Symbolic Control of Stochastic Switched Systems via Finite Abstractions*

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Abstract. Stochastic switched systems are a class of continuous-time dynamical models with probabilistic evolution over a continuous domain and control-dependent discrete dynamics over a finite set of modes. As such, they represent a subclass of general stochastic hybrid systems. While the literature has witnessed recent progress in the dynamical analysis and controller synthesis for the stability of stochastic switched systems, more complex and challenging objectives related to the verification of and the synthesis for logic specifications (properties expressed as formulas in linear temporal logic or as automata on infinite strings) have not been formally investigated as of yet. This paper addresses these complex objectives by constructively deriving approximately equivalent (bisimilar) symbolic models of stochastic switched systems. More precisely, a finite symbolic model that is approximately bisimilar to a stochastic switched system is constructed under some dynamical stability assumptions on the concrete model. This allows to formally synthesize controllers (switching signals) over the finite symbolic model that are valid for the concrete system, by means of mature techniques in the literature.

1 Introduction

Stochastic hybrid systems are general dynamical systems comprising continuous and discrete dynamics interleaved with probabilistic noise and stochastic events [4]. Because of their versatility and generality they carry great promise in many safety critical applications [4], including power networks, automotive and financial engineering, air traffic control, biology, telecommunications, and embedded systems. Stochastic *switched* systems are a relevant class of stochastic hybrid systems: they consist of a finite set of modes of operation, each of which is associated to a probabilistic dynamical behavior; further, their discrete dynamics, in the form of mode changes, are governed by a deterministic control signal. However, unlike general stochastic hybrid systems,

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they do not present probabilistic discrete dynamics (random switch of modes), nor continuous resets upon mode change.

It is known [12] that switched systems can be endowed with global dynamics that are not characteristic of the behavior of any of their single modes: for instance, global instability can arise by proper choice of the discrete switches between a set of stable dynamical modes. This global emergent behavior is one of the many features that makes switched systems theoretically interesting. With focus on *stochastic* switched systems, despite recent progress on basic dynamical analysis focused on stability properties [6], there are no notable results in terms of more complex objectives, such as those dealing with verification or (controller) synthesis for logical specifications. Specifications of interest are expressed as formulas in linear temporal logic or via automata on infinite strings, and as such they are not amenable to be handled by classical techniques for stochastic processes.

A promising direction to investigate these general properties is the use of *symbolic models*. Symbolic models are abstract descriptions of the original dynamics, where each abstract state (or symbol) corresponds to an aggregate of states in the concrete system. When a finite symbolic model is obtained and formally is in relationship with the original system, one can leverage mature techniques for controller synthesis over the discrete model [14] to automatically synthesize controllers for the original system. Towards this goal, a relevant approach is the construction of finite-state symbolic models that are *bisimilar* to the original system. Unfortunately, the class of continuous (time and space) dynamical systems admitting exactly bisimilar finite-state symbolic models is quite restrictive and in particular it covers mostly non-probabilistic models. The results in [5] provide a notion of exact stochastic bisimulation for a class of stochastic hybrid systems, however [5] does not provide any abstraction algorithm, nor looks at the synthesis problem. Therefore, rather than requiring exact equivalence, one can resort to *approximate bisimulation* relations [8], which introduce metrics between the trajectories of the abstract and the concrete models, and further require boundedness in time of these distances.

The construction of approximately bisimilar symbolic models has been recently studied for non-probabilistic continuous control systems, possibly endowed with non-determinism [13,18, and references therein], as well as for non-probabilistic switched systems [9]. However stochastic systems, particularly when endowed with switched dynamics, have only been partially explored. With focus on these models, only a few existing results deal with abstractions of discrete-time processes [2, and references therein]. Results for continuous-time models cover models with specific dynamics: probabilistic rectangular hybrid automata [20] and stochastic dynamical systems under contractivity assumptions [1]. Further, the results in [10] only *check* the (approximate) relationship between an uncountable abstraction and a class of stochastic hybrid systems via a notion of stochastic (bi)simulation function, however, these results do not provide any *construction* of the approximation, nor do they deal with *finite* abstractions, and appear to be computationally tractable only in the case of no inputs. In summary, to the best of our knowledge, there is no comprehensive work on the construction of finite bisimilar abstractions for continuous-time stochastic systems with control actions or with switched dynamics. A recent result [22] by the authors investigates this goal over stochastic control systems, however without any hybrid dynamics.

The main contribution of this work consists in showing the existence and the construction of approximate bisimilar symbolic models for incrementally stable stochastic switched systems. Incremental stability is a stability assumption applied to the stochastic switched systems under study: it can be described in terms of a so-called Lyapunov function (which can either be a single global function or correspond to a set of mode-dependent ones). It is an extension of a similar notion developed for non-probabilistic switched systems [9] in the sense that the results for non-probabilistic switched systems represent a special case of the results in this paper when the continuous dynamics are degenerate (they present no noise). The effectiveness of the results is illustrated with the synthesis of a controller (switching signal) for a room temperature regulation problem (admitting a global – or common – Lyapunov function), which is further subject to a constraint expressed by a finite automaton. More precisely, we display a switched controller synthesis for the purpose of temperature regulation toward a desired level, subject to the discrete constraint.

2 Stochastic Switched Systems

2.1 Notation

The identity map on a set A is denoted by 1_A. If A is a subset of B, we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$ denote the set of natural, nonnegative integer, integer, real, positive, and nonnegative real numbers, respectively. The symbols I_n, 0_n, and $0_{n\times m}$ denote the identity matrix, the zero vector, and the zero matrix in $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, and by $\|x\|$ the infinity norm of x, namely, $\|x\| = \max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ denotes the absolute value of x_i. Given a matrix $M = \{m_{ij}\} \in \mathbb{R}^{n\times m}$, we denote by $\|M\|$ the infinity norm of M, namely, $\|M\| = \max_{1\leq i\leq n} \sum_{j=1}^{m} |m_{ij}|$, and by $\|M\|_F$ the Frobenius norm of M, namely, $\|M\|_F = \sqrt{\text{Tr}\,(MM^T)}$, where $\text{Tr}(P) = \sum_{i=1}^{n} p_{ii}$ for any $P = \{p_{ij}\} \in \mathbb{R}^{n\times n}$. The notations $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ stand for the minimum and maximum eigenvalues of matrix A, respectively.

The closed ball centered at $x \in \mathbb{R}^n$ with radius ε is defined by $\mathcal{B}_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \varepsilon\}$. A set $B \subseteq \mathbb{R}^n$ is called a *box* if $B = \prod_{i=1}^{n} [c_i, d_i]$, where $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$ for each $i \in \{1, \ldots, n\}$. The *span* of a box B is defined as $span(B) = \min\{|d_i - c_i| \mid i = 1, \ldots, n\}$. By defining $[\mathbb{R}^n]_\eta = \{a \in \mathbb{R}^n \mid a_i = k_i\eta, k_i \in \mathbb{Z}, i = 1, \cdots, n\}$, the set $\bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\lambda(p)$ is a countable covering of $\mathbb{R}^n$ for any $\eta \in \mathbb{R}^+$ and $\lambda \geq \eta$. For a box B and $\eta \leq span(B)$, define the η-approximation $[B]_\eta = [\mathbb{R}^n]_\eta \cap B$. Note that $[B]_\eta \neq \varnothing$ for any $\eta \leq span(B)$. Geometrically, for any $\eta \in \mathbb{R}^+$ with $\eta \leq span(B)$ and $\lambda \geq \eta$, the collection of sets $\{\mathcal{B}_\lambda(p)\}_{p\in[B]_\eta}$ is a finite covering of B, i.e., $B \subseteq \bigcup_{p\in[B]_\eta} \mathcal{B}_\lambda(p)$. We extend the notions of $span$ and approximation to finite unions of boxes as follows. Let $A = \bigcup_{j=1}^{M} A_j$, where each A_j is a box. Define $span(A) = \min\{span(A_j) \mid j = 1, \ldots, M\}$, and for any $\eta \leq span(A)$, define $[A]_\eta = \bigcup_{j=1}^{M} [A_j]_\eta$.

