Chapter 38
American Option Pricing with Time-Varying Parameters

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Abstract In this paper, we provide an explicit formula for American option pricing on a dividend-paying equity when the parameters in Black-Scholes equation are time dependent. By using a general transformation, the option value is shown as an explicit formula which is based on the value of American option with constant parameters. Finally, the optimal boundary of American option is given.

Keywords Option pricing · Time-varying parameters · American put option · Black-scholes equation

38.1 Introduction

Options have been traded on public exchanges since 1973. There are American and European options and a variety of exotic options in the market. American option entitles the holder to buy or sell, respectively, at any time prior to a specified expiration date. The existence of derivative securities leads to the mathematical question: *pricing*. Our paper presents the valuation of an American put option by solving the Black-Scholes partial differential equation (for short, PDE) with time dependent parameters. Since some investors do not want to corporate the market's view on the direction of the future behavior which the option price depends on, we concern on American option with time-varying parameters in order to give a mathematical model to handle the concrete affairs in practice.

By using the Feynman-Kac theorem [11], the popular method of matingale pricing for contingent claims (see [2, 3, 5–7]) is equivalent to the PDE technique. We

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shall only focus on the transformation of PDEs to obtain solutions to the valuation problems.

The original derivation of the Black-Scholes equation with time-varying parameters can be found in [8]. A method of reducing this PDE into the heat equation was described in [10] and an alternative approach to solving this PDE with time-varying parameters was given in [9]. In [10], it has to keep track of how the terminal condition and solve the problem based on the heat equation. In [9], Rodrigo and Mamon provided a simple derivation of an explicit formula for pricing a European option on a dividend-paying equity when the parameters in Black-Scholes PDE are time dependent. The approach of [9] is to transform the Black-Scholes equation with time-varying parameters directly into a Black-Scholes equation with time independent but arbitrary parameters.

Although Merton [8], Rodrigo and Mamon [9], Wilmott et al [10] introduced different approaches to solve the Black-Scholes equation with time-varying parameters, they only considered European option but not American option. American option is quite different with European option in that the buyer of an American option can opt, at any time of his choice, for a lump-sum settlement of the option. In [3] and [4], American option's price has been decomposed to a European option's price (see [1]) plus another part due to the extra premium required by early exercising the contract. In this paper, we show the price of American put option with time-varying parameters by using the approach of [9] and the decomposition of American option.

38.2 Preliminaries

Let S be the price of a stock, $V(S,t)$ be the value of an American option on a dividend-paying equity at time t and K be the exercise price or strike price in the contract. We assume that $\sigma(t)$ denotes the volatility of the equity at time t, $r(t)$ and $q(t)$ are riskless interest rate and dividend yield at time t, respectively. It is clear that $r(t)$, $q(t)$ and $\sigma(t)$ are time dependent parameters which are different with constant parameters. From [8], American put option satisfies the following equation:

$$\begin{cases} \min\{-\mathscr{L}V, V-g(S)\}=0, \\ V(S,t)=g(S), \end{cases} \tag{38.1}$$

where

$$\mathscr{L}V=\frac{\partial V}{\partial t}+\frac{\sigma^2(t)}{2}S^2\frac{\partial^2 V}{\partial S^2}+(r(t)-q(t))S\frac{\partial V}{\partial S}-r(t)V$$

and

$$g(S)=\begin{cases} (S-K)^+, & \text{call option,} \\ (K-S)^+, & \text{put option.} \end{cases} \tag{38.2}$$

Definition 38.1. In mathematical theory, American option pricing is a free boundary problem. The free boundary is a boundary curve (to be determined) which divides the domain $\{0 \le S < \infty, 0 \le t \le T\}$ into two parts: the continuation region and the stopping region. To be specific, e.g., consider an American put option. The two sets of American put option are:

$$\begin{aligned}
\Sigma_1 &= \{(S,t) \in \mathbb{R}^+ \times [0,T) | V(S,t) > (K-S)^+\} \\
&= \{(S,t) | S(t) \le S < \infty, 0 \le t \le T\}, \\
\Sigma_2 &= \{(S,t) \in \mathbb{R}^+ \times [0,T) | V(S,t) = (K-S)^+\} \\
&= \{(S,t) | 0 \le S \le S(t), 0 \le t \le T\},
\end{aligned}$$

where $S(t) < K$ for $0 \le t < T$. It is called that Σ_1 is the continuation region which means it is possible to hold an American put option and find an exercise policy that gives riskless profits and Σ_2 is the stopping region which means it is possible to sell the American put option and can make riskless profits for every exercise policy option of the buyer. $\Gamma : S = S(t)$ is called the optimal exercise boundary and it must be determined simultaneously with the option price $V(S,t)$.

