On the Total Perimeter of Homothetic Convex Bodies in a Convex Container*

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Abstract. For two convex bodies, C and D, consider a packing S of n positive homothets of C contained in D. We estimate the total perimeter of the bodies in S, denoted $\text{per}(S)$, in terms of n. When all homothets of C touch the boundary of the container D, we show that either $\text{per}(S) = O(\log n)$ or $\text{per}(S) = O(1)$, depending on how C and D "fit together," and these bounds are the best possible apart from the constant factors. Specifically, we establish an optimal bound $\text{per}(S) = O(\log n)$ unless D is a convex polygon and every side of D is parallel to a corresponding segment on the boundary of C (for short, D is *parallel to* C). When D is parallel to C but the homothets of C may lie anywhere in D, we show that $\text{per}(S) = O((1 + \text{esc}(S)) \log n/ \log\log n)$, where $\text{esc}(S)$ denotes the total distance of the bodies in S from the boundary of D. Apart from the constant factor, this bound is also the best possible.

Keywords: Convex body, perimeter, maximum independent set, homothet, traveling salesman, approximation algorithm.

1 Introduction

A finite set $S = \{C_1, \ldots, C_n\}$ of convex bodies is a *packing* in a convex body (*container*) $D \subset \mathbb{R}^2$ if the bodies $C_1, \ldots, C_n \in S$ are contained in D and they have pairwise disjoint interiors. The term *convex body* above refers to a compact convex set with nonempty interior in $\mathbb{R}^2$. The perimeter of a convex body $C \subset \mathbb{R}^2$ is denoted $\text{per}(C)$, and the total perimeter of a packing S is denoted $\text{per}(S) = \sum_{i=1}^{n} \text{per}(C_i)$. Our interest is estimating $\text{per}(S)$ in terms of n.

We start with a few immediate observations. (1) If the convex bodies in the packing S are arbitrary, then we can assume that the packing S is in fact a tiling of the container, that is, $D = \bigcup_{i=1}^{n} C_i$. It is then easy to show that $\text{per}(S) \leq \text{per}(D) + 2(n-1)\,\text{diam}(D)$, where $\text{diam}(D)$ is the diameter of D. This bound can be achieved by subdividing D into n compact convex tiles via $n-1$ near diameter segments. (2) If all bodies in S are congruent to a convex body C, then $\text{per}(S) =$

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$n\,\mathrm{per}(C)$, and bounding $\mathrm{per}(S)$ from above reduces to the classical problem of determining the maximum number of interior-disjoint congruent copies of C that fit in D [2].

In this paper, we consider packings S that consist of positive homothets of a convex body C. We establish an easy general bound in this case.

Proposition 1. *For every pair of convex bodies, C and D, and every packing S of n positive homothets of C in D, we have* $\mathrm{per}(S) \le \rho(C,D)\sqrt{n}$, *where $\rho(C,D)$ depends on C and D. Apart from this multiplicative constant, this bound is the best possible.*

Motivated by applications to the traveling salesman problem with neighborhoods (TSPN), we would like to bound $\mathrm{per}(S)$ in terms of n if all homothets in S touch the boundary of D (see Fig. 1). Specifically, for a pair of convex bodies, C and D, let $f_{C,D}(n)$ denote the maximum perimeter $\mathrm{per}(S)$ of a packing of n positive homothet of C in the container D, where each element of S touches the boundary of D. We would like to estimate the growth rate of $f_{C,D}(n)$ as n goes to infinity. We prove a logarithmic upper bound $f_{C,D}(n) = O(\log n)$ for every pair of convex bodies, C and D.

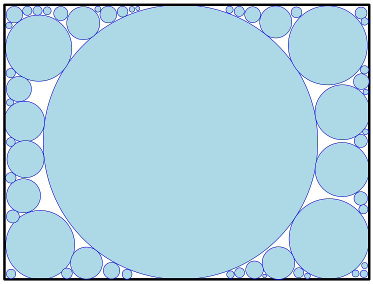

Fig. 1. A packing of disks in a rectangle container, where all disks touch the boundary of the container

Proposition 2. *For every pair of convex bodies, C and D, and every packing S of n positive homothets of C in D, where each element of S touches the boundary of D, we have* $\mathrm{per}(S) \le \rho(C,D)\log n$, *where $\rho(C,D)$ depends on C and D.*

The upper bound $f_{C,D}(n) = O(\log n)$ is asymptotically tight for some pairs C and D, and not so tight for others. For example, it is not hard to attain an $\Omega(\log n)$ lower bound when C is an axis-aligned square, and D is a triangle (Fig. 2, left). However, $f_{C,D}(n) = \Theta(1)$ when both C and D are axis-aligned squares. We start by establishing a logarithmic lower bound in the simple setting where C is a circular disk and D is a unit square.

Theorem 1. *The total perimeter of n pairwise disjoint disks lying in the unit square $U = [0,1]^2$ and touching the boundary of U is $O(\log n)$. Apart from the constant factor, this bound is the best possible.*

We determine $f_{C,D}(n)$ up to constant factors for all pairs of convex bodies of bounded description complexity. (A planar set has *bounded description complexity* if its boundary consists of a finite number of algebraic curves of bounded degrees.) We show that either $f_{C,D} = \Theta(\log n)$ or $f_{C,D}(n) = \Theta(1)$ depending on how C and D "fit together". To distinguish these cases we need the following definitions.

