On the Bergqvist Approach to the Penrose Inequality

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Abstract The Penrose inequality in terms of the Bondi mass at past null infinity can be approached with a method due to Ludvigsen and Vickers and clarified later on by Bergqvist (Ludvigsen and Vickers, J. Phys. A: Math. Gen. 16:3349–3353, 1983; Bergqvist, Class. Quantum Grav. 14:2577–2583, 1997). In this work, we apply the method to the special case of null shells of dust collapsing in a four-dimensional Minkowski background (Penrose construction, 1973). Our main conclusion is that the class of surfaces covered by the method is severely restricted. We provide afterwards a wide family of surfaces satisfying the Penrose inequality which includes the ones determined by the Bergqvist method.

1 Introduction

The Penrose inequality [3] bounds from below the total mass of a spacetime in terms of the area of suitable surfaces that represent black holes. There are several versions of the Penrose inequality (see [4] for a relatively recent review). For asymptotically flat four-dimensional spacetimes with a regular past null infinity, the inequality reads $16\pi M_B^2 \geq |S_0|$, where $|S_0|$ is the area of any marginally outer trapped surface $|S_0|$ whose outer directed past null cone is smooth and M_B is the Bondi mass on the cut defined by the intersection of the outer past null cone of S_0 and past null infinity. Ludvigsen and Vickers [1] proposed an argument to prove this inequality which used an implicit assumption that does not hold in general [2]. Moreover, it is not easy to write down conditions directly on S_0 which ensure that this extra assumption holds true. Therefore the Penrose inequality for the Bondi mass is still an open problem. The Penrose inequality was originally put forward by Penrose in 1973 [3].

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His strategy consisted of arranging incoming null shells of dust in the Minkowski spacetime. After the shell has passed, there are two well-differenciated regions with different geometries separated by a null hypersurface and the energy on the shell can be arranged so that a trapped surface S_0 forms with respect to the exterior geometry. One of the beauties of the construction is that the Penrose inequality becomes a geometric inequality in the Minkowski spacetime, with no reference to the shell construction (see [3, 5] for details).

2 Bergqvist Method

Let M be a four-dimensional asymptotically flat spacetime at past null infinity satisfying the dominant energy condition. Consider a spacelike two-surface S_0 of spherical topology. The normal bundle NS_0 of S_0 admits a global basis of future null vectors k and ℓ. Consider the normalization $\langle k, \ell \rangle = -1$. As usual, the null extrinsic curvatures are defined by $K^{\ell}(X, Y) = -\langle \ell, \nabla_X Y \rangle$, with X, Y tangent vectors to S_0, and similarly for K^k. The traces of these tensors define the null expansions θ_ℓ, θ_k.

Assume S_0 to be a marginally outer trapped surface (MOTS), i.e. $\theta_\ell = 0$. If we choose any real number r_0, we can consider the unique past directed null geodesics $\alpha_p(r)$ starting at $\alpha_p(r = r_0) = p \in S_0$, with tangent vector $\alpha'_p = -k|_p$ and $\nabla_k k = 0$. Let Ω be the null hypersurface generated by these geodesics. We will refer to k as the inner future null direction. In general Ω will become singular due to the development of caustics. However, for suitable S_0 and appropriate choice of inner direction, Ω will be regular everywhere, with no caustics developing even at past null infinity. Any such S_0 will be called *spacetime convex*.

