The Δ_2-Condition and φ-Families of Probability Distributions

Rui F. Vigelis[1] and Charles C. Cavalcante[2]

[1] Computer Engineering, Campus Sobral,
Federal University of Ceará, Sobral-CE, Brazil
rfvigelis@ufc.br
[2] Wireless Telecommunication Research Group, Department of Teleinformatics
Engineering, Federal University of Ceará, Fortaleza-CE, Brazil
charles@ufc.br

Abstract. In this paper, we provide some results related to the Δ_2-condition of Musielak–Orlicz functions and φ-families of probability distributions, which are modeled on Musielak–Orlicz spaces. We show that if two φ-families are modeled on Musielak–Orlicz spaces generated by Musielak–Orlicz functions satisfying the Δ_2-condition, then these φ-families are equal as sets. We also investigate the behavior of the normalizing function near the boundary of the set on which a φ-family is defined.

1 Introduction

In [10], φ-families of probability distributions are introduced as a generalization of exponential families of probability distributions [8,7]. The main idea leading to this generalization is the replacement of the exponential function with a φ-function (a definition is given below). These families (of probability distributions) are subsets of the collection $\mathcal{P}_\mu$ of all μ-a.e. strictly positive probability densities. What the papers [8,7,10] provide is a framework endowing $\mathcal{P}_\mu$ with a structure of C^∞-Banach manifold [5], where a family constitutes a connected component of $\mathcal{P}_\mu$. These families are modeled on Musielak–Orlicz spaces (exponential families are modeled on exponential Orlicz spaces) [6,4,9]. In many properties of these spaces, the Δ_2-condition of Musielak–Orlicz functions plays a central role. For example, a Musielak–Orlicz space L^Φ is equal to the Musielak–Orlicz class $\tilde{L}^\Phi$ if and only if the Musielak–Orlicz function Φ satisfies the Δ_2-condition. In this paper we investigate the Δ_2-condition in the context of φ-families. In Sect. 2, we show that if two φ-families are modeled on Musielak–Orlicz spaces generated by Musielak–Orlicz functions satisfying the Δ_2-condition, then these φ-families are equal as sets. In Sect. 3, we investigate the behavior of the normalizing function near the boundary of the set on which a φ-family is defined. In the rest of this section, φ-families are exposed.

A φ-family is the image of a mapping whose domain is a subset of a Musielak–Orlicz space. In what follows, this statement will be made more precise. Musielak–Orlicz spaces are just briefly introduced here. These spaces are thoroughly exposed in [6,4,9].

F. Nielsen and F. Barbaresco (Eds.): GSI 2013, LNCS 8085, pp. 729–736, 2013.

Let (T, Σ, μ) be a σ-finite, non-atomic measure space. A function $\Phi\colon T \times [0,\infty) \to [0,\infty]$ is said to be a *Musielak–Orlicz function* if

(i) $\Phi(t,\cdot)$ is convex and lower semi-continuous for μ-a.e. $t \in T$,
(ii) $\Phi(t,0) = \lim_{u\downarrow 0} \Phi(t,u) = 0$ and $\lim_{u\to\infty} \Phi(t,u) = \infty$ for μ-a.e. $t \in T$,
(iii) $\Phi(\cdot,u)$ is measurable for each $u \geq 0$.

We notice that $\Phi(t,\cdot)$, by (i)–(ii), is not equal to 0 or ∞ on the interval $(0,\infty)$. A Musielak–Orlicz function Φ is said to be an *Orlicz function* if the functions $\Phi(t,\cdot)$ are the same for μ-a.e. $t \in T$.