On the optimal exercise boundary Γ,

$$\begin{aligned}
&V(S,t)\big|_{S=S(t)} = K - S(t), \\
&\lim_{S \to S(t)} \frac{\partial V(S,t)}{\partial S} = -1 = \frac{\partial (K-S)^+}{\partial S}\bigg|_{S=S(t)}.
\end{aligned} \tag{38.3}$$

The free boundary condition (38.3) indicates the option price's derivative is continuous at crossing the optimal exercise boundary. This fact expresses the principle of American option pricing. In order to value an American put option, we should get $V(S,t) \in C^1_\Sigma$ where $\Sigma = \{(S,t) | 0 \le S < \infty, 0 \le t \le T\} = \Sigma_1 \cup \Sigma_2 \cup \Gamma$.

Within the framework of constant risk-free rate r_c, constant dividend yield q_c and time independent volatility σ_c, the following definition and lemma are given.

Definition 38.2. $G(S,t;\xi,T)$ is called the **fundamental solution of the Black-Scholes equation**, if it satisfies the following terminal value problem to the Black-Scholes equation:

$$\begin{cases}
\mathscr{L}V = \dfrac{\partial V}{\partial t} + \dfrac{\sigma_c^2}{2} S^2 \dfrac{\partial^2 V}{\partial S^2} + (r_c - q_c) S \dfrac{\partial V}{\partial S} - r_c V = 0, \\
V(S,t) = \delta(S - \xi),
\end{cases}$$

where $0 < S < \infty$, $0 < \xi < \infty$, $0 < t < T$ and $\delta(x)$ is a Dirac function.

Lemma 38.1. *[4] Assume that the volatility of the equity, riskless interest rate and dividend rate are constant, the price of American put option $V_c(S,t)$ satisfies:*

$$V_c(S,t) = V_{Ec}(S,t) + e_c(S,t),$$

where $V_{Ec}(S,t)$ is the European put option price contract and $e_c(S,t)$ is the early exercise premium with constant parameters,

$$\begin{aligned} V_{Ec}(S,t) &= Ke^{-r_c(T-t)}N(-d_2) - Se^{-q_c(T-t)}N(-d_1), \\ e_c(S,t) &= \int_t^T d\eta \int_0^{S(\eta)} (Kr_c - \xi q_c)G(S,t;\xi,\eta)d\xi \end{aligned}$$

and

$$\begin{cases} d_1 = \dfrac{\ln\frac{S}{K} + (r_c - q_c + \frac{\sigma_c^2}{2})(T-t)}{\sigma_c\sqrt{T-t}}, \\ d_2 = d_1 - \sigma_c\sqrt{T-t}. \end{cases}$$

38.3 American Put Option Pricing with Time-Varying Parameters

In this section, we investigate the American put option pricing problem with time varying parameters.

From Equations (38.1) and (38.2), we know that the price process of American put option $V(S,t)$ satisfies:

$$\begin{cases} \min\{-\mathscr{L}V, V - g(S)\} = 0, \\ V(S,t) = (K-S)^+. \end{cases}$$

Following the methodology of Lemma 38.1, we separate the variational inequality equation into a European put option and early exercise premium.

38.3.1 European Put Option with Time-Varying Parameters

At first, we investigate the price of the European put option which satisfies:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{\sigma^2(t)}{2}S^2\dfrac{\partial^2 V}{\partial S^2} + \big(r(t) - q(t)\big)S\dfrac{\partial V}{\partial S} - r(t)V = 0, \\ V(S,t) = (K-S)^+. \end{cases} \tag{38.4}$$

Within the framework of constant parameters which includes risk-free rate, constant dividend yield and time-independent volatility, the Black-Scholes PDE of the option pricing $\overline{V}(\overline{S},\overline{t})$ at time $\overline{t}$ is given by:

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \overline{t}} + \dfrac{\sigma_c^2}{2}\overline{S}^2\dfrac{\partial^2 \overline{V}}{\partial \overline{S}^2} + (r_c - q_c)\overline{S}\dfrac{\partial \overline{V}}{\partial \overline{S}} - r\overline{V} = 0, \\ \overline{V}(\overline{S},\overline{T}) = (\overline{K} - \overline{S})^+, \end{cases} \tag{38.5}$$

where $\overline{K}$ is the exercise price at time $\overline{T}$. The parameters σ_c, r_c, q_c and $\overline{K}$ are assumed to be positive. Following the methodology of [9], we transform Equation (38.4) into the Black-Scholes PDE with constant parameters Equation (38.5) directly. Therefore, $\bar{t} = \overline{T}$ when $t = T$.