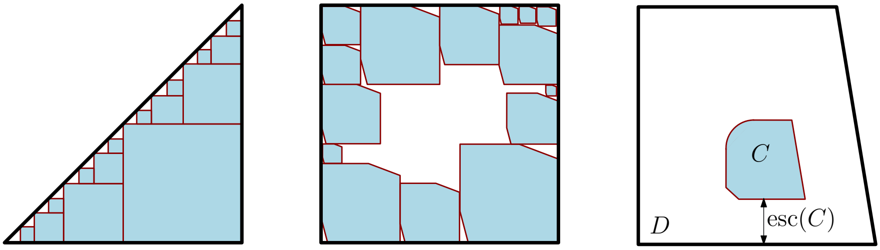

Fig. 2. Left: a square packing in a triangle where every square touches the boundary of the triangle. Middle: a packing of homothetic hexagons in a square where every hexagon touches the boundary of the square. Right: a convex body C in the interior of a trapezoid D at distance $\text{esc}(C)$ from the boundary of D. The trapezoid D is *parallel to* C: every side of D is parallel and "corresponds" to a side of C.

Definitions. For a direction vector $\mathbf{d} \in \mathbb{S}^1$ and a convex body C, the *supporting line* $\ell_{\mathbf{d}}(C)$ is a directed line of direction $\mathbf{d}$ such that $\ell_{\mathbf{d}}(C)$ is tangent to C, and the closed halfplane on the left of $\ell_{\mathbf{d}}(C)$ contains C. If $\ell_{\mathbf{d}}(C) \cap C$ is a nondegenerate line segment, we refer to it as a *side* of C.

We say that a convex polygon (container) D is *parallel to* a convex body C when for every direction $\mathbf{d} \in \mathbb{S}^1$ if $\ell_{\mathbf{d}}(D) \cap D$ is a side of D, then $\ell_{\mathbf{d}}(C) \cap C$ is also a side of C. Figure 2(right) depicts a trapezoid D parallel to a convex body C. For example, every positive homothet of a convex polygon P is parallel to P; and all axis-aligned rectangles are parallel to each other.

Classification. We generalize the lower bound construction in Theorem 1 to arbitrary convex bodies, C and D, of bounded description complexity, where D is not parallel to C.

Theorem 2. *Let C and D be two convex bodies of bounded description complexity. For every packing S of n positive homothets of C in D, where each element of S touches the boundary of D, we have* $\text{per}(S) \leq \rho(C, D) \log n$, *where $\rho(C, D)$ depends on C and D. Apart from the factor $\rho(C, D)$, this bound is the best possible unless D is a convex polygon parallel to C.*

If D is a convex polygon parallel to C, and every homothet of C in a packing S touches the boundary of D, then it is not difficult to see that $\text{per}(S)$ is bounded.

Proposition 3. *Let C and D be convex bodies such that D is a convex polygon parallel to C. Then every packing S of n positive homothets of C in D, where each element of S touches the boundary of D, we have* $\text{per}(S) \leq \rho(C, D)$, *where $\rho(C, D)$ depends on C and D.*

In the special case that D is a convex polygon parallel to C, it is also of interest to establish asymptotically tight upper bounds for $\text{per}(S)$ without the assumption that the bodies in S touch the boundary of the container D. The desired dependence is in terms of n and the total distance of the bodies in S from the boundary of D. Specifically, for two convex bodies, $C \subset D \subset \mathbb{R}^2$, let the *escape distance* $\mathrm{esc}(C)$ be the distance between C and the boundary of D (Fig. 2, right); and for a packing $S = \{C_1, \ldots, C_n\}$ in a container D, let $\mathrm{esc}(S) = \sum_{i=1}^{n} \mathrm{esc}(C_i)$. We prove the following bound for pairs of convex bodies C and D, where D is a convex polygon parallel to C.

Theorem 3. *Let C and D be two convex bodies such that D is a convex polygon parallel to C. For every packing S of n positive homothets of C in D, we have*

$$\mathrm{per}(S) \leq \rho(C, D)\,(\mathrm{per}(D) + \mathrm{esc}(S))\;\frac{\log n}{\log\log n},$$

where $\rho(C, D)$ depends on C and D. Apart from the constant factor $\rho(C, D)$, this bound is the best possible.

Motivation. In the *Euclidean Traveling Salesman Problem* (ETSP), given a set S of n points in $\mathbb{R}^d$, we wish to find a closed polygonal chain (*tour*) of minimum Euclidean length whose vertex set is S. The Euclidean TSP is known to be NP-hard, but it admits a PTAS in $\mathbb{R}^d$, where $d \in \mathbb{N}$ is constant [1]. In the *TSP with Neighborhoods* (TSPN), given a set of n sets (neighborhoods) in $\mathbb{R}^d$, we wish to find a closed polygonal chain of minimum Euclidean length that has a vertex in each neighborhood. The neighborhoods are typically simple geometric objects (of bounded description complexity) such as disks, rectangles, line segments, or lines. Since ETSP is NP-hard, TSPN is also NP-hard. TSPN admits a PTAS for certain types of neighborhoods [10], but is hard to approximate for others [4].

For n connected (possibly overlapping) neighborhoods in the plane, TSPN can be approximated with ratio $O(\log n)$ by an algorithm of Mata and Mitchell [9]. See also the survey by Bern and Eppstein [3] for a short outline of this algorithm. At its core, the $O(\log n)$-approximation relies on the following early result by Levcopoulos and Lingas [8]: every (simple) rectilinear polygon P with n vertices, r of which are reflex, can be partitioned into rectangles of total perimeter $O(\mathrm{per}(P) \log r)$ in $O(n \log n)$ time.

One approach to approximate TSPN (in particular, it achieves a constant-ratio approximation for unit disks) is the following [5,7]. Given a set S of n neighborhoods, compute a maximal subset $I \subseteq S$ of pairwise disjoint neighborhoods (i.e., a packing), compute a good tour for I, and then augment it by traversing the boundary of each set in I. Since each neighborhood in $S \setminus I$ intersects some neighborhood in I, the augmented tour visits all members of S. This approach is particularly appealing since good approximation algorithms are often available for pairwise disjoint neighborhoods [10]. The bottleneck of this approach is the length increase incurred by extending a tour of I by the total perimeter of the neighborhoods in I. An upper bound $\mathrm{per}(I) = o(\mathrm{OPT}(I) \log n)$ would immediately imply an improved $o(\log n)$-factor approximation ratio for TSPN.