Let L^0 denote the linear space of all real-valued, measurable functions on T, with equality μ-a.e. Given any Musielak–Orlicz function Φ, we denote the functional $I_\Phi(u) = \int_T \Phi(t,|u(t)|)d\mu$, for any $u \in L^0$. The *Musielak–Orlicz space*, *Musielak–Orlicz class*, and *Morse–Transue space* generated by a Musielak–Orlicz function Φ are defined by

$$L^\Phi = \{u \in L^0 : I_\Phi(\lambda u) < \infty \text{ for some } \lambda > 0\},$$
$$\tilde{L}^\Phi = \{u \in L^0 : I_\Phi(u) < \infty\},$$

and

$$E^\Phi = \{u \in L^0 : I_\Phi(\lambda u) < \infty \text{ for all } \lambda > 0\},$$

respectively. The Musielak–Orlicz space L^Φ is a Banach space when it is equipped with the *Luxemburg norm*

$$\|u\|_\Phi = \inf\Big\{\lambda > 0 : I_\Phi\Big(\frac{u}{\lambda}\Big) \leq 1\Big\},$$

or the *Orlicz norm*

$$\|u\|_{\Phi,0} = \sup\bigg\{\bigg|\int_T uv d\mu\bigg| : v \in \tilde{L}^{\Phi^*} \text{ and } I_{\Phi^*}(v) \leq 1\bigg\},$$

where $\Phi^*(t,v) = \sup_{u\geq 0}(uv - \Phi(t,u))$ is the *Fenchel conjugate* of $\Phi(t,\cdot)$. These norms are equivalent and the inequalities $\|u\|_\Phi \leq \|u\|_{\Phi,0} \leq 2\|u\|_\Phi$ hold for all $u \in L^\Phi$.

Whereas exponential families are based on the exponential function, φ-families are based on φ-functions. A function $\varphi\colon T \times \mathbb{R} \to (0,\infty)$ is said to be a *φ-function* if the following conditions are satisfied:

(a1) $\varphi(t,\cdot)$ is convex for μ-a.e. $t \in T$,
(a2) $\lim_{u\to-\infty} \varphi(t,u) = 0$ and $\lim_{u\to\infty} \varphi(t,u) = \infty$ for μ-a.e. $t \in T$,
(a3) $\varphi(\cdot,u)$ is measurable for each $u \in \mathbb{R}$.

In addition, we assume a positive, measurable function $u_0\colon T \to (0,\infty)$ can be found such that, for every measurable function $c\colon T \to \mathbb{R}$ for which $\varphi(t,c(t))$ is in $\mathcal{P}_\mu$, we have that (a4) $\varphi(t, c(t) + \lambda u_0(t))$ is μ-integrable for all $\lambda > 0$.

The exponential function is an example of φ-function, since $\varphi(t,u) = \exp(u)$ satisfies conditions (a1)–(a3) and (a4) with $u_0 = \mathbf{1}_T$, where $\mathbf{1}_A$ is the indicator function of a subset $A \subseteq T$. Another example of φ-function is the Kaniadakis' κ-exponential (see [2] and [10, Example 1]). Let $\varphi'_+(t,\cdot)$ denote the right derivative of $\varphi(t,\cdot)$. In what follows, $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}'_+$ denote the function operators $\boldsymbol{\varphi}(u)(t) := \varphi(t,u(t))$ and $\boldsymbol{\varphi}'_+(u)(t) := \varphi'_+(t,u(t))$, respectively, for any real-valued function $u\colon T \to \mathbb{R}$.

A φ-family is defined to be a subset of the collection

$$\mathcal{P}_\mu = \{p \in L^0 : p > 0 \text{ and } \mathbb{E}[p] = 1\},$$

where $\mathbb{E}[\cdot] = \int_T (\cdot) d\mu$ denotes integration with respect to μ. For each probability density $p \in \mathcal{P}_\mu$, we associate a φ-family $\mathcal{F}^\varphi_c \subset \mathcal{P}_\mu$ centered at p, where $c\colon T \to \mathbb{R}$ is a measurable function such that $p = \boldsymbol{\varphi}(c)$. The Musielak–Orlicz space L^{Φ_c} on which the φ-family $\mathcal{F}^\varphi_c$ is modeled is given in terms of the Musielak–Orlicz function

$$\Phi_c(t,u) = \varphi(t, c(t) + u) - \varphi(t, c(t)). \tag{1}$$

We will use the notation L^φ_c, $\tilde{L}^\varphi_c$ and E^φ_c in the place of L^{Φ_c}, $\tilde{L}^{\Phi_c}$ and E^{Φ_c}, respectively, to indicate that Φ_c is given by (1). Because $\boldsymbol{\varphi}(c)$ is μ-integrable, the Musielak–Orlicz space L^φ_c corresponds to the set of all functions $u \in L^0$ for which there exists $\varepsilon > 0$ such that $\boldsymbol{\varphi}(c + \lambda u)$ is μ-integrable for all $\lambda \in (-\varepsilon, \varepsilon)$.