Given a set X, a function $\mathbf{d} : X \times X \rightarrow \mathbb{R}_0^+$ is a metric on X if for any $x, y, z \in X$, the following three conditions are satisfied: i) $\mathbf{d}(x, y) = 0$ if and only if $x = y$; ii) $\mathbf{d}(x, y) = \mathbf{d}(y, x)$; and iii) (triangle inequality) $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z)$. A continuous function $\gamma : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \rightarrow \infty$ as $r \rightarrow \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed nonzero r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \rightarrow 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.

2.2 Stochastic Switched Systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0}$ satisfying the usual conditions of completeness and right-continuity [11, p. 48]. Let $(W_s)_{s\geq 0}$ be a $\widehat{q}$-dimensional $\mathbb{F}$-Brownian motion [17].

Definition 1. *A stochastic switched system is a tuple $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$, where*

- $\mathbb{R}^n$ *is the continuous state space;*
- $\mathsf{P} = \{1, \cdots, m\}$ *is a finite set of modes;*
- $\mathcal{P}$ *is a subset of $\mathcal{S}(\mathbb{R}_0^+, \mathsf{P})$, which denotes the set of piecewise constant functions (by convention continuous from the right) from $\mathbb{R}_0^+$ to P, and characterized by a finite number of discontinuities on every bounded interval in $\mathbb{R}_0^+$;*
- $F = \{f_1, \cdots, f_m\}$ *such that, for all $p \in \mathsf{P}$, $f_p : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function satisfying the following Lipschitz assumption: there exists a constant $L \in \mathbb{R}^+$ such that, for all $x, x' \in \mathbb{R}^n$: $\|f_p(x) - f_p(x')\| \leq L\|x - x'\|$;*
- $G = \{g_1, \cdots, g_m\}$, *such that for all $p \in \mathsf{P}$, $g_p : \mathbb{R}^n \rightarrow \mathbb{R}^{n\times\widehat{q}}$ is a continuous function satisfying the following Lipschitz assumption: there exists a constant $Z \in \mathbb{R}^+$ such that for all $x, x' \in \mathbb{R}^n$: $\|g_p(x) - g_p(x')\| \leq Z\|x - x'\|$.*

Let us discuss the semantics of model Σ. For any given $p \in \mathsf{P}$, we denote by Σ_p the subsystem of Σ defined by the stochastic differential equation

$$\mathrm{d}\,\xi = f_p(\xi)\,\mathrm{d}\,t + g_p(\xi)\,\mathrm{d}\,W_t, \tag{1}$$

where f_p is known as the drift, g_p as the diffusion, and again W_t is Brownian motion. A solution process of Σ_p exists and is uniquely determined owing to the assumptions on f_p and on g_p [17, Theorem 5.2.1, p. 68].

For the global model Σ, a continuous-time stochastic process $\xi : \Omega \times \mathbb{R}_0^+ \rightarrow \mathbb{R}^n$ is said to be a *solution process* of Σ if there exists a switching signal $\upsilon \in \mathcal{P}$ satisfying

$$\mathrm{d}\,\xi = f_\upsilon(\xi)\,\mathrm{d}\,t + g_\upsilon(\xi)\,\mathrm{d}\,W_t, \tag{2}$$

$\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.) at each time $t \in \mathbb{R}_0^+$ when υ is constant. Let us emphasize that υ is a piecewise constant function defined over $\mathbb{R}_0^+$ and taking values in P, which simply dictates which mode the solution process ξ is in at any time $t \in \mathbb{R}_0^+$. Notice that the mode changes are non-probabilistic in that they are fully encompassed by a given function υ in $\mathcal{P}$ and that, whenever a mode is changed (discontinuity in υ), the value of the process ξ is not reset on $\mathbb{R}^n$ – thus ξ is a continuous function of time.

We further write $\xi_{a\upsilon}(t)$ to denote the value of the solution process at time $t \in \mathbb{R}_0^+$ under the switching signal υ from initial condition $\xi_{a\upsilon}(0) = a$ $\mathbb{P}$-a.s., in which a is a random variable that is measurable in $\mathcal{F}_0$. Note that in general the stochastic switched system Σ may start from a random initial condition.

Finally, note that a solution process of Σ_p is also a solution process of Σ corresponding to the constant switching signal $\upsilon(t) = p$, for all $t \in \mathbb{R}_0^+$. We also use $\xi_{ap}(t)$ to denote the value of the solution process of Σ_p at time $t \in \mathbb{R}_0^+$ from the initial condition $\xi_{ap}(0) = a$ $\mathbb{P}$-a.s.

3 Notions of Incremental Stability

This section introduces some stability notions for stochastic switched systems, which generalize the concepts of incremental global asymptotic stability (δ-GAS) [3] for dynamical systems and of incremental global uniform asymptotic stability (δ-GUAS) [9] for non-probabilistic switched systems. The main results presented in this work rely on the stability assumptions discussed in this section.

Definition 2. *The stochastic subsystem Σ_p is incrementally globally asymptotically stable in the qth moment (δ-GAS-M$_q$), where $q \geq 1$, if there exists a $\mathcal{KL}$ function β_p such that for any $t \in \mathbb{R}_0^+$, and any $\mathbb{R}^n$-valued random variables a and a' that are measurable in $\mathcal{F}_0$, the following condition is satisfied:*

$$\mathbb{E}\left[\|\xi_{ap}(t) - \xi_{a'p}(t)\|^q\right] \leq \beta_p\left(\mathbb{E}\left[\|a - a'\|^q\right], t\right). \tag{3}$$

Intuitively, the notion requires (a higher moment of) the distance between trajectories to be bounded and decreasing in time. It can be easily checked that a δ-GAS-M$_q$ stochastic subsystem Σ_p is δ-GAS [3] in the absence of any noise. Further, note that when $f_p(0_n) = 0_n$ and $g_p(0_n) = 0_{n\times\widehat{q}}$ (drift and diffusion terms vanish at the origin), then δ-GAS-M$_q$ implies global asymptotic stability in the qth moment (GAS-M$_q$) [6], which means that all the trajectories of Σ_p converge in the qth moment to the (constant) trajectory $\xi_{0_n p}(t) = 0_n$, for all $t \in \mathbb{R}_0^+$, (the equilibrium point). We extend the notion of δ-GAS-M$_q$ to stochastic switched systems as follows.