Theorem 2 confirms that this approach cannot beat the $O(\log n)$ approximation ratio for most types of neighborhoods (e.g., circular disks). In the current formulation, Proposition 2 yields the upper bound $\mathrm{per}(I) = O(\log n)$ assuming a convex container, so in order to use this bound, a tour of I needs to be augmented into a convex partition; this may increase the length by a $\Theta(\log n/\log\log n)$-factor in the worst case [6,8]. For convex polygonal neighborhoods, the bound $\mathrm{per}(I) = O(1)$ in Proposition 3 is applicable after a tour for I has been augmented into a convex partition with *parallel* edges (e.g., this is possible for axis-aligned rectangle neighborhoods, and an axis-aligned approximation of the optimal tour for I). The convex partition of a polygon with $O(1)$ distinct orientations, however, may increase the length by a $\Theta(\log n)$-factor in the worst case [8]. Overall our results confirm that we cannot beat the current $O(\log n)$ ratio for TSPN for any type of homothetic neighborhoods if we start with an arbitrary independent set I and an arbitrary near-optimal tour for I.

An improved approximation for TSPN may require additional properties of I or the initial tour for I. Alternatively, it may not be necessary to traverse the entire perimeter of all elements in I to obtain a tour for S. The escape distance $\mathrm{esc}(C)$ is a tool for measuring the necessary detour to visit a neighborhood $C \in S \setminus I$. Theorem 3 indicates that the total perimeter $\mathrm{per}(I')$ of a *second* independent set $I' \subset S \setminus I$ may be significantly larger than $\mathrm{per}(I)$.

2 Preliminaries: A Few Easy Pieces

Proof of Proposition 1. Let $\mu_i > 0$ denote the homothety factor of C_i, i.e., $C_i = \mu_i C$, for $i = 1, \ldots, n$. Since S is a packing we have $\sum_{i=1}^{n} \mu_i^2 \mathrm{area}(C) \le \mathrm{area}(D)$. By the Cauchy-Schwarz inequality we have $(\sum_{i=1}^{n} \mu_i)^2 \le n \sum_{i=1}^{n} \mu_i^2$. It follows that

$$\begin{aligned}\mathrm{per}(S) &= \sum_{i=1}^{n} \mathrm{per}(C_i) = \mathrm{per}(C) \sum_{i=1}^{n} \mu_i \\ &\le \mathrm{per}(C)\sqrt{n}\sqrt{\left(\sum_{i=1}^{n} \mu_i^2\right)} \le \mathrm{per}(C)\sqrt{\frac{\mathrm{area}(D)}{\mathrm{area}(C)}}\,\sqrt{n}.\end{aligned}$$

Set now $\rho(C, D) := \mathrm{per}(C)\sqrt{\mathrm{area}(D)/\mathrm{area}(C)}$, and the proof of the upper bound is complete.

For the lower bound, consider two convex bodies, C and D. Let U be a maximal axis-aligned square inscribed in D, and let μC be the largest positive homothet of C that fits into U. Note that $\mu = \mu(C, D)$ is a constant that depends on C and D only. Subdivide U into $\lceil\sqrt{n}\rceil^2$ congruent copies of the square $\frac{1}{\lceil\sqrt{n}\rceil}U$. Let S be the packing of n copies of $\frac{\mu}{\lceil\sqrt{n}\rceil}C$ (i.e., n translates), with at most one in each square $\frac{1}{\lceil\sqrt{n}\rceil}U$. The total perimeter of the packing is $\mathrm{per}(S) = n \cdot \frac{\mu}{\lceil\sqrt{n}\rceil}\mathrm{per}(C) = \Theta(\sqrt{n})$, as claimed. □

Proof of Proposition 2. Let $S = \{C_1, \ldots, C_n\}$ be a packing of n homothets of C in D where each element of S touches the boundary of D. Observe that $\mathrm{per}(C_i) \le \mathrm{per}(D)$ for all $i = 1, \ldots, n$. Partition the elements of S into subsets as follows. For $k = 1, \ldots, \lceil \log_2 n \rceil$, let S_k denote the set of homothets C_i such that $\mathrm{per}(D)/2^k < \mathrm{per}(C_i) \le \mathrm{per}(D)/2^{k-1}$; and let S_0 be the set of homothets C_i of perimeter less than $\mathrm{per}(D)/2^{\lceil \log_2 n \rceil}$. Then the sum of perimeters of the elements in S_0 is $\mathrm{per}(S_0) \le n\,\mathrm{per}(D)/2^{\lceil \log_2 n \rceil} \le \mathrm{per}(D)$ since $S_0 \subseteq S$ contains at most n elements altogether.