The elements of the φ-family $\mathcal{F}^\varphi_c \subset \mathcal{P}_\mu$ centered at $p = \boldsymbol{\varphi}(c) \in \mathcal{P}_\mu$ are given by the one-to-one mapping

$$\boldsymbol{\varphi}_c(u) := \boldsymbol{\varphi}(c + u - \psi(u)u_0), \qquad \text{for each } u \in \mathcal{B}^\varphi_c, \tag{2}$$

where the set $\mathcal{B}^\varphi_c \subseteq L^\varphi_c$ is defined as the intersection of the convex set

$$\mathcal{K}^\varphi_c = \{u \in L^\varphi_c : \mathbb{E}[\boldsymbol{\varphi}(c + \lambda u)] < \infty \text{ for some } \lambda > 1\}$$

with the closed subspace

$$B^\varphi_c = \{u \in L^\varphi_c : \mathbb{E}[u\boldsymbol{\varphi}'_+(c)] = 0\},$$

and the *normalizing function* $\psi\colon \mathcal{B}^\varphi_c \to [0,\infty)$ is introduced so that expression (2) defines a probability distribution in $\mathcal{P}_\mu$. By [10, Lemma 2], the set $\mathcal{K}^\varphi_c$ is open in L^φ_c, and hence $\mathcal{B}^\varphi_c$ is open in B^φ_c.

Its is clear that the collection $\{\mathcal{F}^\varphi_c : \boldsymbol{\varphi}(c) \in \mathcal{P}_\mu\}$ covers the whole family $\mathcal{P}_\mu$. Moreover, φ-families are maximal in the sense that if two φ-families have a non-empty intersection, then they coincide as sets. Let $\mathcal{F}^\varphi_{c_1}$ and $\mathcal{F}^\varphi_{c_2}$ be two φ-families centered at $\boldsymbol{\varphi}(c_1) \in \mathcal{P}_\mu$ and $\boldsymbol{\varphi}(c_2) \in \mathcal{P}_\mu$, for some measurable functions $c_1, c_2\colon T \to \mathbb{R}$. If the φ-families $\mathcal{F}^\varphi_{c_1}$ and $\mathcal{F}^\varphi_{c_2}$ have non-empty intersection, then $\mathcal{F}^\varphi_{c_1} = \mathcal{F}^\varphi_{c_2}$ and the spaces $L^\varphi_{c_1}$ and $L^\varphi_{c_2}$ are equal as sets, and have equivalent norms. Because the transition map $\boldsymbol{\varphi}^{-1}_{c_2} \circ \boldsymbol{\varphi}_{c_1} : \mathcal{B}^\varphi_{c_1} \to \mathcal{B}^\varphi_{c_2}$ is an affine transformation, the collection of charts $\{(\mathcal{B}^\varphi_c, \boldsymbol{\varphi}_c)\}_{\boldsymbol{\varphi}(c) \in \mathcal{P}_\mu}$ is an atlas of class C^∞, endowing $\mathcal{P}_\mu$ with a structure of C^∞-Banach manifold. A verification of these claims is found in [10].

