Kasner Solution in Brans–Dicke Theory and Its Corresponding Reduced Cosmology

Seyed M.M. Rasouli

Abstract We present a brief review of the modified Brans–Dicke theory (MBDT) in arbitrary dimensions, whereby the $(N+1)$-dimensional field equations reduce to the N-dimensional (ND) configuration with sources and an effective induced scalar potential. We then investigate a generalized Bianchi type I anisotropic cosmology in $5D$ BD theory that leads to an extended Kasner solution. By employing the original equations of MBDT, we probe the reduced Kasner cosmology on the hypersurface with proceeding the investigations for a few cosmological quantities, explaining their properties for some cosmological models.

1 Dimensional Reduction of Brans–Dicke Theory in Arbitrary Dimensions

The original motivation of the induced-matter theory (IMT) [1] was to achieve the unification of matter and geometry. Recently, the idea of the IMT has been employed for generalizing BD theory, as a fundamental underlying theory, in four [2] and arbitrary dimensions [3]. In the following, we present only a brief review of the latter.

The variation of the $(N+1)D$ BD action in vacuum with respect to metric and BD scalar field, ϕ, give the equations

$$G_{ab}^{(N+1)} = \frac{\omega}{\phi^2}\left[(\nabla_a\phi)(\nabla_b\phi) - \frac{1}{2}\gamma_{ab}(\nabla^c\phi)(\nabla_c\phi)\right] + \frac{1}{\phi}\Big(\nabla_a\nabla_b\phi - \gamma_{ab}\nabla^2\phi\Big), \quad (1)$$

$$\nabla^2\phi = 0, \quad (2)$$

S.M.M. Rasouli (✉)
Departamento de Física, Universidade da Beira Interior, 6200 Covilhã, Portugal
e-mail: mrasouli@ubi.pt

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where the Latin indices run from 0 to N; γ_{ab} is the metric associated to the $(N+1)D$ space-time, $\nabla^2 \equiv \nabla_a \nabla^a$ and ω is a dimensionless parameter. Here, we have chosen $c = 1$.

In the following, we only employ the equations of the MBDT in arbitrary dimensions, which convey relations between the $(N + 1)D$ field equations to the corresponding ones with sources in ND space-time in the context of BD theory [3].

We can find the reduced field equations on the ND hypersurface by employing the BD field Eqs. (1), (2) and a $(N + 1)D$ space-time with a line element

$$dS^2 = \gamma_{ab}(x^c)dx^a dx^b = g_{\mu\nu}(x^\alpha, l)dx^\mu dx^\nu + \epsilon\psi^2 (x^\alpha, l)\, dl^2, \tag{3}$$

where the Greek indices run from zero to $(N - 1)$, l is a non-compact coordinate associated to $(N + 1)$th dimension, the parameter $\epsilon = \pm 1$ allows we to choose the extra dimension to be either time-like or space-like, and ψ is the another scalar field taken as a function of all the coordinates.

Here, we only present some of the reduced equations on the ND hypersurface. These equations will be described in two separated parts with a short interpretation.

1. We can construct the Einstein tensor on the hypersurface as

$$G^{(N)}_{\mu\nu} = \frac{8\pi}{\phi}T^{(\mathrm{BD})}_{\mu\nu} + \frac{\omega}{\phi^2}\left[(\mathscr{D}_\mu\phi)(\mathscr{D}_\nu\phi) - \frac{1}{2}g_{\mu\nu}(\mathscr{D}_\alpha\phi)(\mathscr{D}^\alpha\phi)\right] + \frac{1}{\phi}\left[\mathscr{D}_\mu\mathscr{D}_\nu\phi - g_{\mu\nu}\mathscr{D}^2\phi\right] - g_{\mu\nu}\frac{V(\phi)}{2\phi}, \tag{4}$$

where $\mathscr{D}_\alpha$ is the covariant derivative on ND hypersurface, which is calculated with $g_{\alpha\beta}$, and $\mathscr{D}^2 \equiv \mathscr{D}^\alpha\mathscr{D}_\alpha$.

The above equations correspond to the BD equations, obtained from the standard BD action containing a scalar potential, but here there are some differences which we clarify them in the following

- The quantity introduced by $V(\phi)$ is actually the effective induced scalar potential on the hypersurface which will be determined by a relation in part 2.
- The quantity $T^{(\mathrm{BD})}_{\mu\nu}$, can be interpreted as an induced energy-momentum tensor (EMT) for a BD theory in N-dimensions and it, in turn, contains three components as

$$T^{(\mathrm{BD})}_{\mu\nu} \equiv T^{(\mathrm{I})}_{\mu\nu} + T^{(\phi)}_{\mu\nu} + \frac{1}{16\pi}g_{\mu\nu}V(\phi), \tag{5}$$

where $8\pi/\phi T^{(\mathrm{I})}_{\mu\nu}$ is the same as the induced EMT appearing in IMT, while

$$\frac{8\pi}{\phi}T^{(\phi)}_{\mu\nu} \equiv \frac{\epsilon\phi_{,N}}{2\psi^2\phi}\left[g_{\mu\nu,N} + g_{\mu\nu}\left(\frac{\omega\phi_{,N}}{\phi} - g^{\alpha\beta}g_{\alpha\beta,N}\right)\right], \tag{6}$$

where $A_{,N}$ is the partial derivative of the quantity A with respect to l.