For $k = 1, \ldots, \lceil \log_2 n \rceil$, the diameter of each $C_i \in S_k$ is bounded above by

$$\mathrm{diam}(C_i) < \mathrm{per}(C_i)/2 \le \mathrm{per}(D)/2^k. \tag{1}$$

Consequently, every point of a body $C_i \in S_k$ lies at distance at most $\mathrm{per}(D)/2^k$ from the boundary of D, denoted ∂D. Let R_k be the set of points in D at distance at most $\mathrm{per}(D)/2^k$ from ∂D. Then

$$\mathrm{area}(R_k) \le \mathrm{per}(D)\,\frac{\mathrm{per}(D)}{2^k} = \frac{(\mathrm{per}(D))^2}{2^k}. \tag{2}$$

Since S consists of homothets, the area of any element $C_i \in S_k$ is bounded from below by

$$\mathrm{area}(C_i) = \mathrm{area}(C)\left(\frac{\mathrm{per}(C_i)}{\mathrm{per}(C)}\right)^2 \ge \mathrm{area}(C)\left(\frac{\mathrm{per}(D)}{2^k\,\mathrm{per}(C)}\right)^2. \tag{3}$$

By a volume argument, (2) and (3) yield

$$|S_k| \le \frac{\mathrm{area}(R_k)}{\min_{C_i \in S_k} \mathrm{area}(C_i)} \le \frac{(\mathrm{per}(D))^2/2^k}{\mathrm{area}(C)(\mathrm{per}(D))^2/(2^k\,\mathrm{per}(C))^2} = \frac{(\mathrm{per}(C))^2}{\mathrm{area}(C)} \cdot 2^k.$$

Since for $C_i \in S_k$, $k = 1, \ldots, \lceil \log_2 n \rceil$, we have $\mathrm{per}(C_i) \le \mathrm{per}(D)/2^{k-1}$, it follows that

$$\mathrm{per}(S_k) \le |S_k| \cdot \frac{\mathrm{per}(D)}{2^{k-1}} \le 2\,\frac{(\mathrm{per}(C))^2}{\mathrm{area}(C)}\,\mathrm{per}(D).$$

Hence the sum of perimeters of all elements in S is bounded by

$$\mathrm{per}(S) = \sum_{k=0}^{\lceil \log_2 n \rceil} \mathrm{per}(S_k) \le \left(1 + 2\,\frac{(\mathrm{per}(C))^2}{\mathrm{area}(C)}\lceil \log_2 n \rceil\right)\mathrm{per}(D),$$

as required. □

Proof of Proposition 3. Let $\rho'(C)$ denote the ratio between $\mathrm{per}(C)$ and the length of a shortest side of C. Recall that each $C_i \in S$ touches the boundary of polygon D. Since D is parallel to C, the side of D that supports C_i must contain a side of C_i. Let a_i denote the length of this side.

$$\mathrm{per}(S) = \sum_{i=1}^{n} \mathrm{per}(C_i) = \sum_{i=1}^{n} a_i\,\frac{\mathrm{per}(C_i)}{a_i} \le \rho'(C)\sum_{i=1}^{n} a_i \le \rho'(C)\,\mathrm{per}(D).$$

Set now $\rho(C, D) := \rho'(C)\,\mathrm{per}(D)$, and the proof is complete. □

3 Disks Touching the Boundary of a Square: Proof of Theorem 1

Let S be a set of n interior-disjoint disks in the unit square $U = [0,1]^2$ that touch the boundary of U. From Proposition 2 we deduce the upper bound $\text{per}(S) = O(\log n)$, as required.

To prove the lower bound, it remains to construct a packing of $O(n)$ disks in the unit square $[-\frac{1}{2}, \frac{1}{2}] \times [0,1]$ such that every disk touches the x-axis, and the sum of their diameters is $\Omega(\log n)$. To each disk we associate its vertical *projection interval* (on the x-axis). The algorithm greedily chooses disks of monotonically decreasing radii such that (1) every diameter is $1/16^k$ for some $k \in \mathbb{N}$; and (2) if the projection intervals of two disks overlap, then one interval contains the other.

For $k = 0, 1, \ldots, \lfloor \log_{16} n \rfloor$, denote by S_k the set of disks of diameter $1/16^k$, constructed by our algorithm. We recursively allocate a set of intervals $X_k \subset [-\frac{1}{2}, \frac{1}{2}]$ to S_k, and then choose disks in S_k such that their projection intervals lie in X_k. Initially, $X_0 = [-\frac{1}{2}, \frac{1}{2}]$, and S_0 contains the disk of diameter 1 inscribed in $[-\frac{1}{2}, \frac{1}{2}] \times [0,1]$. The length of each maximal interval $I \subseteq X_k$ will be a multiple of $1/16^k$, so I can be covered by projection intervals of interior-disjoint disks of diameter $1/16^k$ touching the x-axis. Every interval $I \subseteq X_k$ will have the property that any disk of diameter $1/16^k$ whose projection interval is in I is disjoint from any (larger) disk in S_j, $j < k$.

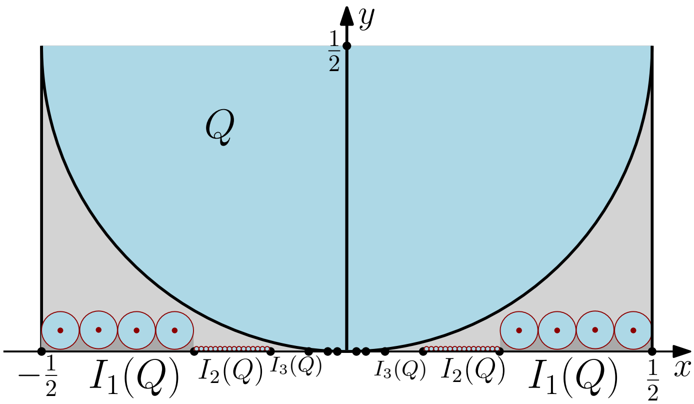

Fig. 3. Disk Q and the exponentially decreasing pairs of intervals $I_k(Q)$, $k = 1, 2, \ldots$.

Consider the disk Q of diameter 1, centered at $(0, \frac{1}{2})$, and tangent to the x-axis (see Fig. 3). It can be easily verified that:

(i) the locus of centers of disks tangent to both Q and the x-axis is the parabola $y = \frac{1}{2}x^2$; and

(ii) any disk of diameter $1/16$ and tangent to the x-axis whose projection interval is in $I_1(Q) = [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$ is disjoint from Q.