Using the transformations:

$$V(S,t) = \varphi(t)\overline{V}(\overline{S},\bar{t}), \quad \overline{S} = \phi(t)S, \quad \bar{t} = \psi(t), \tag{38.6}$$

we can get the following equations by chain rule:

$$\begin{aligned} \frac{\partial V}{\partial t} &= \varphi(t)\left(\frac{\partial \overline{V}}{\partial \overline{S}}\phi'(t)S + \psi'(t)\frac{\partial \overline{V}}{\partial \bar{t}}\right) + \varphi'(t)\overline{V}, \\ \frac{\partial V}{\partial S} &= \varphi(t)\phi(t)\frac{\partial \overline{V}}{\partial \overline{S}}, \frac{\partial^2 V}{\partial S^2} = \varphi(t)\phi(t)^2\frac{\partial^2 \overline{V}}{\partial \overline{S}^2}. \end{aligned} \tag{38.7}$$

Substituting Equation (38.7) to (38.4), then regrouping and comparing it with Equation (38.5), we have:

$$\begin{aligned} &\frac{\partial \overline{V}}{\partial \bar{t}} + \frac{\sigma^2(t)\phi^2(t)S^2}{2\psi'(t)}\frac{\partial^2 \overline{V}}{\partial \overline{S}^2} + \frac{\big(r(t)-q(t)\big)\phi(t)S + \phi'(t)S}{\psi'(t)}\frac{\partial \overline{V}}{\partial \overline{S}} \\ &\quad + \frac{\varphi'(t) - r(t)\varphi(t)}{\varphi(t)\psi'(t)}\overline{V} = 0, \\ &\frac{\sigma^2(t)\phi^2(t)S^2}{2\psi'(t)} = \frac{\sigma_c^2}{2}\phi^2(t)S^2, \\ &\frac{\big(r(t)-q(t)\big)\phi(t)S + \phi'(t)S}{\psi'(t)} = (r_c - q_c)\phi(t)S, \\ &\frac{\varphi'(t) - r(t)\varphi(t)}{\varphi(t)\psi'(t)} = -r_c. \end{aligned}$$

Integrating the above formulas, we can get $\varphi(t), \phi(t)$ and $\psi(t)$ as follows:

$$\begin{aligned} \varphi(t) &= A\exp\left\{-\int_t^T \Big(r(\tau) - r_c\psi'(\tau)\Big)d\tau\right\}, \\ \phi(t) &= B\exp\left\{-\int_t^T \Big((r_c - q_c)\psi'(\tau) + q(\tau) - r(\tau)\Big)d\tau\right\}, \\ \psi(t) &= -\frac{1}{\sigma_c^2}\int_t^T \sigma^2(\tau)d\tau + C, \end{aligned}$$

where A, B are positive constants and C is an arbitrary constant. From the terminal condition of Equations (38.4), (38.5) and the transformation Equation (38.6), we have:

$$(K-S)^+ = V(S,t) = \varphi(T)\overline{V}(\overline{S},\overline{T}) = \varphi(T)(\overline{K} - \overline{S})^+$$

$$= \varphi(T)\phi(T)\left(\frac{\overline{K}}{\phi(T)} - S\right)^+.$$

Thus, $\varphi(T)\phi(T) = 1$, $\frac{\overline{K}}{\phi(T)} = K$, $A = \varphi(T) = \frac{K}{\overline{K}}$, $B = \phi(T) = \frac{\overline{K}}{K}$, and $C = \psi(T) = \overline{T}$. Then, the price of the European put option with time-varying parameters Equation (38.4) is given as follows:

$$\begin{aligned}
V(S,t) &= \frac{K}{\overline{K}} \exp\left\{-\int_t^T \left(r(\tau) - r_c \frac{\sigma^2(\tau)}{\sigma_c^2}\right) d\tau\right\} \overline{V}(\overline{S},\overline{t}),\\
\overline{S} &= \frac{\overline{K}}{K} \exp\left\{-\int_t^T \left((r_c - q_c)\frac{\sigma^2(\tau)}{\sigma_c^2} + q(\tau) - r(\tau)\right) d\tau\right\} S,\\
\overline{t} &= -\frac{1}{\sigma_c^2}\int_t^T \sigma^2(\tau)d\tau + \overline{T},
\end{aligned}$$

where $\overline{V}(\overline{S},\overline{t})$ is the price of the classic European put option Equation (38.5) which is given by:

$$\overline{V}(\overline{S},\overline{t}) = \overline{K}\exp\{-r_c(\overline{T}-\overline{t})\}N(-\overline{d}_2) - \overline{S}\exp\{-q_c(\overline{T}-\overline{t})\}N(-\overline{d}_1),$$

with:

$$\begin{aligned}
\overline{d}_1 &= \frac{\ln\overline{S} - \ln\overline{K} + (r_c - q_c + \frac{\sigma_c^2}{2})(\overline{T}-\overline{t})}{\sigma_c\sqrt{\overline{T}-\overline{t}}},\\
\overline{d}_2 &= \overline{d}_1 - \sigma_c\sqrt{\overline{T}-\overline{t}}
\end{aligned} \tag{38.8}$$

and $N(x)$ is the cumulative distribution function of a standard normal variable.

38.3.2 Early Exercise Premium with Time-Varying Parameters

In order to get $e(S,t)$, we solve the fundamental solution of the following Black-Scholes equation first.

$$\begin{cases}
\mathscr{L}V = \dfrac{\partial V}{\partial t} + \dfrac{\sigma^2(t)}{2}S^2\dfrac{\partial^2 V}{\partial S^2} + \big(r(t) - q(t)\big)S\dfrac{\partial V}{\partial S} - r(t) = 0,\\
V(S,t) = \delta(S-\xi).
\end{cases} \tag{38.9}$$

Within the framework of constant parameters, the fundamental solution $\overline{G}(\overline{S},\overline{t};\overline{\xi},\overline{T})$ at time $\overline{t}$ is given by:

$$\begin{cases}
\dfrac{\partial\overline{V}}{\partial\overline{t}} + \dfrac{\sigma_c^2}{2}\overline{S}^2\dfrac{\partial^2\overline{V}}{\partial\overline{S}^2} + (r_c - q_c)\overline{S}\dfrac{\partial\overline{V}}{\partial\overline{S}} - r_c\overline{V} = 0,\\
\overline{V}(\overline{S},\overline{T}) = \delta(\overline{S}-\overline{\xi}),
\end{cases} \tag{38.10}$$

where $0 < \overline{S} < \infty$, $0 < \overline{\xi} < \infty$, $0 < \bar{t} < \overline{T}$ and $\delta(x)$ is a Dirac function.

It is clear that the only difference between Equations (38.4) and (38.9) is the terminal condition. Thus, by Equation (38.6) and terminal conditions of Equations (38.9) and (38.10), we have:

$$\begin{aligned}\delta(S-\xi) = V(S,t) &= \varphi(T)\overline{V}(\overline{S},\overline{T}) = \varphi(T)\delta(\phi(T)S-\overline{\xi}) \\ &= \frac{\varphi(T)}{\phi(T)}\delta\left(S-\frac{\overline{\xi}}{\phi(T)}\right).\end{aligned}$$

Therefore, $\frac{\varphi(T)}{\phi(T)} = 1$, $\frac{\overline{\xi}}{\phi(T)} = \xi, A = \varphi(T) = \frac{\overline{\xi}}{\xi}, B = \phi(T) = \frac{\overline{\xi}}{\xi}$ and $C = \psi(T) = \overline{T}$. Then, the fundamental solution of Equation (38.9) is:

$$\begin{aligned}G(S,t;\xi,T) &= \frac{\overline{\xi}}{\xi}\exp\left\{-\int_t^T\left(r(\tau)-r_c\frac{\sigma^2(\tau)}{\sigma_c^2}\right)d\tau\right\}\overline{G}(\overline{S},\bar{t};\overline{\xi},\overline{T}),\\ \overline{S} &= \frac{\overline{\xi}}{\xi}\exp\left\{-\int_t^T\left((r_c-q_c)\frac{\sigma^2(\tau)}{\sigma_c^2}+q(\tau)-r(\tau)\right)d\tau\right\}S, \quad (38.11)\\ \bar{t} &= -\frac{1}{\sigma_c^2}\int_t^T\sigma^2(\tau)d\tau+\overline{T}, \quad (38.12)\end{aligned}$$