Definition 3. *A stochastic switched system $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$ is incrementally globally uniformly asymptotically stable in the qth moment (δ-GUAS-M$_q$), where $q \geq 1$, if there exists a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $\mathbb{R}^n$-valued random variables a and a' that are measurable in $\mathcal{F}_0$, and any switching signal $\upsilon \in \mathcal{P}$, the following condition is satisfied:*

$$\mathbb{E}\left[\|\xi_{a\upsilon}(t) - \xi_{a'\upsilon}(t)\|^q\right] \leq \beta\left(\mathbb{E}\left[\|a - a'\|^q\right], t\right). \tag{4}$$

Essentially Definition 3 extends Definition 2 uniformly over any possible switching signal υ. As expected, the notion generalizes known ones in the literature: it can be easily seen that a δ-GUAS-M$_q$ stochastic switched system Σ is δ-GUAS [9] in the absence of any noise and that, whenever $f_p(0_n) = 0_n$ and $g_p(0_n) = 0_{n\times\widehat{q}}$ for all $p \in \mathsf{P}$, then δ-GUAS-M$_q$ implies global uniform asymptotic stability in the qth moment (GUAS-M$_q$) [6].

For non-probabilistic systems the δ-GAS property can be characterized by scalar functions defined over the state space, known as Lyapunov functions [3]. Similarly, we describe δ-GAS-M$_q$ in terms of the existence of *incremental Lyapunov functions*.

Definition 4. *Define the* diagonal set Δ *as:* $\Delta = \{(x,x) \mid x \in \mathbb{R}^n\}$. *Consider a stochastic subsystem Σ_p and a continuous function $V_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ that is twice continuously differentiable on $\{\mathbb{R}^n \times \mathbb{R}^n\}\backslash\Delta$. Function V_p is called an incremental global asymptotic stability in the* qth *moment (δ-GAS-M$_q$) Lyapunov function for Σ_p, where $q \geq 1$, if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}_p$, $\overline{\alpha}_p$, and a constant $\kappa_p \in \mathbb{R}^+$, such that*

(i) $\underline{\alpha}_p$ *(resp.* $\overline{\alpha}_p$*) is a convex (resp. concave) function;*
(ii) for any $x, x' \in \mathbb{R}^n$, $\underline{\alpha}_p\left(\|x - x'\|^q\right) \leq V_p(x,x') \leq \overline{\alpha}_p\left(\|x - x'\|^q\right)$;
(iii) for any $x, x' \in \mathbb{R}^n$, *such that* $x \neq x'$,

$$\mathcal{L}V_p(x,x') := [\partial_x V_p \;\; \partial_{x'} V_p] \begin{bmatrix} f_p(x) \\ f_p(x') \end{bmatrix} + \frac{1}{2} Tr \left(\begin{bmatrix} g_p(x) \\ g_p(x') \end{bmatrix} \left[g_p^T(x) \;\; g_p^T(x') \right] \begin{bmatrix} \partial_{x,x} V_p & \partial_{x,x'} V_p \\ \partial_{x',x} V_p & \partial_{x',x'} V_p \end{bmatrix} \right) \leq -\kappa_p V_p(x,x').$$

The operator $\mathcal{L}$ is the infinitesimal generator associated to the stochastic subsystem (1) [17, Section 7.3], which characterizes the derivative of the expected value of functions of the process with respect to time. For non-probabilistic systems, $\mathcal{L}$ allows computing the conventional functional derivative with respect to time. The symbols ∂_x and $\partial_{x,x'}$ denote first- and second-order partial derivatives with respect to x and x', respectively. Note that condition (i) is not required in the context of non-probabilistic systems [3].

The following theorem describes δ-GAS-M$_q$ in terms of the existence of a δ-GAS-M$_q$ Lyapunov function.

Theorem 1. *A stochastic subsystem Σ_p is δ-GAS-M$_q$ if it admits a δ-GAS-M$_q$ Lyapunov function.*

As qualitatively stated in the Introduction, it is known that a non-probabilistic switched system, whose subsystems are all δ-GAS, may exhibit some unstable behaviors under fast switching signals [9] and, hence, may not be δ-GUAS. The same occurrence can affect stochastic switched systems endowed with δ-GAS-M$_q$ subsystems. The δ-GUAS property of non-probabilistic switched systems can be established by using a common (or global) Lyapunov function, or alternatively via multiple functions that are mode dependent [9]. This leads to the following extensions for δ-GUAS-M$_q$ of stochastic switched systems.

Assume that for any $p \in \mathsf{P}$, the stochastic subsystem Σ_p admits a δ-GAS-M$_q$ Lyapunov function V_p, satisfying conditions (i)-(iii) in Definition 4 with $\mathcal{K}_\infty$ functions $\underline{\alpha}_p$,

$\overline{\alpha}_p$, and a constant $\kappa_p \in \mathbb{R}^+$. Let us introduce functions $\underline{\alpha}$ and $\overline{\alpha}$ and constant κ for use in the rest of the paper. Let the $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and the constant κ be defied as $\underline{\alpha} = \min\{\underline{\alpha}_1, \cdots, \underline{\alpha}_m\}$, $\overline{\alpha} = \max\{\overline{\alpha}_1, \cdots, \overline{\alpha}_m\}$, and $\kappa = \min\{\kappa_1, \cdots, \kappa_m\}$. First we show a result based on the existence of a common Lyapunov function, characterized by functions $\underline{\alpha} = \underline{\alpha}_1 = \cdots = \underline{\alpha}_m$ and $\overline{\alpha} = \overline{\alpha}_1 = \cdots = \overline{\alpha}_m$, and parameter κ.

Theorem 2. *Consider a stochastic switched system $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$. If there exists a function V that is a common δ-GAS-M$_q$ Lyapunov function for all the subsystems $\{\Sigma_1, \cdots, \Sigma_m\}$, then Σ is δ-GUAS-M$_q$.*

The condition conservatively requires the existence of a single function V that is valid for all the subsystems Σ_p, where $p \in \mathsf{P}$. When this common δ-GAS-M$_q$ Lyapunov function V fails to exist, the δ-GUAS-M$_q$ property of Σ can still be established by resorting to multiple δ-GAS-M$_q$ Lyapunov functions (one per mode) over a restricted set of switching signals. More precisely, from Definition 1, let $\mathcal{S}_{\tau_d}\left(\mathbb{R}_0^+, \mathsf{P}\right)$ denote the set of switching signals υ with *dwell time* $\tau_d \in \mathbb{R}_0^+$, meaning that $\upsilon \in \mathcal{S}\left(\mathbb{R}_0^+, \mathsf{P}\right)$ has dwell time τ_d if the switching times $t_1, t_2, \ldots$ (occurring at the discontinuity points of υ) satisfy $t_1 > \tau_d$ and $t_i - t_{i-1} \geq \tau_d$, for all $i \geq 2$. We now show a result based on multiple Lyapunov functions.