Indeed, the center of any such disk is $(x_1, \frac{1}{16})$, for $x_1 \leq -\frac{5}{16}$ or $x_1 \geq \frac{5}{16}$, and hence lies below the parabola $y = \frac{1}{2}x^2$. Similarly, for all $k \in \mathbb{N}$, any disk of diameter $1/16^k$ and tangent to the x-axis whose projection interval is in $I_k(Q) = [-\frac{1}{2^k}, -\frac{1}{2^{k+1}}] \cup [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$ is disjoint from Q. For an arbitrary disk D tangent to the x-axis, and an integer $k \geq 1$, denote by $I_k(D) \subseteq [-\frac{1}{2}, \frac{1}{2}]$ the pair of intervals corresponding to $I_k(Q)$; for $k = 0$, $I_k(D)$ consists of only one interval.

We can now recursively allocate intervals in X_k and choose disks in S_k ($k = 0, 1, \ldots, \lfloor \log_{16} n \rfloor$) as follows. Recall that $X_0 = [-\frac{1}{2}, \frac{1}{2}]$, and S_0 contains a single disk of unit diameter inscribed in the unit square $[-\frac{1}{2}, \frac{1}{2}] \times [0, 1]$. Assume that we have already defined the intervals in X_{k-1}, and selected disks in S_{k-1}. Let X_k be the union of the interval pairs $I_{k-j}(D)$ for all $D \in S_j$ and $j = 0, 1, \ldots, k-1$. Place the maximum number of disks of diameter $1/16^k$ into S_k such that their projection intervals are contained in X_k. For a disk $D \in S_j$ ($j = 0, 1, \ldots, k-1$) of diameter $1/16^j$, the two intervals in X_{k-j} each have length $\frac{1}{2} \cdot \frac{1}{2^{k-j}} \cdot \frac{1}{16^j} = \frac{8^{k-j}}{2} \cdot \frac{1}{16^k}$, so they can each accommodate the projection intervals of $\frac{8^{k-j}}{2}$ disks in S_k.

We prove by induction on k that the length of X_k is $\frac{1}{2}$, and so the sum of the diameters of the disks in S_k is $\frac{1}{2}$, $k = 1, 2, \ldots, \lfloor \log_{16} n \rfloor$. The interval $X_0 = [-\frac{1}{2}, \frac{1}{2}]$ has length 1. The pair of intervals $X_1 = [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$ has length $\frac{1}{2}$. For $k = 2, \ldots, \lfloor \log_{16} n \rfloor$, the set X_k consists of two types of (disjoint) intervals: (a) The pair of intervals $I_1(D)$ for every $D \in S_{k-1}$ covers half of the projection interval of D. Over all $D \in S_{k-1}$, they jointly cover half the length of X_{k-1}. (b) Each pair of intervals $I_{k-j}(D)$ for $D \in S_{k-j}$, $j = 0, \ldots, k-2$, has half the length of $I_{k-j-1}(D)$. So the sum of the lengths of these intervals is half the length of X_{k-1}; although they are disjoint from X_{k-1}. Altogether, the sum of lengths of all intervals in X_k is the same as the length of X_{k-1}. By induction, the length of X_{k-1} is $\frac{1}{2}$, hence the length of X_k is also $\frac{1}{2}$, as claimed. This immediately implies that the sum of diameters of the disks in $\bigcup_{k=0}^{\lfloor \log_{16} n \rfloor} S_k$ is $1 + \frac{1}{2}\lfloor \log_{16} n \rfloor$. Finally, one can verify that the total number of disks used is $O(n)$. Write $K = \lfloor \log_{16} n \rfloor$. Indeed, $|S_0| = 1$, and $|S_k| = |X_k|/16^{-k} = 16^k/2$, for $k = 1, \ldots, K$, where $|X_k|$ denotes the total length of the intervals in X_k. Consequently, $|S_0| + \sum_{k=1}^{K} |S_k| = O(16^k) = O(n)$, as required. □

4 Homothets Touching the Boundary: Proof of Theorem 2

The upper bound $\text{per}(S) = O(\log n)$ follows from Proposition 2. It remains to construct a packing S of perimeter $\text{per}(S) = \Omega(\log n)$ for given C and D. Let C and D be two convex bodies with bounded description complexity. We wish to argue analogously to the case of disks in a square. Therefore, we choose an arc $\gamma \subset \partial D$ that is smooth and sufficiently "flat," but contains no side parallel to a corresponding side of C. Then we build a hierarchy of homothets of C touching the arc γ, so that the depth of the hierarchy is $O(\log n)$, and the homothety factors decrease by a constant between two consecutive levels.

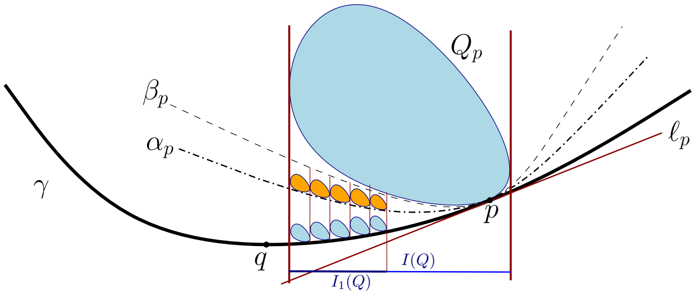

Fig. 4. If a homothet C_p is tangent to $\gamma \subset \partial D$ at point p, then there are polynomials α_p and β_p that separate γ from C_p. We can place a constant number of congruent homothets of C between α_p and β_p whose vertical projections cover $I_1(Q)$. These homothets can be translated vertically down to touch γ.