2 The Δ_2-Condition and φ-Families

A Musielak–Orlicz function Φ is said to satisfy the Δ_2-*condition*, or to belong to the Δ_2-*class* (denoted by $\Phi \in \Delta_2$), if a constant $K > 0$ and a non-negative function $f \in \tilde{L}^\Phi$ can be found such that

$$\Phi(t, 2u) \leq K\Phi(t, u), \qquad \text{for all } u \geq f(t), \quad \text{and } \mu\text{-a.e. } t \in T. \tag{3}$$

It is easy to see that, if a Musielak–Orlicz function Φ satisfies the Δ_2-condition, then $I_\Phi(u) < \infty$ for every $u \in L^\Phi$. In this case, L^Φ, $\tilde{L}^\Phi$ and E^Φ are equal as sets. On the other hand, if the Musielak–Orlicz function Φ does not satisfy the Δ_2-condition, then E^Φ is a proper subspace of L^Φ. In addition, we can state:

Lemma 1. *Let Φ be a Musielak–Orlicz function not satisfying the Δ_2-condition and such that $\Phi(t, b_\Phi(t)) = \infty$ for μ-a.e. $t \in T$, where $b_\Phi(t) = \sup\{u \geq 0 : \Phi(t, u) < \infty\}$. Then we can find functions u_* and u^* in L^Φ such that*

$$\begin{cases} I_\Phi(\lambda u_*) < \infty, & \text{for } 0 \leq \lambda \leq 1, \\ I_\Phi(\lambda u_*) = \infty, & \text{for } 1 < \lambda, \end{cases} \tag{4}$$

and

$$\begin{cases} I_\Phi(\lambda u^*) < \infty, & \text{for } 0 \leq \lambda < 1, \\ I_\Phi(\lambda u^*) = \infty, & \text{for } 1 \leq \lambda. \end{cases} \tag{5}$$

This lemma is a well established result for Orlicz functions (see [4, Sect. 8.4]). A proof of Lemma 1 is given in [11]. The next result shows that we can always find a φ-family modeled on a Musielak–Orlicz space generated by a Musielak–Orlicz function not satisfying the Δ_2-condition.

Proposition 1. *Given any φ-function φ, we can find a measurable function $c\colon T \to \mathbb{R}$ with $\mathbb{E}[\boldsymbol{\varphi}(c)] = 1$ such that the Musielak–Orlicz function $\Phi_c(t, u) = \varphi(t, c(t) + u) - \varphi(t, c(t))$ does not satisfy the Δ_2-condition.*

Proof. Let A and B be two disjoint, measurable sets satisfying $0 < \mu(A) < \infty$ and $0 < \mu(B) < \infty$. Fixed any measurable function $\widetilde{c}$ such that $\mathbb{E}[\boldsymbol{\varphi}(\widetilde{c})] = 1$, we take any non-integrable function f supported on A such that $\boldsymbol{\varphi}(\widetilde{c})\mathbf{1}_A \leq f\mathbf{1}_A < \infty$. Let $u\colon T \to [0, \infty)$ be a measurable function supported on A such that $\boldsymbol{\varphi}(\widetilde{c} + u)\mathbf{1}_A = f\mathbf{1}_A$. If $\beta > 0$ is such that $\mathbb{E}[\boldsymbol{\varphi}(\widetilde{c} - u)\mathbf{1}_A] + \beta\mu(B) + \mathbb{E}[\boldsymbol{\varphi}(\widetilde{c})\mathbf{1}_{T\setminus(A\cup B)}] = 1$, then we define

$$c = (\widetilde{c} - u)\mathbf{1}_A + \overline{c}\mathbf{1}_B + \widetilde{c}\mathbf{1}_{T\setminus(A\cup B)},$$

where $\overline{c}\colon T \to \mathbb{R}$ is a measurable function supported on B such that $\varphi(t, \overline{c}(t)) = \beta$, for μ-a.e. $t \in B$. Because the function u is supported on A, we can write

$$\mathbb{E}[\boldsymbol{\varphi}(c + u)] = \mathbb{E}[\boldsymbol{\varphi}(\widetilde{c})\mathbf{1}_A] + \mathbb{E}[\boldsymbol{\varphi}(\overline{c})\mathbf{1}_B] + \mathbb{E}[\boldsymbol{\varphi}(\widetilde{c})\mathbf{1}_{T\setminus(A\cup B)}] < \infty.$$

On the other hand, since f is non-integrable, we have

$$\mathbb{E}[\boldsymbol{\varphi}(c + 2u)] > \mathbb{E}[\boldsymbol{\varphi}(\widetilde{c} + u)\mathbf{1}_A] = \mathbb{E}[f] = \infty.$$

Therefore, the Musielak–Orlicz function Φ_c does not satisfy the Δ_2-condition.