Theorem 3. *Let $\tau_d \in \mathbb{R}_0^+$, and consider a stochastic switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}_{\tau_d}, F, G)$ with $\mathcal{P}_{\tau_d} \subseteq \mathcal{S}_{\tau_d}\left(\mathbb{R}_0^+, \mathsf{P}\right)$. Assume that for any $p \in \mathsf{P}$, there exists a δ-GAS-M$_q$ Lyapunov function V_p for subsystem $\Sigma_{\tau_d,p}$ and that in addition there exits a constant $\mu \geq 1$ such that*

$$\forall x, x' \in \mathbb{R}^n, \ \ \forall p, p' \in \mathsf{P}, \ \ V_p(x, x') \leq \mu V_{p'}(x, x'). \tag{5}$$

If $\tau_d > \log \mu / \kappa$, then Σ_{τ_d} is δ-GUAS-M$_q$.

The above result can be practically interpreted as the following fact: global stability is preserved under subsystem stability and enough time spent in each mode. Theorems 1, 2, and 3 provide sufficient conditions for certain stability properties, however they all hinge on finding proper Lyapunov functions.

For stochastic switched systems Σ (resp. Σ_{τ_d}) with f_p and g_p of the form of polynomials, for any $p \in \mathsf{P}$, one can resort to available software tools, such as SOSTOOLS [19], to search for appropriate δ-GAS-M$_q$ Lyapunov functions.

We look next into special instances where these functions are known explicitly or can be easily computed based on the model dynamics. The first result provides a sufficient condition for a particular function V_p to be a δ-GAS-M$_q$ Lyapunov function for a stochastic subsystem Σ_p, when $q = 1, 2$ (first or second moment).

Lemma 1. *Consider a stochastic subsystem Σ_p. Let $q \in \{1, 2\}$, $P_p \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, and the function $V_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ be defined as follows:*

$$V_p(x, x') := \left(\widetilde{V}(x, x')\right)^{\frac{q}{2}} = \left(\frac{1}{q}\left(x - x'\right)^T P_p \left(x - x'\right)\right)^{\frac{q}{2}}, \tag{6}$$

and satisfies

$$(x-x')^T P_p(f_p(x)-f_p(x'))+\frac{1}{2}\left\|\sqrt{P_p}\left(g_p(x)-g_p(x')\right)\right\|_F^2 \leq -\kappa_p\left(V_p(x,x')\right)^{\frac{2}{q}}, \quad (7)$$

or, if f_p is differentiable, satisfies

$$(x-x')^T P_p\partial_x f_p(z)(x-x')+\frac{1}{2}\left\|\sqrt{P_p}\left(g_p(x)-g_p(x')\right)\right\|_F^2 \leq -\kappa_p\left(V_p(x,x')\right)^{\frac{2}{q}}, \quad (8)$$

for all x, x', z in $\mathbb{R}^n$, and for some constant $\kappa_p \in \mathbb{R}^+$. Then V_p is a δ-GAS-M$_q$ Lyapunov function for Σ_p.

The next result provides a condition that is equivalent to (7) or to (8) for affine stochastic subsystems Σ_p (that is, for subsystems with affine drift and linear diffusion terms) in the form of a linear matrix inequality (LMI), which can be easily solved numerically.

Corollary 1. *Consider a stochastic subsystem Σ_p, where for any $x \in \mathbb{R}^n$, $f_p(x) := A_p x + b_p$ for some $A_p \in \mathbb{R}^{n\times n}, b_p \in \mathbb{R}^n$, and $g_p(x) := \left[\sigma_{1,p}x \;\; \sigma_{2,p}x \;\; \ldots \;\; \sigma_{\widehat{q},p}x\right]$ for some $\sigma_{i,p} \in \mathbb{R}^{n\times n}$, where $i = 1,\ldots,\widehat{q}$. Then, function V_p in* (6) *is a δ-GAS-M$_q$ Lyapunov function for Σ_p if there exists a positive constant $\widehat{\kappa}_p \in \mathbb{R}^+$ satisfying the following LMI:*

$$P_pA_p + A_p^T P_p + \sum_{i=1}^{\widehat{q}} \sigma_{i,p}^T P_p \sigma_{i,p} \prec -\widehat{\kappa}_p P_p. \quad (9)$$

Notice that Corollary 1 allows obtaining tighter upper bounds for the inequalities (3) and (4) for any $p \in \mathsf{P}$, by selecting appropriate matrices P_p satisfying the LMI in (9).

4 Symbolic Models and Approximate Equivalence Relations

We employ the notion of *system* [21] to provide (in Sec. 5) an alternative description of stochastic switched systems that can be later directly related to their symbolic models.

Definition 5. *A system S is a tuple $S = (X, X_0, U, \longrightarrow, Y, H)$, where X is a set of states, $X_0 \subseteq X$ is a set of initial states, U is a set of inputs, $\longrightarrow \subseteq X \times U \times X$ is a transition relation, Y is a set of outputs, and $H : X \rightarrow Y$ is an output map.*

We write $x \xrightarrow{u} x'$ if $(x, u, x') \in \longrightarrow$. If $x \xrightarrow{u} x'$, we call state x' a u-successor, or simply a successor, of state x. For technical reasons, we assume that for each $x \in X$, there is some u-successor of x, for some $u \in U$ – let us remark that this is always the case for the considered systems later in this paper. A system S is said to be

- *metric*, if the output set Y is equipped with a metric $\mathbf{d} : Y \times Y \rightarrow \mathbb{R}_0^+$;
- *finite*, if X is a finite set;
- *deterministic*, if for any state $x \in X$ and any input u, there exists at most one u-successor.

For a system $S = (X, X_0, U, \longrightarrow, Y, H)$ and given any state $x_0 \in X_0$, a finite state run generated from x_0 is a finite sequence of transitions:

$$x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-2}} x_{n-1} \xrightarrow{u_{n-1}} x_n, \tag{10}$$

such that $x_i \xrightarrow{u_i} x_{i+1}$ for all $0 \leq i < n$. A finite state run can be trivially extended to an infinite state run as well. A finite output run is a sequence $\{y_0, y_1, \ldots, y_n\}$ such that there exists a finite state run of the form (10) with $y_i = H(x_i)$, for $i = 1, \ldots, n$. A finite output run can also be directly extended to an infinite output run as well.

Now, we recall the notion of approximate (bi)simulation relation, introduced in [8], which is useful when analyzing or synthesizing controllers for deterministic systems.

Definition 6. *Let $S_a = (X_a, X_{a0}, U_a, \underset{a}{\longrightarrow}, Y_a, H_a)$ and $S_b = (X_b, X_{b0}, U_b, \underset{b}{\longrightarrow}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $\mathbf{d}$. For $\varepsilon \in \mathbb{R}_0^+$, a relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate simulation relation from S_a to S_b if the following three conditions are satisfied:*

(i) for every $x_{a0} \in X_{a0}$, there exists $x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$;
(ii) for every $(x_a, x_b) \in R$ we have $\mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
(iii) for every $(x_a, x_b) \in R$ we have that $x_a \underset{a}{\xrightarrow{u_a}} x'_a$ in S_a implies the existence of $x_b \underset{b}{\xrightarrow{u_b}} x'_b$ in S_b satisfying $(x'_a, x'_b) \in R$.

A relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate bisimulation relation between S_a and S_b if R is an ε-approximate simulation relation from S_a to S_b and R^{-1} is an ε-approximate simulation relation from S_b to S_a.