We choose an arc $\gamma \subset \partial D$ as follows. If D has a side with some direction $\mathbf{d} \in \mathbb{S}^1$ such that C has no parallel side of the same direction $\mathbf{d}$, then let γ be this side of D. Otherwise, ∂D contains an algebraic curve γ_1 of degree 2 or higher. Let $q \in \gamma_1$ be an interior point of this curve such that γ_1 is twice differentiable at q. Assume, after a rigid transformation of D if necessary, that $q = (0, 0)$ is the origin and the supporting line of D at q is the x-axis. By the inverse function theorem, there is an arc $\gamma_2 \subseteq \gamma_1$, containing q, such that γ_2 is the graph of a twice differentiable function of x. Finally, let $\gamma \subset \gamma_2$ be an arc such that the part of ∂C that has the same tangent lines as γ_2 contains no segments (sides).

For every point $p \in \gamma$, let $p = (x_p, y_p)$, and let s_p be the slope of the tangent line of D at p. Then the tangent line of D at $p \in \gamma$ is $\ell_p(x) = s_p(x - x_p)$. For any homothet Q of C, let Q_p denote a translate of Q tangent to ℓ_p at point p (Fig. 4). If both C and D have bounded description complexity, then there are constants $\rho_0 > 0$, $\kappa, \in \mathbb{N}$ and $A < B$, such that for every point $p \in \gamma$ and every homothety factor ρ, $0 < \rho < \rho_0$, the polynomials

$$\alpha_p(x) = A|x - x_p|^\kappa + s_p(x - x_p) \quad \text{and} \quad \beta_p(x) = B|x - x_p|^\kappa + s_p(x - x_p)$$

separate γ from the convex body $Q_p = (\rho C)_p$.

Similarly to the proof of Theorem 1, the construction is guided by nested *projection intervals*. Let $Q = (\rho C)_p$ be a homothet of C that lies in D and is tangent to γ at point $p \in \gamma$. Denote by $I(Q)$ the vertical projection of Q to the x-axis. For $k = 1, \ldots$, we recursively define disjoint intervals or interval pairs $I_k(Q) \subset I(Q)$ of length $|I_k(Q)| = |I(Q)|/2^k$. During the recursion, we maintain the invariant that the set $J_k(Q) = I(Q) \setminus \bigcup_{j<k} I_j(Q)$ is an interval of length $|I(Q)|/2^{k-1}$ that contains x_p. Assume that $I_1(Q), \ldots, I_{k-1}(Q)$ have been defined, and we need to choose $I_k(Q) \subset J_k(Q)$. If x_p lies in the central one quarter of $J_k(Q)$, then let $I_k(Q)$ be a pair of intervals that consists of the left and right *quarters* of $J_k(Q)$. If x_p lies to the left (right) of the central one quarter of $J_k(Q)$, then let $I_k(Q)$ be the right (left) *half* of $J_k(Q)$. It is now an easy matter to check (by induction on k) that $|x - x_p| \geq |I(Q)|/8^k$ for all $x \in I_k(Q)$.

Consequently,

$$\beta_p(x) - \alpha_p(x) \geq (B - A) \cdot \left(\frac{|I(Q)|}{8^k} \right)^{\kappa} \tag{4}$$

for all $x \in I_k(Q)$. There is a constant $\mu > 0$ such that a homothet $\mu^k Q$ with arbitrary projection interval in $I_k(Q)$ fits between the curves α_p and β_p. Refer to Fig. 4. Therefore we can populate the region between curves α_p and β_p and above $I_k(Q)$ with homothets ρQ, of homotety factors $\mu^k/2 < \rho \leq \mu^k$, such that their projection intervals are pairwise disjoint and cover $I_k(Q)$. By translating these homothets vertically until they touch γ, they remain disjoint from Q and preserve their projection intervals. We can now repeat the construction of the previous section and obtain $\lceil \log_{(2/\mu)} n \rceil$ layers of homothets touching γ, such that the total length of the projections of the homothets in each layer is $\Theta(1)$. Consequently, the total perimeter of the homothets in each layer is $\Theta(1)$, and the overall perimeter of the packing is $\Theta(\log n)$, as required. □

5 Homothets in a Parallel Container: Proof of Theorem 3

Upper bound. Let $S = \{C_1, \ldots, C_n\}$ be a packing of n homothets of a convex body C in a container D such that D is a convex polygon parallel to C. For each element $C_i \in S$, $\text{esc}(C_i)$ is the distance between a side of D and a corresponding side of C_i. For each side a of D, let $S_a \subseteq S$ denote the set of $C_i \in S$ for which a is the closest side of D (ties are broken arbitrarily). Since D has finitely many sides, it is enough to show that for each side a of D, we have

$$\text{per}(S_a) \leq \rho_a(C, D)\,(\text{per}(D) + \text{esc}(S))\,\frac{\log |S_a|}{\log \log |S_a|},$$

where $\rho_a(C, D)$ depends on a, C and D only.

Suppose that $S_a = \{C_1, \ldots, C_n\}$ is a packing of n homothets of C such that $\text{esc}(C_i)$ equals the distance between C_i and side a of D. Assume for convenience that a is horizontal. Let $c \subset \partial C$ be the side of C corresponding to the side a of D. Let $\rho_1 = \text{per}(C)/|c|$, and then we can write $\text{per}(C) = \rho_1 |c|$. Refer to Fig. 5(left).

Denote by $b \subset c$ the line segment of length $|b| = |c|/2$ with the same midpoint as c. Since C is a convex body, the two vertical lines though the two endpoints of b intersect C in two line segments denoted h_1 and h_2, respectively. Let $\rho_2 = \min(|h_1|, |h_2|)/|b|$, and then $\min(|h_1|, |h_2|) = \rho_2 |b|$. By convexity, every vertical line that intersects segment b intersects C in a vertical segment of length at least $\rho_2 |b|$. Note that ρ_1 and ρ_2 are constants depending on C and D. For each homothet $C_i \in S_a$, let $b_i \subset \partial C_i$ be the homothetic copy of segment $b \subset \partial C$.