where

$$\overline{G}(\overline{S},\bar{t};\overline{\xi},\overline{T}) = \frac{e^{-r_c(\overline{T}-\bar{t})}}{\overline{\xi}\sigma_c\sqrt{2\pi(\overline{T}-\bar{t})}} \exp\left\{-\frac{\left(\ln\frac{\overline{S}}{\overline{\xi}}+(r_c-q_c-\frac{\sigma_c^2}{2})(\overline{T}-\bar{t})\right)^2}{2\sigma_c^2(\overline{T}-\bar{t})}\right\} \quad (38.13)$$

is the fundamental solution of the classic Black-Sholes Equation (38.10).

Theorem 38.1. *If the fundamental solution $G(S,t;\xi,\eta)$ of Black-Scholes equation is regarded as a function of ξ,η, then it is the fundamental solution of the adjoint equation of the Black-Scholes equation. That is, let $\upsilon(\xi,\eta) = G(S,t;\xi,\eta)$, then $\upsilon(\xi,\eta)$ satisfies:*

$$\begin{cases}\mathscr{L}^*\upsilon = -\dfrac{\partial\upsilon}{\partial\eta}+\dfrac{\sigma^2(\eta)}{2}\dfrac{\partial^2(\xi^2\upsilon)}{\partial\xi^2}-\Big(r(\eta)-q(\eta)\Big)\dfrac{\partial(\xi\upsilon)}{\partial\xi}\\ \qquad -r(\eta)\upsilon = 0,\\ \upsilon(\xi,t) = \delta(\xi-S),\end{cases} \quad (38.14)$$

where $0 < \xi < \infty$, $0 < S < \infty$, $t < \eta$. If the fundamental solution of Equation (38.14) is $G^*(\xi,\eta;S,t)$, then $G(S,t;\xi,\eta) = G^*(\xi,\eta;S,t)$.

Proof. Consider the integral:

$$
\begin{aligned}
0 &= \int_0^\infty \int_{t+\varepsilon}^{\eta-\varepsilon} \{G^*(x,y;S,t)\mathscr{L}G(x,y;\xi,\eta) - G(x,y;\xi,\eta)\mathscr{L}^*G^*(x,y;S,t)\}\,dxdy \\
&= \int_0^\infty dx \int_{t+\varepsilon}^{\eta-\varepsilon} \left\{ \frac{\partial}{\partial y}(G^*G) + \frac{\sigma^2(y)}{2}\frac{\partial}{\partial x}\left(x^2G^*\frac{\partial G}{\partial x}\right) - \frac{\sigma^2(y)}{2}\frac{\partial}{\partial x}\left(G\frac{\partial}{\partial x}(x^2G^*)\right) \right. \\
&\quad \left. + \Big(r(y)-q(y)\Big)\frac{\partial}{\partial x}(xGG^*) \right\} dy.
\end{aligned}
$$

When $x \to 0$ and/or ∞, $x^2G^*\frac{\partial G}{\partial x} \to 0, G\frac{\partial}{\partial x}\left(x^2G^*\right) \to 0, xGG^* \to 0$. Thus

$$
\int_0^\infty G^*(x,\eta-\varepsilon;S,t)G(x,\eta-\varepsilon;\xi,\eta)dx = \int_0^\infty G^*(x,t+\varepsilon;S,t)G(x,t+\varepsilon;\xi,\eta)dx.
$$

Letting $\varepsilon \to 0$, from the initial conditions of Equations (38.9) and (38.14), we have:

$$
\int_0^\infty G^*(x,\eta;S,t)\delta(x-\xi)dx = \int_0^\infty \delta(x-S)G(x,t;\xi,\eta)dx,
$$

which implies $G^*(\xi,\eta;S,t) = G(S,t;\xi,\eta)$. This completes the proof. □

Theorem 38.2. *For time varying parameters, the value of an American put option $V(S,t)$ is:*

$$
V(S,t) = V_E(S,t) + e(S,t), \tag{38.15}
$$

where $V_E(S,t)$ is the value of the European put option and $e(S,t)$ is the early exercise premium of the American put option.