Let S_r be the surfaces obtained by dragging the initial surface S_0 along the null geodesics after a parameter "r". Let η_{S_r} be the volume form of S_r. The method used by Ludvigsen and Vickers [1] and later on by Bergqvist [2] uses as hypothesis the following conditions at infinity:

$$\lim_{r\to\infty} \frac{\eta_{S_r}}{r^2} = \eta_{\mathbb{S}^2}, \qquad \theta_k = \frac{-2}{r} + O(r^{-3}), \qquad \theta_\ell = \frac{1}{r} + \frac{a}{r^2} + O(r^{-2}), \tag{1}$$

where $\eta_{\mathbb{S}^2}$ is the volume form of a limiting metric of Gauss curvature one (which may be defined on any of the S_r as they are all diffeomorphic to each other via the geodesics). Let E_B be the Bondi energy on the cut defined by the intersection of Ω and past null infinity with respect to the reference frame defined by the flow of the surfaces S_r. With a suitable choice of scaling in k and a choice of r_0 the form for θ_k given above can always be accomplished. However imposing the rest of the conditions does in general restrict the original surface S_0. The Bondi energy can be expressed as $-8\pi E_B = \int_{\mathbb{S}^2} a\eta_{\mathbb{S}^2}$. The method involves two functions of r:

$$M_b(r) := 8\pi E_B + \int_{S_r} \theta_\ell(r)\eta_{S_r} - 4\pi r, \qquad D(r) := \sqrt{4\pi|S_r|} - 4\pi r.$$

The Penrose inequality takes the form $M_b(r_0) \geq D(r_0)$ and the method proves this by showing $M_b \geq 0$ and $D \leq 0$ for all r. The function M_b (often called Bergqvist mass) is nonincreasing as a consequence of the spherical topology of S_0 and the dominant energy condition. The asymptotic conditions (1) imply that M_b approaches zero and that D is nowhere positive, which establishes the inequality (under assumptions (1)).

3 Bergqvist Method in $\mathscr{M}^{1,3}$

We focus now on the Penrose construction of null shells in the four-dimensional Minkowski spacetime $\mathscr{M}^{1,3}$. Let S_0 be any embedded spacetime convex surface in $\mathscr{M}^{1,3}$. Let ξ' be the future directed, unit generator of a time translation and $\{k', \ell'\}$ the future null basis of the normal bundle of S_0 satisfying $\langle k', \xi' \rangle = -1$ and $\langle k', \ell' \rangle = -1$, with k' inner. Objects defined with respect to the geometry exterior and interior to the shell will be distinguished with signs $+$ and $-$ respectively. The energy density ρ' of the shell satisfies the equation $k'(\rho') = -\theta_{k'}\rho'$ and is adjusted so that S_0 is a MOTS with respect to the outer geometry. The jump of $\theta_{\ell'}$ across the shell satisfies (see e.g. [4]) $\theta^+_{\ell'} - \theta^-_{\ell'} = -8\pi\rho'_0$ (we use $'$ for all objects depending on ξ'). The integral of the energy density on any spatial section of Ω equals the Bondi energy E'_B with respect to the reference frame determined by the flow of surfaces generated by k'. The Penrose inequality can be rewritten [3, 5] as

$$\int_{S_0} \theta^-_{\ell'} \eta_{S_0} \geq \sqrt{4\pi|S_0|}.$$

For any other inner future null section k of NS_0, let $f := -\langle k, \xi' \rangle$. Define ℓ as the null normal vector satisfying $\langle k, \ell \rangle = -1$. S_0 being spacetime convex, the intersection of Ω with a constant time hyperplane Σ'_0 orthogonal to ξ' and completely to the past of S_0 is a (strictly) convex hypersurface of Euclidean space, which we will denote by $\widehat{S_0}'$. We define also τ'_r ('time height' to Σ'_0) as the orthogonal distance of any point of each S_r to Σ'_0. Since $\widehat{S_0}'$ is convex, we can endow it with the standard two-sphere metric $\overline{\gamma}'$ via the Gauss map and introduce the support function h', which measures the signed distance from the euclidean origin to each tangent plane of $\widehat{S_0}'$. All geometric objects on S_r can be expressed in terms of the geometry of the standard two sphere $(\mathbb{S}^2, \overline{\gamma}')$, and in terms of h', f, $\tau'_0 = \tau'_r|_{r=r_0}$ and r_0. A straightforward calculation shows that the asymptotic behaviour at $r = +\infty$ of the null expansions is

$$\theta_k = \frac{-2}{r} + \frac{C}{r^2} + O(r^{-3}), \qquad \theta^-_\ell = \frac{-1}{f^3}(\triangle_{\overline{\gamma}'} f - f(1 + \frac{1}{f^2}|Df|^2_{\overline{\gamma}'}))\frac{1}{r} + O(r^{-2}),$$