Put $\lambda = 2\lceil \log n / \log \log n \rceil$. Partition S_a into two subsets $S_a = S_{\text{far}} \cup S_{\text{close}}$ as follows. For each $C_i \in S_a$, let $C_i \in S_{\text{close}}$ if $\text{esc}(C_i) < \rho_2 |b_i| / \lambda$, and $C_i \in S_{\text{far}}$ otherwise. For each homothet $C_i \in S_{\text{close}}$, let $\text{proj}_i \subseteq a$ denote the vertical projection of segment b_i onto the horizontal side a (refer to Fig. 5, right). The perimeter of each $C_i \in S_a$ is $\text{per}(C_i) = \rho_1 |c_i| = 2\rho_1 |b_i| = 2\rho_1 |\text{proj}_i|$. We have

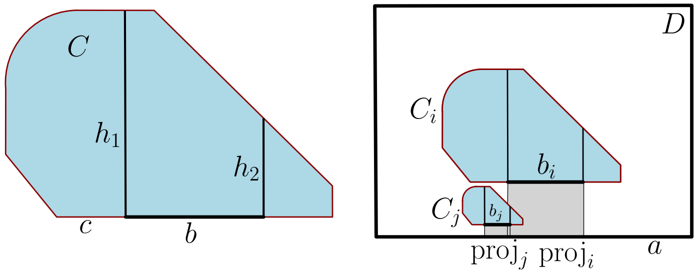

Fig. 5. Left: A convex body C with a horizontal side c. The segment $b \subset c$ has length $|b| = |c|/2$, and the vertical segments h_1 and h_2 are incident to the endpoints of b. Right: Two homothets, C_i and C_j, in a convex container D. The vertical projections of b_i and b_j onto the horizontal side a are proj_i and proj_j.

$$\text{per}(S_{\text{far}}) = \sum_{C_i \in S_{\text{far}}} \text{per}(C_i) = \sum_{C_i \in S_{\text{far}}} 2\rho_1 |b_i| \leq \sum_{C_i \in S_{\text{far}}} 2\rho_1 \frac{\text{esc}(C_i)\,\lambda}{\rho_2}$$
$$\leq \frac{2\rho_1 \text{esc}(S)}{\rho_2}\,\lambda. \quad (5)$$

It remains the estimate $\text{per}(S_{\text{close}})$ as an expression of λ.

$$\sum_{C_i \in S_{\text{close}}} \text{per}(C_i) = 2\rho_1 \sum_{C_i \in S_{\text{close}}} |\text{proj}_i|. \quad (6)$$

Define the *depth* function for every point of the horizontal side a by

$$d : a \to \mathbb{N}, \qquad d(x) = |\{C_i \in S_{\text{close}} : x \in \text{proj}_i\}|.$$

That is, $d(x)$ is the number of homothets such that the vertical projection of segment b_i contains point x. For every positive integer $k \in \mathbb{N}$, let

$$I_k = \{x \in a : d(x) \geq k\},$$

that is, I_k is the set of points of depth at least k. Since S_{close} is finite, the set $I_k \subseteq a$ is measurable. Denote by $|I_k|$ the measure (total length) of I_k. By definition, we have $|a| \geq |I_1| \geq |I_2| \geq \ldots$. A standard double counting for the integral $\int_{x \in a} d(x)\, \mathrm{d}x$ yields

$$\sum_{C_i \in S_{\text{close}}} |\text{proj}_i| = \sum_{k=1}^{\infty} |I_k|. \quad (7)$$

If $d(x) = k$ for some point $x \in a$, then k segments b_i, lie above x. Each $C_i \in S_{\text{close}}$ is at distance $\text{esc}(C_i) < \rho_2 |b_i|/\lambda$ from a. Suppose that proj_i and proj_j intersect for $C_i, C_j \in S_{\text{close}}$ (Fig. 5, right). Then one of them has to be closer to a than the other: we may assume w.l.o.g. $\text{esc}(C_j) < \text{esc}(C_i)$. Now a vertical segment between $b_i \subset C_i$ and $\text{proj}_i \subset a$ intersects b_j. The length of this intersection segment satisfies $\rho_2|b_j| \leq \text{esc}(C_i) < \rho_2|b_i|/\lambda$. Consequently, $|b_j| < |b_i|/\lambda$ (or, equivalently, $|\text{proj}_j| < |\text{proj}_i|/\lambda$) holds for any consecutive homothets above point $x \in a$. In particular, for the k-th smallest projection containing $x \in a$, we have $|\text{proj}_k| \leq |a|/\lambda^{k-1} = |a|\lambda^{1-k}$.

We claim that

$$|I_k| \leq |a|\lambda^{\lambda-k} \qquad \text{for } k \geq \lambda + 1. \tag{8}$$

Suppose, to the contrary, that $|I_k| > |a|\lambda^{\lambda-k}$ for some $k \geq \lambda+1$. Then there are homothets $C_i \in S_{\text{close}}$ of side lengths at most $|a|/\lambda^{k-1}$, that jointly project into I_k. Assuming that $|I_k| > |a|\lambda^{\lambda-k}$, it follows that the number of these homothets is at least

$$\frac{|a|\lambda^{\lambda-k}}{|a|\lambda^{1-k}} = \lambda^{\lambda-1} = \left(2\left\lceil\frac{\log n}{\log\log n}\right\rceil\right)^{2\lceil\frac{\log n}{\log\log n}\rceil-1} > n,$$

contradicting the fact that $S_{\text{close}} \subseteq S$ has at most n elements. Combining (6), (7), and (8), we conclude that

$$\begin{aligned}\text{per}(S_{\text{close}}) &= 2\rho_1\sum_{k=1}^{\infty}|I_k| \leq 2\rho_1\left(\lambda|I_1| + \sum_{k=\lambda+1}^{\infty}|I_k|\right) \leq 2\rho_1\left(\lambda + \sum_{j=1}^{\infty}\frac{1}{\lambda^j}\right)|a| \\ &\leq 2\rho_1(\lambda+1)\,\text{per}(D). \end{aligned} \tag{9}$$