$$
\begin{aligned}
V_E(S,t) &= \frac{K}{\overline{K}}\exp\left\{-\int_t^T \left(r(\tau) - r_c\frac{\sigma^2(\tau)}{\sigma_c^2}\right)d\tau\right\}\overline{V}_E(\overline{S},\overline{T}), \\
e(S,t) &= \int_t^T d\eta \int_0^{S(\eta)} (Kr(\eta) - q(\eta)\xi)G(S,t;\xi,\eta)d\xi,
\end{aligned} \tag{38.16}
$$

where $G(S,t;\xi,\eta)$ is the fundamental solution of the Black-Scholes equation with time-varying parameters,

$$
\begin{aligned}
\overline{V}_E(\overline{S},\overline{T}) &= \overline{K}\exp\left[-r_c(\overline{T}-\overline{t})\right]N(-\overline{d}_2) - \overline{S}\exp\left[-q_c(\overline{T}-\overline{t})\right]N(-\overline{d}_1), \\
G(S,t;\xi,\eta) &= \frac{\overline{\xi}}{\xi}\exp\left\{-\int_t^\eta \left(r(\tau) - r_c\frac{\sigma^2(\tau)}{\sigma_c^2}\right)d\tau\right\}\overline{G}(\overline{S},\overline{t};\overline{\xi},\overline{\eta})
\end{aligned} \tag{38.17}
$$

and $\overline{d}_1, \overline{d}_2$ are defined as Equation (38.8).

Proof. For American put option's domain $\Sigma = \{(S,t)|0 \le S < \infty, 0 \le t \le T\}$, $V(S,t)$ has continuous second derivative in each region. Thus $V(S,t)$ satisfies

$$
-\mathscr{L}V(S,t) = \begin{cases} 0, & (S,t) \in \Sigma_1, \\ Kr(t) - q(t)S, & (S,t) \in \Sigma_2, \end{cases} \tag{38.18}
$$

where $\mathscr{L}$ is the Black-Scholes operator.

Multiplying $G^*(\xi,\eta;S,t)$ to the both sides of Equation (38.18) and integrating it on domain $\{(\xi,\eta)|0 \le \xi < \infty, t+\varepsilon \le \eta \le T\}$, we have:

$$= -\int_{t+\varepsilon}^{T} d\eta \int_0^\infty \Big\{ \frac{\partial}{\partial \eta}(G^* V) + \frac{\sigma^2(\eta)}{2}\frac{\partial}{\partial \xi}\Big(\xi^2 G^* \frac{\partial V}{\partial \xi}\Big) - \frac{\sigma^2(\eta)}{2}\frac{\partial}{\partial \xi}\Big(V\frac{\partial}{\partial \xi}(\xi^2 G^*)\Big) + \Big(r(\eta)-q(\eta)\Big)\frac{\partial}{\partial \xi}(\xi V G^*)\Big\} d\xi.$$

When $\xi \to 0$ and/or ∞, $\xi^2 G^* \frac{\partial V}{\partial \xi} \to 0, V\frac{\partial}{\partial \xi}(\xi^2 G^*) \to 0, \xi V G^* \to 0$. Thus,

$$\int_{t+\varepsilon}^{T} d\eta \int_0^{S(\eta)} \Big(Kr(\eta) - q(\eta)\xi\Big) G^*(\xi,\eta;S,t) d\xi$$
$$= \int_0^\infty G^*(\xi,t+\varepsilon;S,t)V(\xi,t+\varepsilon)d\xi - \int_0^\infty G^*(\xi,T;S,t)V(\xi,T)d\xi.$$

Letting $\varepsilon \to 0$, from Theorem 38.1 and the terminal condition of Equation (38.14), we have:

$$\begin{aligned} V(S,t) &= \int_0^\infty G(S,t;\xi,T)(K-\xi)^+ d\xi + \int_t^T d\eta \int_0^{S(\eta)} \Big(Kr(\eta)-q(\eta)\xi\Big) G(S,t;\xi,\eta) d\xi, \\ &= \frac{K}{\overline{\overline{K}}} \exp\Big\{-\int_t^T \Big(r(\tau) - r_c \frac{\sigma^2(\tau)}{\sigma_c^2}\Big) d\tau\Big\} \overline{V}_E(\overline{S},\overline{T}) \\ &\quad + \int_t^T d\eta \int_0^{S(\eta)} \Big(Kr(\eta) - q(\eta)\xi\Big) \cdot \frac{\overline{\overline{\xi}}}{\xi} \\ &\quad \cdot \exp\Big\{-\int_t^\eta \Big(r(\tau) - r_c \frac{\sigma^2(\tau)}{\sigma_c^2}\Big) d\tau\Big\} \overline{G}(\overline{S},\overline{t};\overline{\overline{\xi}},\overline{\eta}) d\xi, \\ &= V_E(S,t) + e(S,t). \end{aligned}$$