$C = u'/f - 2r_0$ with $u' = \triangle_{\overline{\gamma}'} h' + 2(h' - \tau'_0)$. A fundamental input of the Bergqvist method is that $\theta_\ell^+ = \frac{1}{r} + O(r^{-2})$. It can be checked that the leading term of θ_ℓ does not jump across the shell, and hence the leading coefficient of θ_ℓ^- must equal 1 for the method to apply. This happens if and only if f satisfies $\triangle_{\overline{\gamma}'} \log f + f^2 = 1$. We can characterize the solutions of this equation as follows:

Theorem 1 (Choice of the Killing). *f satisfies $\triangle_{\overline{\gamma}'} \log f + f^2 = 1$ if and only if there is a new unit time translation ξ satisfying $\langle k, \xi \rangle = -1$.*

In terms of the new Killing ξ, the explicit expressions for θ_k and θ_ℓ simplify notably, even though the objects themselves remain unaltered (note that neither k, nor the parametrization of the geodesics has been changed). From $\langle k, \xi \rangle = -1$ and using the jump equation for θ_ℓ and $\int_S \rho \eta_S = E_B$, the expression for M_b becomes $M_b(r) = \int_{S_r} \theta_\ell^-(r) \eta_{S_r} - 4\pi r$. It is easy to see that the limits of M_b and D as $r \to \infty$ coincide and are equal to $L_{r_0} := \frac{1}{2} \int_{\mathbb{S}^2} C \eta_{\mathbb{S}^2}$, where C (which, recall, has not changed) can now be written in the form $u - 2r_0$, with $u = \triangle_{\overline{\gamma}} h + 2(h - \tau_0)$ and all quantities are determined with respect to the geometry of the plane Σ_0 orthogonal to ξ. Comparing with Sect. 2 the Bergqvist approach requires setting $C = 0$, i.e. $u = 2r_0$. This is equivalent to $\tau_0 = \frac{H(\widehat{S_0})}{\mathrm{Scal}(\widehat{S_0})} - \beta$, $\beta > 0$, where τ_0 is the 'time height' from S_0 to Σ_0, and $H(\widehat{S_0})$ and $\mathrm{Scal}(\widehat{S_0})$ are, respectively, the mean and the scalar curvature of the projected surface $\widehat{S_0}$ in Σ_0. It follows that the class of surfaces for which the method applies depends on a single parameter for each choice of convex $\widehat{S_0}$ and hence it is severely restricted, as claimed.

If we completely relax the condition $C = 0$ we can still apply a suitable modification of the method. Recall that $M_b \geq L_{r_0}$. A different way of obtaining $M_b \geq D$ is imposing conditions so that $D \leq L_{r_0}$. Since $\lim_{r \to \infty} D(r) = L_{r_0}$, we can ask D to satisfy $\frac{dD}{dr}(r) \geq 0$. This leads to the following condition (see Theorem 6 in [6]):

$$4\pi \int_{\mathbb{S}^2} \left((\triangle_{\overline{\gamma}} h)^2 + 2h \triangle_{\overline{\gamma}} h \right) \eta_{\mathbb{S}^2} \geq 4\pi \int_{\mathbb{S}^2} u^2 \eta_{\mathbb{S}^2} - \left(\int_{\mathbb{S}^2} u \eta_{\mathbb{S}^2} \right)^2 .$$

The class of surfaces satisfying this inequality is quite large as it depends on arbitrary functions for each choice of $\widehat{S_0}$. It is also immediate to check that it includes the class covered by the Bergqvist method.

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