Putting (5) and (9) together yields

$$\begin{aligned}\text{per}(S_a) &= \text{per}(S_{\text{close}}) + \text{per}(S_{\text{far}}) \leq 2\rho_1\left((\lambda+1)\,\text{per}(D) + \frac{\text{esc}(S)}{\rho_2}\,\lambda\right) \\ &\leq \rho(C,D)\,(\text{per}(D)+\text{esc}(S))\;\lambda = \rho(C,D)\,(\text{per}(D)+\text{esc}(S))\;\frac{\log n}{\log\log n},\end{aligned}$$

for a suitable $\rho(C,D)$ depending on C and D, as required; here we set $\rho(C,D) = 2\rho_1\max(2, 1/\rho_2)$.

Lower bound for squares. We first confirm the given lower bound for squares, i.e., we construct a packing S of $O(n)$ axis-aligned squares in the unit square $U = [0,1]^2$ with total perimeter $\Omega((\text{per}(U)+\text{esc}(S))\log n/\log\log n)$.

Let $n \geq 4$, and put $\lambda = \lfloor\log n/\log\log n\rfloor/2$. We arrange each square $C_i \in S$ such that $\text{per}(C_i) = \lambda\,\text{esc}(C_i)$. We construct S as the union of λ subsets $S = \bigcup_{j=1}^{\lambda} S_j$, where S_j is a set of congruent squares, at the same distance from the bottom side of U.

Let S_1 be a singleton set consisting of one square of side length $1/4$ (and perimeter 1) at distance $1/\lambda$ from the bottom side of U. Let S_2 be a set of 2λ squares of side length $1/(4\cdot 2\lambda)$ (and perimeter $1/(2\lambda)$), each at distance $1/(2\lambda^2)$ from the bottom side of U. Note that these squares lie strictly below the first square in S_1, since $1/(8\lambda) + 1/(2\lambda^2) < 1/\lambda$. The total length of the vertical projections of the squares in S_2 is $2\lambda \cdot 1/(8\lambda) = 1/4$.

Similarly, for $j = 3\dots,\lambda$, let S_j be a set of $(2\lambda)^{j-1}$ squares of side length $\frac{1}{4\cdot(2\lambda)^{j-1}}$ (and perimeter $1/(2\lambda)^{j-1}$), each at distance $1/(2^{j-1}\lambda^j)$ from the bottom side of U. These squares lie strictly below any square in S_{j-1}; and the total length of their vertical projections onto the x-axis is $(2\lambda)^{j-1}\cdot\frac{1}{4\cdot(2\lambda)^{j-1}} = 1/4$.

The number of squares in $S = \bigcup_{j=1}^{\lambda} S_j$ is

$$\sum_{j=1}^{\lambda}(2\lambda)^{j-1} = \Theta\left((2\lambda)^{\lambda}\right) = O(n).$$

The total distance from the squares to the boundary of U is

$$\mathrm{esc}(S) = \sum_{j=1}^{\lambda}(2\lambda)^{j-1}\frac{1}{2^{j-1}\lambda^j} = \lambda\,\frac{1}{\lambda} = 1.$$

The total perimeter of all squares in S is

$$4\cdot\sum_{j=1}^{\lambda}\frac{1}{4} = \lambda = \Omega\left(\frac{\log n}{\log\log n}\right) = \Omega\left((\mathrm{per}(U)+\mathrm{esc}(S))\,\frac{\log n}{\log\log n}\right),$$

as required.

General lower bound. We now use establish the lower bound in the general setting. Given a convex body C and a convex polygon D parallel to C, we construct a packing S of $O(n)$ positive homothets of C in D with total perimeter $\Omega((\mathrm{per}(D)+\mathrm{esc}(S))\log n/\log\log n)$.

Let a be an arbitrary side of D. Assume w.l.o.g. that a is horizontal. Let U_C be the minimum axis-aligned square containing C. Clearly, we have $\frac{1}{2}\mathrm{per}(U_C) \le \mathrm{per}(C) \le \mathrm{per}(U_C)$. We first construct a packing S_U of $O(n)$ axis-aligned squares in D such that for each square $U_i \in S_U$, $\mathrm{esc}(U_i)$ equals the distance from the horizontal side a. We then obtain the packing S by inscribing a homothet C_i of C in each square $U_i \in S_U$ such that C_i touches the bottom side of U_i. Consequently, we have $\mathrm{per}(S) \ge \mathrm{per}(S_U)/2$ and $\mathrm{esc}(S) = \mathrm{esc}(S_U)$, since $\mathrm{esc}(C_i) = \mathrm{esc}(U_i)$ for each square $U_i \in S_U$.

It remains to construct the square packing S_U. Let $U(a)$ be a maximal axis-aligned square contained in D such that its bottom side is contained in a. S_U is a packing of squares in $U(a)$ that is homothetic with the packing of squares in the unit square U described previously. Put $\rho_1 = \mathrm{per}(U(a))/\mathrm{per}(U) = \mathrm{per}(U(a))/4$. We have $\mathrm{per}(S) \ge \frac{1}{4}\,\rho_1\ \Omega\left((\mathrm{per}(U)+\mathrm{esc}(S))\,\frac{\log n}{\log\log n}\right)$, or

$$\mathrm{per}(S) \ge \rho(C,D)\left((\mathrm{per}(D)+\mathrm{esc}(S))\,\frac{\log n}{\log\log n}\right),$$

where $\rho(C,D)$ is a factor depending on C and D, as required. □

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