System S_a is ε-approximately simulated by S_b, or S_b ε-approximately simulates S_a, denoted by $S_a \preceq_{\mathcal{S}}^{\varepsilon} S_b$, if there exists an ε-approximate simulation relation from S_a to S_b. System S_a is ε-approximate bisimilar to S_b, denoted by $S_a \cong_{\mathcal{S}}^{\varepsilon} S_b$, if there exists an ε-approximate bisimulation relation between S_a and S_b.

Note that when $\varepsilon = 0$, the condition (ii) in the above definition is changed to $(x_a, x_b) \in R$ if and only if $H_a(x_a) = H_b(x_b)$, and R becomes an exact simulation relation, as introduced in [16]. Similarly, when $\varepsilon = 0$ and whenever applicable, R translates into an exact bisimulation relation.

5 Symbolic Models for Stochastic Switched Systems

This section contains the main contributions of this work. We show that for any stochastic switched system Σ (resp. Σ_{τ_d} as in Theorem 3), admitting a common (resp. multiple) δ-GAS-M$_q$ Lyapunov function(s), and for any precision level $\varepsilon \in \mathbb{R}^+$, we can construct a finite system that is ε-approximate bisimilar to Σ (resp. Σ_{τ_d}). In order to do so, we use systems as an abstract representation of stochastic switched systems, capturing all the information contained in them. More precisely, given a stochastic switched system $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$, we define an associated metric system $S(\Sigma) = (X, X_0, U, \longrightarrow, Y, H)$, where:

- X is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- X_0 is the set of all $\mathbb{R}^n$-valued random variables that are measurable over the trivial sigma-algebra $\mathcal{F}_0$, i.e., the system starts from a non-probabilistic initial condition, which is equivalently a random variable with a Dirac probability distribution;
- $U = \mathsf{P} \times \mathbb{R}^+$;
- $x \xrightarrow{p,\tau} x'$ if x and x' are measurable in $\mathcal{F}_t$ and $\mathcal{F}_{t+\tau}$, respectively, for some $t \in \mathbb{R}_0^+$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(t) = x$ and $\xi_{xp}(\tau) = x'$ $\mathbb{P}$-a.s.;
- Y is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $H = 1_X$.

We assume that the output set Y is equipped with the natural metric $\mathbf{d}(y, y') = \left(\mathbb{E}\left[\|y - y'\|^q\right]\right)^{\frac{1}{q}}$, for any $y, y' \in Y$ and some $q \geq 1$. Let us remark that the set of states of $S(\Sigma)$ is uncountable and that $S(\Sigma)$ is a deterministic system in the sense of Definition 5, since (cf. Subsection 2.2) its solution process is uniquely determined.

In subsequent developments, we will work with a sub-system of $S(\Sigma)$ obtained by selecting those transitions of $S(\Sigma)$ describing trajectories of duration τ, where τ is a given sampling time. This can be seen as a time discretization or a sampling of $S(\Sigma)$. This restriction is practically motivated by the fact that the switching in the original model Σ has to be controlled by a digital platform with a given clock period (τ). More precisely, given a stochastic switched system $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$ and a sampling time $\tau \in \mathbb{R}^+$, we define the associated system $S_\tau(\Sigma) = \left(X_\tau, X_{\tau 0}, U_\tau, \underset{\tau}{\longrightarrow}, Y_\tau, H_\tau\right)$, where $X_\tau = X$, $X_{\tau 0} = X_0$, $U_\tau = \mathsf{P}$, $Y_\tau = Y$, $H_\tau = H$, and

- $x_\tau \xrightarrow[\tau]{p} x'_\tau$ if x_τ and x'_τ are measurable, respectively, in $\mathcal{F}_{k\tau}$ and $\mathcal{F}_{(k+1)\tau}$ for some $k \in \mathbb{N}_0$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(k\tau) = x_\tau$ and $\xi_{x_\tau p}(\tau) = x'_\tau$ $\mathbb{P}$-a.s..

Note that a finite state run $x_0 \xrightarrow[\tau]{u_0} x_1 \xrightarrow[\tau]{u_1} \ldots \xrightarrow[\tau]{u_{N-1}} x_N$ of $S_\tau(\Sigma)$, where $u_i \in \mathsf{P}$ and $x_i = \xi_{x_{i-1}u_{i-1}}(\tau)$ for $i = 1, \cdots, N$, captures the trajectory of the stochastic switched system Σ at times $t = 0, \tau, \cdots, N\tau$, started from the non-probabilistic initial condition x_0 and resulting from a switching signal υ obtained by the concatenation of the modes u_i $\big($i.e. $\upsilon(t) = u_{i-1}$ for any $t \in [(i-1)\tau, i\,\tau[\big)$, for $i = 1, \cdots, N$.

Before introducing the symbolic model for the stochastic switched system, we proceed with the next lemma, borrowed from [22], which provides an upper bound on the distance (in the qth moment metric) between the solution processes of Σ_p and the corresponding non-probabilistic system obtained by disregarding the diffusion term (g_p).

Lemma 2. *Consider a stochastic subsystem Σ_p such that $f_p(0_n) = 0_n$ and $g_p(0_n) = 0_{n \times \widehat{q}}$. Suppose there exists a δ-GAS-M$_q$ Lyapunov function V_p for Σ_p such that its Hessian is a positive semidefinite matrix in $\mathbb{R}^{2n \times 2n}$ and $q \geq 2$. Then for any x in a compact set $\mathsf{D} \subset \mathbb{R}^n$ and any $p \in \mathsf{P}$, we have*

$$\mathbb{E}\left[\left\|\xi_{xp}(t)-\overline{\xi}_{xp}(t)\right\|^q\right] \le h_p(g_p, t), \tag{11}$$

where $\overline{\xi}_{xp}$ is the solution of the ordinary differential equation (ODE) $\dot{\overline{\xi}}_{xp} = f_p\left(\overline{\xi}_{xp}\right)$ starting from the initial condition x, and the nonnegative valued function h_p tends to zero as $t \to 0$, $t \to +\infty$, or as $Z \to 0$, where Z is the Lipschitz constant, introduced in Definition 1.

Although the result in [22, Lemma 3.7] is based on the existence of δ-ISS-M$_q$ Lyapunov functions, one can similarly show the result in Lemma 2 by using δ-GAS-M$_q$ Lyapunov functions. In particular, one can compute explicitly function h_p using [22, Equation (9.4)] with slight modifications. Moreover, we refer the interested readers to [22, Lemma 3.9 and Corollary 3.10], providing explicit forms of the function h_p for (affine) stochastic subsystems Σ_p admitting a δ-GAS-M$_q$ Lyapunov function V_p as in (6), where $q \in \{1, 2\}$. Note that one does not require the condition $f_p(0_n) = 0_n$ for affine subsystems Σ_p. For later use, we introduce function $h(G, t) = \max\{h_1(g_1, t), \cdots, h_m(g_m, t)\}$ for all $t \in \mathbb{R}_0^+$.

