d^2 + d^3 \leqslant 0$. When these condition holds, for each pixel p, $\mathcal{D}^{CV-3}(\boldsymbol{\phi}_p), \mathcal{D}^{CV-4}(\boldsymbol{\phi}_p), \mu\mathcal{R}^{CV-3\ or\ 4}(\boldsymbol{\phi}_p)$ can be minimized by finding the min cuts on graphs defined on Fig.4(a), Fig.4(b) and Fig.4(c), respectively.*



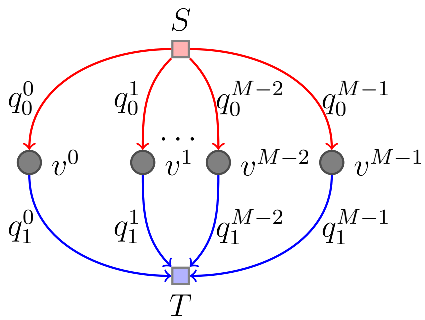
(a) $\mathcal{D}^{CV-3}(\boldsymbol{\phi}_p)$



(b) $\mathcal{D}^{CV-4}(\boldsymbol{\phi}_p)$



(c) $\mu\mathcal{R}^{CV-3\ or\ 4}(\boldsymbol{\phi}_p)$



(d) The continuous relaxation max-flow for CV (data term).

Fig. 4. Graph construction for 3,4-phase CV model. In Fig.4(c), all the weights of red and blue edges (t-link) are 4μ, and all the weights of black edges (n-link) are μ.

Therefore, to minimize the 3 or 4-phase discrete model can be implemented by solving the min cut of the graph which is constituted by putting the graphs defined by data term and regularization term together.

In 4-phase case, the corollary 2 coincides to result in [1, 8]. Here, we extend the results to any phases CV model, which is not easy to handle with the method of [1, 8].

In fact, when the condition $\sum_{t_0=0}^{1} \cdots \sum_{t_{m-1}=0}^{1} c_{t_{m-1},\cdots,t_0}^{i_{m-1},\cdots,i_0} \leqslant 0$ in proposition 4 holds, the non-convex term $-\phi_p^{i_0}\phi_p^{i_1}\cdots\phi_p^{i_{m-1}}$ can be replaced by its convex envelope $\max\{-\phi^{i_0}, -\phi^{i_1}, \cdots, -\phi^{i_{m-1}}\}$ since these two functionals have the same minimizer in the binary case. As in [1], we call this condition as convex condition. To get a convex model, one can relax the non-convex constraint $\phi^i \in \{0,1\}$ to an interval $[0,1]$. In the continuous case, we can get a convex model for K-phase CV model.

As for the convex condition in proposition 4, it depends on the segmentation data. In the 4-phase case, it have been theoretically analyzed in [8]. In many real image segmentation problems, it may hold that 4 cluster are sufficient. However, we note that this condition would become stricter when the number of the phases increases, thus in the discrete case, the K-phase CV model may not be exactly optimized by graph cut if the segmentation data is very bad (fails to satisfy the convex condition). To overcome this flaw, one may use an approximation graph cut method by cutting off the coefficient $\sum_{t_0=0}^{1}\cdots\sum_{t_{m-1}=0}^{1} c_{t_{m-1},\cdots,t_0}^{i_{m-1},\cdots,i_0}$ as 0 when the condition fails. However, such an approximation method may not be the original CV model. Here, we shall propose another continuous relaxation max-flow method, which is also a convex model but without any convex conditions. Refer to [18] in this proceedings for another method to derive a convex relaxation for the multiphase CV model. The relaxation of [18] is tight for any phases, but need conditions to guarantee convexity.

4 Continuous Convex Relaxation Max-Flow for CV

The continuous max-flow method for 2-phase CV model and Potts model has been proposed by Yuan *etc.* in [15, 23]. It has been extended to 4-phase CV model by Bae-Tai in [1] with the earlier mentioned convex condition. Here, we propose a convex continuous max-flow method for any phases CV model but without any convex conditions.