The main result of this section is a consequence of the following proposition:

Proposition 2. *Let $b : T \to \mathbb{R}$ be a measurable function such that $\mathbb{E}[\boldsymbol{\varphi}(b)] = 1$. Then $L_b^\varphi \subseteq L_c^\varphi$ for every measurable function $c : T \to \mathbb{R}$ such that $\mathbb{E}[\boldsymbol{\varphi}(c)] = 1$ if, and only if, the Musielak–Orlicz function $\Phi_b(t,u) = \varphi(t, b(t) + u) - \varphi(t, b(t))$ satisfies the Δ_2-condition.*

Proof. Assume that Φ_b satisfies the Δ_2-condition. Let $c : T \to \mathbb{R}$ be any measurable function such that $\mathbb{E}[\boldsymbol{\varphi}(c)] = 1$. Denoting $A = \{t \in T : c(t) \geq b(t)\}$, it is clear that the function $(c - b)\mathbf{1}_A$ is in L_b^φ. Hence, for any function $u \in L_b^\varphi$, we can write

$$\mathbb{E}[\boldsymbol{\varphi}(c + |u|)] = \mathbb{E}[\boldsymbol{\varphi}(b + (c - b) + |u|)] \leq \mathbb{E}[\boldsymbol{\varphi}(b + (c - b)\mathbf{1}_A + |u|)] < \infty,$$

since $(c-b)\mathbf{1}_A + |u|$ is in L_b^φ, and the sets L_b^φ and $\tilde{L}_b^\varphi$ are equal. Thus, $L_b^\varphi \subseteq L_c^\varphi$.

Now we suppose that Φ_b does not satisfy the Δ_2-condition. From Lemma 1, there exists a non-negative function $u \in \tilde{L}^{\Phi_b}$ such that $I_{\Phi_b}(\lambda u) = \infty$ for all $\lambda > 1$. Using the function u, we will provide a measurable function $c : T \to \mathbb{R}$ with $\mathbb{E}[\boldsymbol{\varphi}(c)] = 1$ for which L_b^φ is not contained in L_c^φ. By [1] or [3, Lemma 2], we can find a sequence of non-decreasing, measurable sets $\{T_n\}$, satisfying $\mu(T_n) < \infty$ and $\mu(T \setminus \bigcup_{n=1}^\infty T_n) = 0$, such that

$$\operatorname*{ess\,sup}_{t \in T_n} \Phi_b(t,u) < \infty, \qquad \text{for all } u > 0, \text{ and each } n \geq 1. \tag{6}$$

Thus, for a sufficiently large $n_0 \geq 1$, the set $A = \{t \in T_{n_0} : u(t) \leq n_0\}$ satisfies $\mathbb{E}[\boldsymbol{\varphi}(b + u)\mathbf{1}_{T \setminus A}] < 1$. Observing that

$$I_{\Phi_b}(\lambda u \mathbf{1}_A) \leq \Big[\operatorname*{ess\,sup}_{t \in T_{n_0}} \Phi_b(t, \lambda n_0)\Big] \mu(T_{n_0}) < \infty, \qquad \text{for each } \lambda > 0,$$

we can infer that

$$I_{\Phi_b}(\lambda u \mathbf{1}_{T \setminus A}) = I_{\Phi_b}(\lambda u) - I_{\Phi_b}(\lambda u \mathbf{1}_A) = \infty, \qquad \text{for all } \lambda > 1. \tag{7}$$