This completes the proof. □

Theorem 38.3. *The optimal boundary of American put option $S = S(t)$ satisfies the following nonlinear integral equation:*

$$\begin{aligned} S(t) = K &- \frac{K}{\overline{\overline{K}}} \exp\Big\{-\int_t^T \Big(r(\tau) - r_c \frac{\sigma^2(\tau)}{\sigma_c^2}\Big) d\tau\Big\} \overline{V}(\overline{S}(t),\overline{t}) \\ &- K\int_t^T r(\eta) \exp\Big\{-\int_t^\eta r(\tau)d\tau\Big\} N\Big(-\tilde{d}_1|_{S=S(t)}\Big) d\eta \\ &+ S(t)\int_t^T q(\eta) \exp\Big\{\int_t^\eta -q(\tau)d\tau\Big\} N\Big(-\tilde{d}_2|_{S=S(t)}\Big) d\eta, \end{aligned} \quad (38.19)$$

where

$$\beta = \frac{r_c - q_c}{\sigma_c^2} - \frac{1}{2},$$

$$\begin{aligned}
\tilde{d}_1 &= \frac{1}{\sqrt{\int_t^\eta \sigma^2(\tau)d\tau}} \Bigg(\ln \frac{S}{S(\eta)} + \beta \int_t^\eta \sigma^2(\tau)d\tau \\
&\quad - \int_t^\eta \Big((r_c - q_c)\frac{\sigma^2(\tau)}{\sigma_c^2} + q(\tau) - r(\tau)\Big)d\tau\Bigg), \\
\tilde{d}_2 &= \tilde{d}_1 + \sqrt{\int_t^\eta \sigma^2(\tau)d\tau}, \\
\overline{S}(t) &= \frac{\overline{\xi}}{\xi} \exp\left\{-\int_t^T \Big((r_c - q_c)\frac{\sigma^2(\tau)}{\sigma_c^2} + q(\tau) - r(\tau)\Big)d\tau\right\} S(t).
\end{aligned}$$

Proof. By Equations (38.11), (38.12), (38.13), (38.16) and (38.17), $e(S,t)$ can be expressed as:

$$\begin{aligned}
e(S,t) = & \int_t^T d\eta \int_0^{S(\eta)} \Big(Kr(\eta) - q(\eta)\xi\Big) \cdot \frac{1}{\xi} \cdot \frac{1}{\sqrt{2\pi \int_t^\eta \sigma^2(\tau)d\tau}} \\
& \cdot \exp\left\{-\int_t^\eta r(\tau)d\tau - \frac{1}{2\int_t^\eta \sigma^2(\tau)d\tau} \Bigg(\ln \frac{S}{\xi} + \beta \int_t^\eta \sigma^2(\tau)d\tau \right. \\
& \left. - \int_t^\eta \Big((r_c - q_c)\frac{\sigma^2(\tau)}{\sigma_c^2} + q(\tau) - r(\tau)\Big)d\tau\Bigg)^2\right\} d\xi. \qquad (38.20)
\end{aligned}$$

Changing the variable to:

$$\begin{aligned}
x = & \frac{1}{\sqrt{\int_t^\eta \sigma^2(\tau)d\tau}} \Bigg(\ln \frac{S}{\xi} - \int_t^\eta \Big((r_c - q_c)\frac{\sigma^2(\tau)}{\sigma_c^2} + q(\tau) - r(\tau)\Big)d\tau \\
& + \beta \int_t^\eta \sigma^2(\tau)d\tau\Bigg), \\
\xi = & S \cdot \exp\left\{-x\sqrt{\int_t^\eta \sigma^2(\tau)d\tau} - \int_t^\eta \Big((r_c - q_c)\frac{\sigma^2(\tau)}{\sigma_c^2} + q(\tau) - r(\tau)\Big)d\tau \right. \\
& \left. + \beta \int_t^\eta \sigma^2(\tau)d\tau\right\}, \\
dx = & \frac{-d\xi}{\xi\sqrt{\int_t^\eta \sigma^2(\tau)d\tau}}.
\end{aligned}$$