For multi-phase CV model, the regularization term is convex and to convert it to continuous max-flow would be the same as [1,15,23,24]. Here we only discuss the data term.

Firstly, we construct a graph displayed in Fig.4(d): for K phases CV model, we copy $M = \lceil \log_2 K \rceil$ vertices $v^m, m = 0, \cdots, M-1$ at each pixel x; let us denote the edges between source s and vertex v^m, sink t and vertex v^m as q_0^m and q_1^m, respectively. For $k = 0, 1, \cdots, K-1$, let the binary representation of k be $\Lambda(k) = b_k^{M-1}\cdots b_k^0$. We impose the capabilities of these edges satisfy the following condition

$$\sum_{m=0}^{M-1} q_{b_k^m}^m \leqslant d^k, k = 0, 1\cdots, 2^M - 1, \tag{7}$$

where $d^k = d^{K-1}$ when $K - 1 \leqslant k \leqslant 2^M - 1$.

The max-flow problem is to find the maximum capabilities of the flows which stream from source s to sink t under a flow preserving condition and the maximum capabilities condition (7). Together with the graph defined in Fig.4(d) and the regularization term, the proposed max-flow problem for any K-phase CV model can be written as

$$\begin{cases} \max\limits_{\boldsymbol{q}_0,\boldsymbol{q}_1,\boldsymbol{p}\in\mathbb{C}} \int_{\Omega} \sum\limits_{m=0}^{M-1} q_1^m(x)\mathrm{d}x, \\ q_0^m(x) - q_1^m(x) - \nabla\cdot\boldsymbol{p}^m(x) = 0, m = 0, 1, \cdots, M-1, \\ \sum_{m=0}^{M-1} q_{b_k^m}^m(x) \leqslant d^k(x), k = 0, 1\cdots, 2^M - 1. \end{cases} \tag{8}$$

Here, $\boldsymbol{q}_0 = (q_0^0, \cdots, q_0^{M-1}), \boldsymbol{q}_1 = (q_1^0, \cdots, q_1^{M-1})$, the second equation is the flow preserving condition of vertex v^m, in which $\boldsymbol{p} = (\boldsymbol{p}^0, \cdots, \boldsymbol{p}^{M-1}) \in \mathbb{C} = \{\boldsymbol{p} : ||\boldsymbol{p}^m||_\infty \leqslant \mu, m = 0, 1, \cdots, M-1.\}$ is the associated flow function for the TV term in CV model, and the third inequality is just the maximum capabilities condition (7).

Applying the Lagrange multiplier method and let $\boldsymbol{\phi} = (\phi^0, \cdots, \phi^{M-1})$ be the M Lagrange multipliers, then the problem (8) can be formulated as the following saddle point problem:

$$\begin{cases} \min\limits_{\boldsymbol{\phi}} \max\limits_{\boldsymbol{q}_0,\boldsymbol{q}_1,\boldsymbol{p}\in\mathbb{C}} \int_\Omega \sum\limits_{m=0}^{M-1} (\phi^m(x)q_0^m(x) + (1-\phi^m(x))q_1^m(x) - \phi^m(x)\nabla\cdot\boldsymbol{p}^m(x))\,\mathrm{d}x, \\ \sum_{m=0}^{M-1} q_{b_k^m}^m(x) \leqslant d^k(x), k = 0, 1\cdots, 2^M - 1. \end{cases} \tag{9}$$