2. The wave equation on the hypersurface is given by

$$\mathscr{D}^2\phi = \frac{8\pi T^{(\mathrm{BD})}}{(N-2)\omega + (N-1)} + \frac{1}{(N-2)\omega + (N-1)}\left[\phi\frac{dV(\phi)}{d\phi} - \frac{N}{2}V(\phi)\right], \quad (7)$$

where

$$\phi\frac{dV(\phi)}{d\phi} \equiv -(N-2)(\omega+1)\left[\frac{(\mathscr{D}_\alpha\psi)(\mathscr{D}^\alpha\phi)}{\psi} + \frac{\epsilon}{\psi^2}\left(\phi_{,NN} - \frac{\psi_{,N}\phi_{,N}}{\psi}\right)\right] \quad (8)$$

$$-\frac{(N-2)\epsilon\omega\phi_{,N}}{2\psi^2}\left[\frac{\phi_{,N}}{\phi} + g^{\mu\nu}g_{\mu\nu,N}\right] + \frac{(N-2)\epsilon\phi}{8\psi^2}\left[g^{\alpha\beta}{}_{,N}g_{\alpha\beta,N} + (g^{\alpha\beta}g_{\alpha\beta,N})^2\right].$$

Actually in this approach, the $(N+1)D$ field equations (1) and (2) split naturally into four sets of equations on every ND hypersurface, in which we only have introduced the two sets (4) and (7). Regarding the geometrical interpretation of the other two sets, we will not discuss them and leave them for next paper in this series.

In the following, we investigate the Kasner solution in BD theory in a $5D$ space-time; then, as an application of the MBDT in cosmology, we present the properties of a the reduced cosmology on the hypersurface.

2 Kasner Solution in Brans–Dicke Theory and Its Corresponding Reduced Cosmology

We start with the generalized Bianchi type I anisotropic model in a $5D$ space-time as

$$dS^2 = -dt^2 + \sum_{i=1}^{3} a_i^2(t,l)dx_i^2 + h^2(t,l)dl^2, \quad (9)$$

where t is the cosmic time, (x_1, x_2, x_3) are the Cartesian coordinates, l is the non-compactified extra dimension, and $a_i(t,l)$, $h(t,l)$ are different cosmological scale factors in each of the four directions. We assume that there is no matter in $5D$ space-time and $\phi = \phi(t,l)$. In addition, based on the usual spatial homogeneity, we solve the field equations (1) and (2) by assuming separation of variables as

$$\phi(t,l) = \phi_0 t^{p_0} l^{s_0}, \qquad a_i(t,l) \propto t^{p_i} l^{s_i}, \qquad h(t,l) = h_o t^{p_4} l^{s_4}, \quad (10)$$

where h_0 and ϕ_0 are constants, and the p_a's and s_a's ($a = 0, 2, 3, 4$) are parameters satisfying field equations. By replacing the *ansatz* (10) into Eq. (2) and using (9), we get five classes of solutions. In the following, we are interested to investigate just the solutions that leads to a generalized Kasner relations in five dimensions. Also, we then apply the MBDT to obtain the corresponding reduced cosmology on a $4D$ hypersurface.

In order to have consistency and ignoring the trivial solutions we set $p_4 \neq 1$ and $s_4 \neq -1$. Also for simplicity, we assume $h_o = 1$ in *ansatz* (10). Hence after a little manipulation, we can obtain the following relations among the generalized Kasner parameters [4]

$$\sum_{a=0}^{4} p_a = 1, \quad (\omega+1)p_0^2 + \sum_{m=1}^{4} p_m^2 = 1, \quad \sum_{\mu=0}^{3} s_\mu = 1 + s_4, \tag{11}$$

$$(\omega+1)s_0^2 + \sum_{i=1}^{3} s_i^2 = (1+s_4)^2, \quad (\omega+1)p_0 s_0 + \sum_{i=1}^{3} p_i s_i = p_4(1+s_4)\,.$$

Equation (11) lead to a few constrains, so that there are only five independent relations among the Kasner parameters, designated as the generalized Kasner relations in $5D$ BD theory.

In what follows, for the sake of brevity, we would like to present a brief review of the results of the reduced Kasner cosmology on the $4D$ hypersurface [4]: the pressure and energy density of the specified induced matter on any $4D$ hypersurface can be derived from (5). These results show that, in general, we cannot consider it as a perfect fluid. By applying (8), (10) and (11), the induced scalar potential is obtained to be either in the power law or in the logarithmic form, in which we only investigate the properties of the former. The properties of a few cosmological quantities as well as physical quantities such as the average scale factor, the mean Hubble parameter, the expansion scalar, the shear scalar and the deceleration parameter have been studied. First, these quantities have been derived in terms of the generalized Kasner parameters. Then, we find that the induced EMT satisfies the barotropic equation of state, where the equation of state parameter, w, is a function of the Kasner parameters. And thus, the evolution of all the quantities has been represented with respect to w, ω and the deceleration parameter, q. We then probe the quantities, in the general case, versus q, t and ω for the stiff fluid and the radiation-dominated universe. We have shown that, for both of the fluids, there is an expanding universe commenced with a big bang, and there is a horizon for each of them. Also, we have shown that the rate of expansion slows down by time. By employing the weak energy condition, the allowed (or the well-behaved) ranges of the deceleration and the BD coupling parameters have been obtained for each of the fluids. The behavior of the quantities, in the very early universe and the very large time show that the models yield empty universes when the cosmic time tends to infinity. However, both of the models, in general, do not approach isotropy for large values of the cosmic time.

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References

1. P. S. Wesson and J. Ponce de Leon, *J. Math. Phys.* **33**, 3883 (1992).
2. J. Ponce de Leon, *JCAP* **03**, 030 (2010).
3. S. M. M. Rasouli, M. Farhoudi and P. V. Moniz "Modified Brans-Dicke Theory in Arbitrary Dimensions", work in progress.
4. S. M. M. Rasouli, M. Farhoudi and H. R. Sepangi, *Class. Quantum Grav.* **28**, 155004 (2011).