Let $\alpha > 0$ be such that $\alpha\mu(A) + \mathbb{E}[\boldsymbol{\varphi}(b + u)\mathbf{1}_{T \setminus A}] = 1$. Then we define

$$c = \overline{c}\mathbf{1}_A + (b + u)\mathbf{1}_{T \setminus A},$$

where $\overline{c}\colon T \to \mathbb{R}$ is a measurable function supported on A such that $\varphi(t, \overline{c}(t)) = \alpha$, for μ-a.e. $t \in A$. It is clear that $\mathbb{E}[\boldsymbol{\varphi}(c)] = 1$. According to [10, Proposition 4], if $c_1, c_2\colon T \to \mathbb{R}$ are measurable functions such that $\mathbb{E}[\boldsymbol{\varphi}(c_1)] = 1$ and $\mathbb{E}[\boldsymbol{\varphi}(c_2)] = 1$, then $(c_1 - c_2) \in L_{c_2}^\varphi$ is a necessary and sufficient condition for $L_{c_1}^\varphi \subseteq L_{c_2}^\varphi$. Thus, to show that L_b^φ is not contained in L_c^φ, we have to verify that $(b - c) \notin L_c^\varphi$. Denoting $F = \{t \in T : c(t) \geq b(t)\}$, for any $\lambda > 0$, we can write

$$\begin{aligned}
\mathbb{E}[\boldsymbol{\varphi}(c + \lambda|b - c|)] &\geq \mathbb{E}[\boldsymbol{\varphi}(c + \lambda(c - b))\mathbf{1}_F] \\
&= \mathbb{E}[\boldsymbol{\varphi}(b + (1 + \lambda)(c - b))\mathbf{1}_F] \\
&\geq \mathbb{E}[\boldsymbol{\varphi}(b + (1 + \lambda)u)\mathbf{1}_{T \setminus A}] \qquad (8) \\
&= \infty, \qquad (9)
\end{aligned}$$

where in (8) we used that $T \setminus A \subseteq F$ and $(c-b)\mathbf{1}_{T\setminus A} = u\mathbf{1}_{T\setminus A}$, and (9) follows from (7). We conclude that $(b-c) \notin L_c^\varphi$, and hence L_b^φ is not contained in L_c^φ. Therefore, if $L_b^\varphi \subseteq L_c^\varphi$ for any measurable function $c: T \to \mathbb{R}$ such that $\mathbb{E}[\boldsymbol{\varphi}(c)] = 1$, then the Musielak–Orlicz function Φ_b satisfies the Δ_2-condition.

Now we can state the main result of this section:

Proposition 3. *Let $b, c\colon T \to \mathbb{R}$ be measurable functions such that $\mathbb{E}[\boldsymbol{\varphi}(b)] = 1$ and $\mathbb{E}[\boldsymbol{\varphi}(c)] = 1$. If the Musielak–Orlicz functions $\Phi_b(t,u) = \varphi(t, b(t)+u) - \varphi(t, b(t))$ and $\Phi_c(t,u) = \varphi(t, c(t)+u) - \varphi(t, c(t))$ satisfy the Δ_2-condition, then L_b^φ and L_c^φ are equal as sets. Moreover, $\mathcal{F}_b^\varphi = \mathcal{F}_c^\varphi$.*

Proof. The conclusion that L_b^φ and L_c^φ are equal as sets follows from Proposition 2. By [10, Proposition 4], it is clear that $(c-b) \in \mathcal{K}_b^\varphi$. Let $\alpha \geq 0$ be such that $u = (c-b) + \alpha u_0$ belongs to $\mathcal{B}_b^\varphi$. If ψ_1 is the normalizing function associated with $\mathcal{F}_b^\varphi$, then $\psi_1(u) = \alpha$ and $\boldsymbol{\varphi}_b(u) = \boldsymbol{\varphi}(b + u - \psi_1(u)u_0) = \boldsymbol{\varphi}(c)$. Thus the φ-families $\mathcal{F}_b^\varphi$ and $\mathcal{F}_c^\varphi$ have a non-empty intersection, and hence $\mathcal{F}_b^\varphi = \mathcal{F}_c^\varphi$.

