Matching the Linet–Tian Spacetime with Conformally Flat Cylindrically Symmetric Sources

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Abstract We derive conformally flat cylindrically symmetric solutions for spacetimes with a cosmological constant and investigate the matching problem of these solutions with the exterior Linet–Tian spacetime.

1 Introduction

In General Relativity, cylindrical solutions have been used to study various fields such as cosmic strings, exact models of rotation matched to different sources and models for extragalactic jets and gravitational radiation. The generalization of the Levi–Civita spacetime to include a nonzero cosmological constant Λ was obtained by Linet [1] and Tian [2]. It was shown by da Silva et al. [3] and Griffiths and

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Podolsky [4] that it changes the spacetime properties dramatically. The Linet–Tian (LT) solution has also been used to describe cosmic strings and, in [5], static cylindrical shell sources have been found for the LT spacetime with negative cosmological constant. Considering the extensive interest in cylindrically symmetric solutions it is worthwhile to analyse some further properties of LT spacetimes.

In this note, we summarize the results of [6], which show that it is possible to match static cylindrically symmetric conformally flat solutions of the Einstein field equations with a cosmological constant Λ with the exterior Linet–Tian spacetime if $\Lambda > 0$.

2 Static Cylindrically Symmetric Anisotropic Sources with $\Lambda \neq 0$

Consider static cylindrically symmetric anisotropic matter bounded by a cylindrical surface S and with energy momentum tensor given by

$$T_{ab} = (\mu + P_r)V_a V_b + P_r g_{ab} + (P_z - P_r)S_a S_b + (P_\phi - P_r)K_a K_b,$$

where μ is the energy density, P_r, P_z and P_ϕ are the principal stresses and V_a, S_a and K_a satisfy $V^a V_a = -1$, $S^a S_a = K^a K_a = 1$, $V^a S_a = V^a K_a = S^a K_a = 0$. For the interior to S the static cylindrically symmetric metric can be written as

$$ds^2 = -A^2 dt^2 + B^2(dr^2 + dz^2) + C^2 d\phi^2, \tag{1}$$

where A, B and C are functions of r. The non-zero components of the Einstein Field Equations $G_{ab} = T_{ab} - \Lambda g_{ab} \equiv \bar{T}_{ab}$ are

$$G_{00} = -\left(\frac{A}{B}\right)^2 \left[\left(\frac{B'}{B}\right)' + \frac{C''}{C}\right] = (\mu + \Lambda)A^2 = \bar{\mu}A^2,$$

$$G_{11} = \frac{A'C'}{AC} + \left(\frac{A'}{A} + \frac{C'}{C}\right)\frac{B'}{B} = (P_r - \Lambda)B^2 = \bar{P}_r B^2,$$

$$G_{22} = \frac{A''}{A} + \frac{C''}{C} + \frac{A'}{A}\frac{C'}{C} - \left(\frac{A'}{A} + \frac{C'}{C}\right)\frac{B'}{B} = (P_z - \Lambda)B^2 = \bar{P}_z B^2,$$

$$G_{33} = \left(\frac{C}{B}\right)^2 \left[\frac{A''}{A} + \left(\frac{B'}{B}\right)'\right] = (P_\phi - \Lambda)C^2 = \bar{P}_\phi C^2,$$

where the primes stand for differentiation with respect to r. The regularity conditions at the axis are $A'(0) = B'(0) = C''(0) = C(0) = 0$, $B(0) = C'(0) = 1$ (see [7]).

3 Conformally Flat Solutions

For a conformally flat interior spacetime, all Weyl tensor components vanish and one obtains the following non-linear second order ordinary differential equations for h and S (see [8]):

$$S' + S^2 - \frac{2h'}{h}S + \frac{h''}{h} = 0, \quad S' + S^2 + \frac{h'}{h}S - \frac{2h''}{h} = 0,$$

where $S = \frac{A'}{A} - \frac{B'}{B}$, $h = \frac{C}{B}$.

Integrating these equations and using the regularity conditions gives $h = a_1 \sinh(a_2 r)$ and $A = a_3 \cosh(a_2 r) B$; where a_1, a_2 and a_3 are non-zero integration constants,.