In order to show the main results, we raise the following supplementary assumption on the δ-GAS-M$_q$ Lyapunov functions V_p: for all $p \in \mathsf{P}$, there exists a $\mathcal{K}_\infty$ and concave function $\widehat{\gamma}_p$ such that

$$|V_p(x, y) - V_p(x, z)| \le \widehat{\gamma}_p\left(\|y - z\|\right), \tag{12}$$

for any $x, y, z \in \mathbb{R}^n$. This assumption is not restrictive, provided the function V_p is limited to a compact subset of $\mathbb{R}^n \times \mathbb{R}^n$. For all $x, y, z \in \mathsf{D}$, where D is a compact subset of $\mathbb{R}^n$, by applying the mean value theorem to the function $y \to V_p(x, y)$, one gets $|V_p(x, y) - V_p(x, z)| \le \widehat{\gamma}_p\left(\|y - z\|\right)$, where $\widehat{\gamma}_p(r) = \left(\max_{(x,y)\in\mathsf{D}\setminus\Delta}\left\|\frac{\partial V_p(x,y)}{\partial y}\right\|\right) r$. In particular, for the δ-GAS-M$_1$ Lyapunov function V_p defined in (6), we obtain $\widehat{\gamma}_p(r) = \frac{\lambda_{\max}(P_p)}{\sqrt{\lambda_{\min}(P_p)}} r$ [21, Proposition 10.5]. For later use, let us define the $\mathcal{K}_\infty$ function $\widehat{\gamma}$ such that $\widehat{\gamma} = \max\{\widehat{\gamma}_1, \cdots, \widehat{\gamma}_m\}$. (Note that, for the case of a common Lyapunov function, we have: $\widehat{\gamma} = \widehat{\gamma}_1 = \cdots = \widehat{\gamma}_m$.) We proceed presenting the main results of this work.

5.1 Common Lyapunov Function

We first show a result based on the existence of a common δ-GAS-M$_q$ Lyapunov function for subsystems $\Sigma_1, \cdots, \Sigma_m$. Consider a stochastic switched system $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$ and a pair $\mathsf{q} = (\tau, \eta)$ of quantization parameters, where τ is the sampling time and η is the state space quantization. Given Σ and q, consider the following system: $S_\mathsf{q}(\Sigma) = (X_\mathsf{q}, X_{\mathsf{q}0}, U_\mathsf{q}, \underset{\mathsf{q}}{\longrightarrow}, Y_\mathsf{q}, H_\mathsf{q})$, where $X_\mathsf{q} = [\mathbb{R}^n]_\eta$, $X_{\mathsf{q}0} = [\mathbb{R}^n]_\eta$, $U_\mathsf{q} = \mathsf{P}$, and

- $x_\mathsf{q} \overset{p}{\underset{\mathsf{q}}{\longrightarrow}} x'_\mathsf{q}$ if there exists a $x'_\mathsf{q} \in X_\mathsf{q}$ such that $\left\|\overline{\xi}_{x_\mathsf{q}p}(\tau) - x'_\mathsf{q}\right\| \le \eta$, where $\dot{\overline{\xi}}_{x_\mathsf{q}p} = f_p\left(\overline{\xi}_{x_\mathsf{q}p}\right)$;

– Y_{q} is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
– $H_{\mathsf{q}} = \imath : X_{\mathsf{q}} \hookrightarrow Y_{\mathsf{q}}$.

In order to relate models, the output set Y_{q} is taken to be that of the stochastic switched system $S_\tau(\Sigma)$. Therefore, in the definition of H_{q}, the inclusion map $\imath$ is meant, with a slight abuse of notation, as a mapping from a grid point to a random variable with a Dirac probability distribution centered at the grid point. There is no loss of generality to alternatively assume that $Y_{\mathsf{q}} = X_{\mathsf{q}}$ and $H_{\mathsf{q}} = 1_{X_{\mathsf{q}}}$.

The transition relation of $S_{\mathsf{q}}(\Sigma)$ is well defined in the sense that for every $x_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ and every $p \in \mathsf{P}$ there always exists $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ such that $x_{\mathsf{q}} \overset{p}{\underset{\mathsf{q}}{\longrightarrow}} x'_{\mathsf{q}}$. This can be seen since by definition of $[\mathbb{R}^n]_\eta$, for any $\widehat{x} \in \mathbb{R}^n$ there always exists a state $\widehat{x}' \in [\mathbb{R}^n]_\eta$ such that $\|\widehat{x} - \widehat{x}'\| \leq \eta$. Hence, for $\overline{\xi}_{x_{\mathsf{q}}p}(\tau)$ there always exists a state $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ satisfying $\left\|\overline{\xi}_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\| \leq \eta$.

We can now present one of the main results of the paper, which relates the existence of a common δ-GAS-M$_q$ Lyapunov function for the subsystems $\Sigma_1, \cdots, \Sigma_m$ to the construction of a finite symbolic model that is approximately bisimilar to the original system.

Theorem 4. *Let $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$ be a stochastic switched system admitting a common δ-GAS-M$_q$ Lyapunov function V, of the form of (6) or the one explained in Lemma 2, for subsystems $\Sigma_1, \cdots, \Sigma_m$. For any $\varepsilon \in \mathbb{R}^+$, and any double $\mathsf{q} = (\tau, \eta)$ of quantization parameters satisfying*

$$\overline{\alpha}\left(\eta^q\right) \leq \underline{\alpha}\left(\varepsilon^q\right), \tag{13}$$

$$\mathsf{e}^{-\kappa\tau}\underline{\alpha}\left(\varepsilon^q\right) + \widehat{\gamma}\left(\left(h(G,\tau)\right)^{\frac{1}{q}} + \eta\right) \leq \underline{\alpha}\left(\varepsilon^q\right), \tag{14}$$

we have that $S_{\mathsf{q}}(\Sigma) \cong_{\mathcal{S}}^{\varepsilon} S_\tau(\Sigma)$.

It can be readily seen that when we are interested in the dynamics of Σ, initialized on a compact $\mathsf{D} \subset \mathbb{R}^n$ of the form of finite union of boxes and for a given precision ε, there always exist a sufficiently large value of τ and a small value of η such that $\eta \leq span(\mathsf{D})$ and the conditions in (13) and (14) are satisfied. For a given fixed sampling time τ, the precision ε is lower bounded by:

$$\varepsilon > \left(\underline{\alpha}^{-1}\left(\frac{\widehat{\gamma}\left(\left(h(G,\tau)\right)^{\frac{1}{q}}\right)}{1-\mathsf{e}^{-\kappa\tau}}\right)\right)^{\frac{1}{q}}. \tag{15}$$

One can easily verify that the lower bound on ε in (15) goes to zero as τ goes to infinity or as $Z \rightarrow 0$, where Z is the Lipschitz constant, introduced in Definition 1. Furthermore, one can try to minimize the lower bound on ε in (15) by appropriately choosing a common δ-GAS-M$_q$ Lyapunov function V.

Note that the results in [9, Theorem 4.1] for non-probabilistic models are fully recovered by the statement in Theorem 4 if the stochastic switched system Σ is not affected by any noise, implying that $h_p(g_p, t)$ is identically zero for all $p \in \mathsf{P}$, and that the δ-GAS-M$_q$ common Lyapunov function simply reduces to being the δ-GAS one.

5.2 Multiple Lyapunov Functions

If a common δ-GAS-M$_q$ Lyapunov function does not exist, one can still attempt computing approximately bisimilar symbolic models by seeking mode-dependent Lyapunov functions and by restricting the set of switching signals using a dwell time τ_d. For simplicity and without loss of generality, we assume that τ_d is an integer multiple of τ, i.e. there exists $N \in \mathbb{N}$ such that $\tau_d = N\tau$.