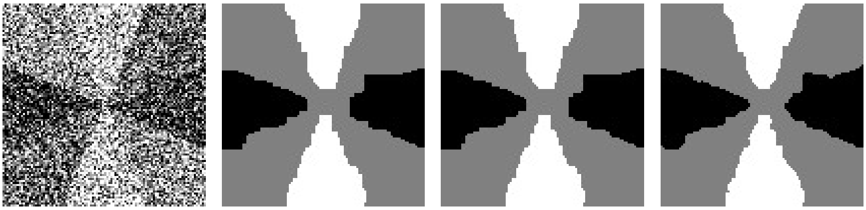
In the above saddle point problem, the term $\mathcal{J}(\boldsymbol{\phi}) = \max\limits_{\boldsymbol{q}_0,\boldsymbol{q}_1,\boldsymbol{p}\in\mathbb{C}} \int_\Omega \sum\limits_{m=0}^{M-1} \phi^m(x)q_0^m(x) + (1-\phi^m(x))q_1^m(x)\mathrm{d}x$ with the constraint condition (7) is associated to the data term $\mathcal{D}^{CV-K}(\boldsymbol{\phi})$ in CV model (6). For this two terms, we can prove that they have the same global minimization.

Proposition 5. *If $\boldsymbol{\phi}^*$ is a binary minimizer of $\mathcal{D}^{CV-K}(\boldsymbol{\phi})$, then $\boldsymbol{\phi}^*$ is also a minimizer of $\mathcal{J}(\boldsymbol{\phi})$ under constraint (7). Moreover, $\mathcal{D}^{CV-K}(\boldsymbol{\phi}^*) = \mathcal{J}(\boldsymbol{\phi}^*) = \int_\Omega \min\{d^0(x), \cdots, d^{2^M-1}(x)\}\mathrm{d}x$.*

As to find the saddle point of (9), it can be done by projection gradient method since the functional is linear and the constraint sets are convex for each variable. We do not plan to list the details of the algorithm here.

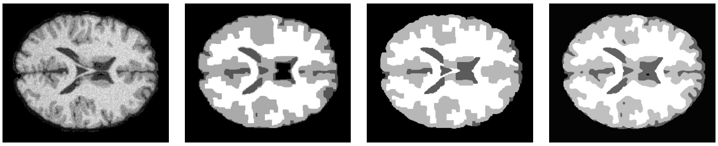
5 Experimental Results

In this section, we shall give some numerical examples with the proposed method. The first one is to partition a synthetic image in Fig. 5(a) into 3 parts. The PCLSM, CV model and continuous convex relaxation max-flow (CCRM) of CV are applied to segment this image. For the PCLSM and CV, we use the constructed graph cut to solve them. The final segmentation results are shown in Fig.5(b)-Fig.5(d), respectively. As can seen from the figure, the result produced by CCRM is smoother than the discrete methods. This is due to the fact that we may employ an isotropic TV in the continuous case but not with graph cuts. At the current CPU program implementation, the discrete graph cut method is faster than CCRM. However, the discrete graph cut algorithm is not easy to work with the parallel program, thus the continuous method CCRM with GPU implementation would be faster than the discrete ones. The second example displayed in Fig. 6 shows the segmentation results of applying the proposed method to a real brain MRI. In this figure, the brain image is partition into 4 parts. As can be found again, the boundary in result of CCRM is smoother than others.



(a) Size: 100×100 (b) PCLSM (c) CV (d) CCRM

Fig. 5. A comparison between discrete graph cut algorithm and continuous convex relaxation max-flow (CCRM). The cost of the CPU time for PCLSM, CV and CCRM are 0.0352s,0.3504s,0.4282s, respectively.



(a) Size591×709 (b) PCLSM (c) CV (d) CCRM

Fig. 6. Different algorithms for real images. The regularization parameters for each method are manually chosen. The clusters is 4.

6 Conclusion and Discussion

We have proposed a unified method to minimize the multiphase image segmentation PCLSM and CV models with discrete graph cut and continuous max-flow. This method is developed by considering the convexification of the multiplying functions with graph cut method. For PCLSM, we construct a graph which is different from the earlier Ishikawa method [7]. For CV models, we extend the result in [1, 6] to any phases and propose a saddle point problem for any phases CV model. Compared the original CV method, the proposed convex relaxation max-flow method is convex and thus it can get the global minimization. Moreover, we show that the key idea of CV model is to use the binary expression to represent a integer, thus one can extend it to any n-decimal numeral system .

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