Then, Equation (38.20) becomes:

$$\begin{aligned}
e(S,t) = & \frac{1}{\sqrt{2\pi}} \int_t^T \exp\left\{-\int_t^\eta r(\tau)d\tau\right\} d\eta \int_{\tilde{d}_1}^\infty \Big(Kr(\eta) - q(\eta)\xi\Big) e^{-\frac{x^2}{2}} dx, \\
= & K \int_t^T r(\eta) \exp\left\{-\int_t^\eta r(\tau)d\tau\right\} N\big(-\tilde{d}_1\big) d\eta
\end{aligned}$$

$$-S\int_t^T q(\eta)\exp\left\{\int_t^\eta -q(\tau)d\tau\right\}N\big(-\tilde{d}_2\big)d\eta. \qquad (38.21)$$

Substituting Equation (38.21) into Equation (38.15), and taking into account:

$$V(S,t)\Big|_{S=S(t)} = V(S(t),t) = K - S(t), \lim_{S\to S(t)} \frac{\partial V(S,t)}{\partial S} = -1 = \frac{\partial (K-S)^+}{\partial S}\Bigg|_{S=S(t)},$$

we have:

$$\begin{aligned} &V_E(S,t)+e(S,t)\Big|_{S=S(t)} = V_E(S(t),t)+e(S(t),t) = K-S(t),\\ &\lim_{S\to S(t)} \frac{\partial}{\partial S}\Big(V_E(S,t)+e(S,t)\Big) = -1 = \frac{\partial (K-S)^+}{\partial S}\Bigg|_{S=S(t)} \end{aligned}$$

and

$$\begin{aligned} S(t) = &\ K - V_E(S(t),t) - e(S(t),t),\\ = &\ K - \frac{K}{\overline{K}}\exp\left\{-\int_t^T \Big(r(\tau) - r_c\frac{\sigma^2(\tau)}{\sigma_c^2}\Big)d\tau\right\}\overline{V}(\overline{S}(t),\overline{t})\\ &-K\int_t^T r(\eta)\exp\left\{-\int_t^\eta r(\tau)d\tau\right\}N\Big(-\tilde{d}_1\big|_{S=S(t)}\Big)d\eta\\ &+S(t)\int_t^T q(\eta)\exp\left\{\int_t^\eta -q(\tau)d\tau\right\}N\Big(-\tilde{d}_1\big|_{S=S(t)}\Big)d\eta. \end{aligned}$$

This completes the proof. □

38.4 Conclusion

In this paper, we give an explicit formula for pricing an American put option on a dividend-paying equity when the parameters in Black-Scholes equation are time dependent. An alternative derivation of the solution is given through the use of a generalized change of variable technique. Our results show that the value of American put option with time-varying parameters can be expressed by that with constant parameters. Further, the optimal boundary of American put option is given. Although it is difficult to solve the nonlinear integral Equation (38.19), numerical methods which are similar to [3] can be employed to handle this problem.

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References

1. Black F, Scholes M (1973) The pricing of options and corporate liabilities. Journal of Political Economy 81:637–659
2. Duffie D (1996) Dynamic asset pricing theory. 2nd Edn, Princeton University Press, New Jersey
3. Elliott RJ, Kopp PE (1999) Mathematics of financial markets. Springer-Verlag, New York
4. Jiang LS (2004) Mathematical modeling and methods of option pricing. Higher Education Press, Beijing (In Chinese)
5. Karatzas I, Shreve SE (1998) Methods of mathematical finance. Springer-Verlag
6. Karouri NEl, Peng S, Quenez MC (1997) Backward stochastic differential equation in finance. Mathematical Finance 7:1–71
7. Merton RC (1990) Continuous time finance. Blackwell Publishers, Cambridege, MA
8. Merton RC (1973) Theory of rational option pricing. Bell Journal of Economics and Management Science 4:141–183
9. Rodrigo MR, Mamon RS (2006) An alternative approach to solving the Black-Scholes equation with time-varying parameters. Applied Mathematics Letters 19:398–402
10. Wilmott P, Howison S, Dewynne J (1999) The mathematics of financial derivatives. Cambridge University Press
11. Øksendal B (1998) Stochastic differential equation, an introduction with applications. 5th Edn, Springer-Verlag, Berlin