Given a stochastic switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}_{\tau_d}, F, G)$ and a sampling time $\tau \in \mathbb{R}^+$, we define the system $S_\tau(\Sigma_{\tau_d}) = (X_\tau, X_{\tau 0}, U_\tau, \underset{\tau}{\longrightarrow}, Y_\tau, H_\tau)$, where:

- $X_\tau = \mathcal{X} \times \mathsf{P} \times \{1, \ldots, N-1\}$, where $\mathcal{X}$ is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $X_{\tau 0} = \mathcal{X}_0 \times \mathsf{P} \times \{0\}$, where $\mathcal{X}_0$ is the set of all $\mathbb{R}^n$-valued random variables that are measurable with respect to the trivial sigma-algebra $\mathcal{F}_0$, i.e., the stochastic switched system starts from a non-probabilistic initial condition;
- $U_\tau = \mathsf{P}$;
- $(x_\tau, p, i) \overset{p}{\underset{\tau}{\longrightarrow}} (x'_\tau, p', i')$ if x_τ and x'_τ are measurable, respectively, in $\mathcal{F}_{k\tau}$ and $\mathcal{F}_{(k+1)\tau}$ for some $k \in \mathbb{N}_0$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(k\tau) = x_\tau$ and $\xi_{x_\tau p}(\tau) = x'_\tau$ $\mathbb{P}$-a.s. and one of the following holds:
  - $i < N-1$, $p' = p$, and $i' = i+1$: switching is not allowed because the time elapsed since the latest switch is strictly smaller than the dwell time;
  - $i = N-1$, $p' = p$, and $i' = N-1$: switching is allowed but no mode switch occurs;
  - $i = N-1$, $p' \neq p$, and $i' = 0$: switching is allowed and a mode switch occurs.
- $Y_\tau = \mathcal{X}$ is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- H_τ is the map taking $(x_\tau, p, i) \in \mathcal{X} \times \mathsf{P} \times \{1, \cdots, N-1\}$ to $x_\tau \in \mathcal{X}$.

We assume that the output set Y_τ is equipped with the natural metric $\mathbf{d}(y, y') = \left(\mathbb{E}\left[\|y - y'\|^q\right]\right)^{\frac{1}{q}}$, for any $y, y' \in Y_\tau$ and some $q \geq 1$. One can readily verify that the (in)finite output runs of $S_\tau(\Sigma_{\tau_d})$ are the (in)finite output runs of $S_\tau(\Sigma)$ corresponding to switching signals with dwell time $\tau_d = N\tau$.

Consider a stochastic switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}_{\tau_d}, F, G)$ and a pair $\mathsf{q} = (\tau, \eta)$ of quantization parameters, where τ is the sampling time and η is the state space quantization. Given Σ_{τ_d} and q, consider the following system: $S_\mathsf{q}(\Sigma_{\tau_d}) = (X_\mathsf{q}, X_{\mathsf{q}0}, U_\mathsf{q}, \underset{\mathsf{q}}{\longrightarrow}, Y_\mathsf{q}, H_\mathsf{q})$, where $X_\mathsf{q} = [\mathbb{R}^n]_\eta \times \mathsf{P} \times \{0, \cdots, N-1\}$, $X_{\mathsf{q}0} = [\mathbb{R}^n]_\eta \times \mathsf{P} \times \{0\}$, $U_\mathsf{q} = \mathsf{P}$, and

- $(x_\mathsf{q}, p, i) \overset{p}{\underset{\mathsf{q}}{\longrightarrow}} (x'_\mathsf{q}, p', i')$ if there exists a $x'_\mathsf{q} \in X_\mathsf{q}$ such that $\left\|\overline{\xi}_{x_\mathsf{q} p}(\tau) - x'_\mathsf{q}\right\| \leq \eta$, where $\dot{\overline{\xi}}_{x_\mathsf{q} p} = f_p\left(\overline{\xi}_{x_\mathsf{q} p}\right)$ and one of the following holds:
  - $i < N-1$, $p' = p$, and $i' = i+1$;
  - $i = N-1$, $p' = p$, and $i' = N-1$;
  - $i = N-1$, $p' \neq p$, and $i' = 0$.
- $Y_\mathsf{q} = \mathcal{X}$ is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;

– H_{q} is the map taking $(x_{\mathsf{q}}, p, i) \in [\mathbb{R}^n]_\eta \times \mathsf{P} \times \{1, \cdots, N-1\}$ to a random variable with a Dirac probability distribution centered at x_{q}.

Similar to what we showed in the case of a common Lyapunov function, the transition relation of $S_{\mathsf{q}}(\Sigma_{\tau_d})$ is well defined in the sense that for every $(x_{\mathsf{q}}, p, i) \in [\mathbb{R}^n]_\eta \times \mathsf{P} \times \{0, \cdots, N-1\}$ there always exists $\left(x'_{\mathsf{q}}, p', i'\right) \in [\mathbb{R}^n]_\eta \times \mathsf{P} \times \{0, \cdots, N-1\}$ such that $(x_{\mathsf{q}}, p, i) \xrightarrow[\mathsf{q}]{p} \left(x'_{\mathsf{q}}, p', i'\right)$.

We present the second main result of the paper, which relates the existence of multiple Lyapunov functions for a stochastic switched system to that of a symbolic model.

Theorem 5. *Consider $\tau_d \in \mathbb{R}_0^+$, and a stochastic switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}_{\tau_d}, F, G)$ such that $\tau_d = N\tau$, for some $N \in \mathbb{N}$. Let us assume that for any $p \in \mathsf{P}$, there exists a δ-GAS-M$_q$ Lyapunov function V_p, of the form in (6) or as the one in Lemma 2, for subsystem $\Sigma_{\tau_d,p}$. Moreover, assume that (5) holds for some $\mu \geq 1$. If $\tau_d > \log \mu/\kappa$, for any $\varepsilon \in \mathbb{R}^+$, and any pair $\mathsf{q} = (\tau, \eta)$ of quantization parameters satisfying*

$$\overline{\alpha}\left(\eta^q\right) \leq \underline{\alpha}\left(\varepsilon^q\right), \tag{16}$$

$$\widehat{\gamma}\left(\left(h(G,\tau)\right)^{\frac{1}{q}} + \eta\right) \leq \frac{\frac{1}{\mu} - \mathsf{e}^{-\kappa\tau_d}}{1 - \mathsf{e}^{-\kappa\tau_d}}\left(1 - \mathsf{e}^{-\kappa\tau}\right)\underline{\alpha}\left(\varepsilon^q\right), \tag{17}$$

we have that $S_{\mathsf{q}}(\Sigma_{\tau_d}) \cong^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma_{\tau_d})$.

It can be readily seen that when we are interested in the dynamics of Σ_{τ_d}, initialized on a compact $\mathsf{D} \subset \mathbb{R}^n$ of the form of finite union of boxes, and for a precision ε, there always exist sufficiently large value of τ and small value of η such that $\eta \leq span(\mathsf{D})$ and the conditions in (16) and (17) are satisfied. For a given fixed sampling time τ, the precision ε is lower bounded by:

$$\varepsilon \geq \left(\underline{\alpha}^{-1}\left(\frac{\widehat{\gamma}\left(\left(h(G,\tau)\right)^{\frac{1}{q}}\right)}{1 - \mathsf{e}^{-\kappa\tau}} \cdot \frac{1 - \mathsf{e}^{-\kappa\tau_d}}{\frac{1}{\mu} - \mathsf{e}^{-\kappa\tau_d}}\right)\right)^{\frac{1}{q}}. \tag{18}$$

The properties of the bound in (18) are analogous to those of the case of a common Lyapunov function. Similarly, Theorem 5 subsumes [9, Theorem 4.2] over non-probabilistic models.