Considering the case $\bar{P}_r = \bar{P}_\phi$, it follows that the metric functions A, B and C in (1) are (see [6])

$$A = \cosh(a_2 r)B, \quad B = \frac{1}{a_4[\cosh(a_2 r) - 1] + 1}, \quad C = \frac{\sinh(a_2 r)}{a_2} B,$$

where a_2 and $a_4 \neq 0$ are constants. The density and pressures are given by

$$\begin{aligned}
\bar{\mu} &= 2\, a_2^2\, a_4\, [(1 - a_4)\cosh(a_2 r) + a_4 + 1] - a_2^2 \\
\bar{P}_r &= 2a_2^2 a_4[(a_4 - 1)\tanh(a_2 r)\sinh(a_2 r) - 1] + a_2^2 \\
\bar{P}_z &= 2a_2^2 a_4 \left[\frac{1 - a_4}{\cosh(a_2 r)} + a_4 - 3\right] + 3a_2^2.
\end{aligned}$$

4 Matching to an Exterior

Consider the exterior Linet–Tian spacetime (M^+, g^+) whose metric is given by

$$\begin{aligned}
ds^{2+} = -a^2 Q^{2/3} P^{-2(1-8\sigma+4\sigma^2)/3\Sigma} dt^2 + d\rho^2 + b^2 Q^{2/3} P^{-2(1+4\sigma-8\sigma^2)/3\Sigma} dz^2 \\
+ c^2 Q^{2/3} P^{4(1-2\sigma-2\sigma^2)/3\Sigma} d\phi^2,
\end{aligned}$$

where $\Sigma = 1 - 2\sigma + 4\sigma^2$, and for $\Lambda > 0$, $Q(\rho) = \frac{1}{\sqrt{3\Lambda}} \sin(2R)$, $P(\rho) = \frac{2}{\sqrt{3\Lambda}} \tan R$, with $R = \frac{\sqrt{3\Lambda}}{2}\rho$. Matching this metric to the interior solution presented in the previous section, one obtains from the equality of the first fundamental forms:

$$\frac{\cosh(a_2 r)}{a_4[\cosh(a_2 r) - 1] + 1} \stackrel{S}{=} a\, Q^{1/3} P^{-(1-8\sigma+4\sigma^2)/3\Sigma}, \tag{2}$$

$$\frac{1}{a_4[\cosh(a_2 r) - 1] + 1} \stackrel{S}{=} b\, Q^{1/3} P^{-(1+4\sigma-8\sigma^2)/3\Sigma}, \tag{3}$$

$$\frac{\sinh(a_2 r)}{a_2[a_4[\cosh(a_2 r) - 1] + 1]} \stackrel{S}{=} c\, Q^{1/3} P^{2(1-2\sigma-2\sigma^2)/3\Sigma}, \tag{4}$$

and from the equality of the second fundamental forms:

$$a_2(1 - a_4)\tanh(a_2 r) \stackrel{S}{=} \sqrt{\frac{\Lambda}{3}}\, \frac{3\sigma - \Sigma \sin^2 R}{\Sigma \sin R \cos R}, \tag{5}$$

$$a_2 a_4 \sinh(a_2 r) \stackrel{S}{=} \sqrt{\frac{\Lambda}{3}}\, \frac{3\sigma(1-2\sigma) + \Sigma \sin^2 R}{\Sigma \sin R \cos R}, \tag{6}$$

$$\frac{a_2[\cosh(a_2 r)(1 - a_4) + a_4]}{\sinh(a_2 r)} \stackrel{S}{=} \sqrt{\frac{\Lambda}{3}}\, \frac{3(1-2\sigma) - 2\Sigma \sin^2 R}{2\Sigma \sin R \cos R}, \tag{7}$$

where $\stackrel{S}{=}$ denotes equality on the matching hypersurface only. From these equations one obtains

$$\sin^2 R \stackrel{S}{=} \frac{3\sigma}{\Sigma}\left[1 + \frac{2(a_4 - 1)(1-\sigma)\sqrt{1-4\sigma}}{a_4\sqrt{1-4\sigma^2} - (a_4 - 1)\sqrt{1-4\sigma}}\right], \tag{8}$$

$$\sinh^2(a_2 r) \stackrel{S}{=} \frac{4\sigma(1-\sigma)}{1-4\sigma} \tag{9}$$

and

$$a_2^2 \stackrel{S}{=} \Lambda\left[2a_4 - 1 - 2a_4(a_4 - 1)\frac{4\sigma(1-\sigma)}{\sqrt{(1-4\sigma)(1-4\sigma^2)}}\right]^{-1}. \tag{10}$$

The inequality $0 \le \sin^2 R_S \le 1$ in (8) and the positivity of the right hand side of (10), for any $0 < \sigma < 1/4$, are satisfied if $1/2 \le a_4 \le 1$.

The parameter a_2 is fixed by (10) whilst ρ_S and r_S are determined from (8) and (9); equations (2)–(4) fix the exterior parameters a, b and c. Thus we conclude that the matching is possible for any $1/2 \le a_4 \le 1$, $0 < \sigma < 1/4$ and $\Lambda > 0$. See [6] for more details.

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