6 Case Study

We experimentally demonstrate the effectiveness of the results. In the example below, the computation of the abstraction $S_{\mathsf{q}}(\Sigma)$ has been performed via the software tool Pessoa [15] on a laptop with CPU 2GHz Intel Core i7. Controller enforcing the specification was found by using standard algorithms from game theory [14], as implemented in Pessoa. The terms $W_t^i, i = 1, 2$, denote the standard Brownian motion.

The stochastic switched system Σ is a simple thermal model of a two-room building, borrowed from [7], affected by noise and described by the following stochastic differential equations:

$$\begin{cases} \mathrm{d}\,\xi_1 = (\alpha_{21}\,(\xi_2-\xi_1)+\alpha_{e1}\,(T_e-\xi_1)+\alpha_f\,(T_f-\xi_1)\,(p-1))\,\mathrm{d}\,t+\sigma_1\xi_1\,\mathrm{d}\,W_t^1, \\ \mathrm{d}\,\xi_2 = (\alpha_{12}\,(\xi_1-\xi_2)+\alpha_{e2}\,(T_e-\xi_2))\,\mathrm{d}\,t+\sigma_2\xi_2\,\mathrm{d}\,W_t^2, \end{cases} \tag{19}$$

where ξ_1 and ξ_2 denote the temperature in each room, $T_e = 10$ (degrees Celsius) is the external temperature and $T_f = 50$ is the temperature of a heater that can be switched off ($p = 1$) or on ($p = 2$): these two operations correspond to the modes P of the model, whereas the state space is $\mathbb{R}^2$. The drifts f_p and diffusion terms g_p, $p = 1, 2$, can be simply written out of (19) and are affine. The parameters of the drifts are chosen based on the ones in [7] as follows: $\alpha_{21} = \alpha_{12} = 5\times10^{-2}$, $\alpha_{e1} = 5\times10^{-3}$, $\alpha_{e2} = 3.3\times10^{-3}$, and $\alpha_f = 8.3\times10^{-3}$. We work on the subset $\mathsf{D} = [20,\ 22]\times[20,\ 22] \subset \mathbb{R}^2$ of the state space of Σ. Within D one can conservatively overapproximate the multiplicative noises in (19) as additive noises with variance between 0.02 and 0.022.

It can be readily verified that the function $V(x_1, x_2) = \sqrt{(x_1-x_2)^T(x_1-x_2)}$ is a common δ-GAS-M$_1$ Lyapunov function for Σ, satisfying the LMI condition (9) with $P_p = I_2$, and $\widehat{\kappa}_p = 0.0083$, for $p \in \{1, 2\}$.

For a given sampling time $\tau = 20$ time units, using inequality (15), the precision ε is lower bounded by the quantity 1.09. While one can reduce this lower bound by increasing the sampling time, as discussed later the empirical bound computed in the experiments is significantly lower than the theoretical bound $\varepsilon = 1.09$. For a selected precision $\varepsilon = 1.1$, the discretization parameter η of $S_{\mathsf{q}}(\Sigma)$, computed from Theorem 4, equals to 0.003. This has lead to a symbolic system $S_{\mathsf{q}}(\Sigma)$ with a resulting number of states equal to 895122. The CPU time employed to compute the abstraction amounted to 506.32 seconds.

Consider the objective to design a controller (switching policy) forcing the first moment of the trajectories of Σ to stay within D. This objective can be encoded via the LTL specification $\Box\mathsf{D}$. Furthermore, to add an additional discrete component to the problem, we assume that the heater has to stay in the off mode ($p = 1$) at most one time slot in every two slots. A time slot is an interval of the form $[k\tau, (k+1)\tau[$, with $k \in \mathbb{N}$ and where τ is the sampling time. Possible switching policies are for instance:

$$|12|12|12|12|12|12|12|\cdots,\ |21|21|21|21|21|21|21|\cdots,\ |12|21|22|12|12|21|22|\cdots,$$

where 2 denotes a slot where the heater is on ($p = 2$) and 1 denotes a slot where heater is off ($p = 1$). This constraint on the switching policies can be represented by the finite system (labeled automaton) in Figure 1, where the allowed initial states are distinguished as targets of a sourceless arrow. The CPU time for synthesizing the controller amounted to 21.14 seconds. In Figure 2, we show several realizations of closed-loop trajectory $\xi_{x_0 \upsilon}$ stemming from initial condition $x_0 = (21, 21)$ (left panel), as well as the corresponding evolution of switching signal υ (right panel), where the finite system is initialized from state q_1. Furthermore, in Figure 2 (middle panels), we show the average value over 100 experiments of the distance in time of the solution process $\xi_{x_0 \upsilon}$ to the set D, namely $\|\xi_{x_0 \upsilon}(t)\|_{\mathsf{D}}$, where the point-to-set distance is defined as $\|x\|_{\mathsf{D}} = \inf_{d\in\mathsf{D}}\|x-d\|$. Notice that the average distance is significantly lower than the precision $\varepsilon = 1.1$, as expected since the conditions based on Lyapunov functions

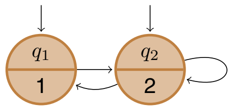

Fig. 1. Finite system describing the constraint over the switching policies. The lower part of the states are labeled with the outputs (2 and 1) denoting whether heater is on ($p = 2$) or off ($p = 1$).

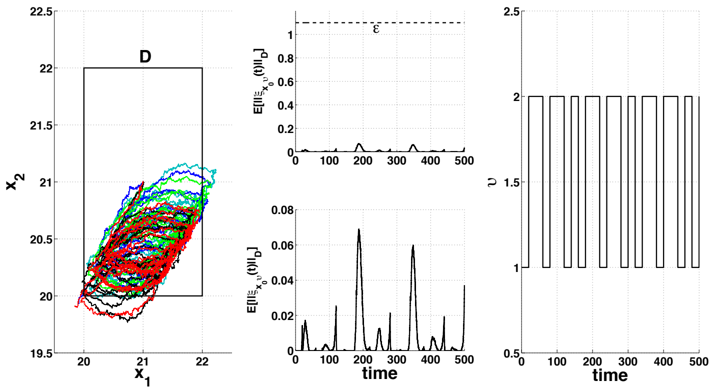

Fig. 2. Several realizations of the closed-loop trajectory $\xi_{x_0 \upsilon}$ with initial condition $x_0 = (21, 21)$ (left panel). Average values (over 100 experiments) of the distance of the solution process $\xi_{x_0 \upsilon}$ to the set D, in different vertical scales (middle panels). Evolution of the synthesized switching signal υ (right panel), where the finite system initialized from state q_1.

can lead to conservative bounds. (As discussed in Corollary 1, bounds can be improved by seeking optimized Lyapunov functions.)

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