3 The Behavior of ψ Near the Boundary of $\mathcal{B}_c^\varphi$

In this section, we investigate the behavior of the normalizing function ψ near the boundary of $\mathcal{B}_c^\varphi$ (with respect to the topology of B_c^φ). More specifically, given any function u in the boundary of $\mathcal{B}_c^\varphi$, which we denote by $\partial\mathcal{B}_c^\varphi$, we want to know whether $\psi(\lambda u)$ converges to a finite value or not as $\lambda \uparrow 1$. For this purpose, we establish under what conditions the set $\mathcal{B}_c^\varphi$ has a non-empty boundary. This result is related to the Δ_2-condition. By definition, a function $u \in L^0$ is in $\mathcal{K}_c^\varphi$ if there exists $\varepsilon > 0$ such that $\mathbb{E}[\boldsymbol{\varphi}(c + \lambda u)] < \infty$ for all $\lambda \in (-\varepsilon, 1+\varepsilon)$. Because the set $\mathcal{B}_c^\varphi = \mathcal{K}_c^\varphi \cap B_c^\varphi$ is open in B_c^φ, we conclude that a function $u \in B_c^\varphi$ belongs to the boundary of $\mathcal{B}_c^\varphi$ if and only if $\mathbb{E}[\boldsymbol{\varphi}(c + \lambda u)] < \infty$ for all $\lambda \in (0,1)$, and $\mathbb{E}[\boldsymbol{\varphi}(c+\lambda u)] = \infty$ for each $\lambda > 1$. If the Musielak–Orlicz function $\Phi_c = \varphi(t, c(t)+u) - \varphi(t, c(t))$ satisfies the Δ_2-condition, then $\mathbb{E}[\boldsymbol{\varphi}(c+u)] < \infty$ for all $u \in L_c^\varphi$. In this case, the set $\mathcal{B}_c^\varphi$ coincides with the closed subspace B_c^φ, and the boundary of $\mathcal{B}_c^\varphi$ is empty. On the other hand, if Φ_c does not satisfies the Δ_2-condition, then the boundary of $\mathcal{B}_c^\varphi$ is non-empty. Moreover, not all functions u in the boundary of $\mathcal{B}_c^\varphi$ satisfy $\mathbb{E}[\boldsymbol{\varphi}(c+u)] < \infty$ (or $\mathbb{E}[\boldsymbol{\varphi}(c+u)] = \infty$). In other words, we can always find functions w_* and w^* in $\partial\mathcal{B}_c^\varphi$ for which $\mathbb{E}[\boldsymbol{\varphi}(c+w_*)] < \infty$ and $\mathbb{E}[\boldsymbol{\varphi}(c+w^*)] = \infty$. This result, which is a consequence of Lemma 1, is provided by the following proposition:

Proposition 4. *The boundary of $\mathcal{B}_c^\varphi$ is non-empty if and only if the Musielak–Orlicz function $\Phi_c = \varphi(t, c(t)+u) - \varphi(t, c(t))$ does not satisfy the Δ_2-condition. Moreover, in any of these cases, there exist functions w_* and w^* in $\partial\mathcal{B}_c^\varphi$ such that $\mathbb{E}[\boldsymbol{\varphi}(c+w_*)] < \infty$ and $\mathbb{E}[\boldsymbol{\varphi}(c+w^*)] = \infty$.*

Proof. Given non-negative functions u_* and u^* in L_c^φ satisfying (4) and (5) in Lemma 1, we consider the functions

$$w_* = u_* - \frac{\mathbb{E}[u_* \boldsymbol{\varphi}'_+(c)]}{\mathbb{E}[u_0 \boldsymbol{\varphi}'_+(c)]} u_0, \qquad \text{and} \qquad w^* = u^* - \frac{\mathbb{E}[u^* \boldsymbol{\varphi}'_+(c)]}{\mathbb{E}[u_0 \boldsymbol{\varphi}'_+(c)]} u_0,$$

which are in B_c^φ. Next we show that w_* is in $\partial\mathcal{B}_c^\varphi$ and satisfies $\mathbb{E}[\boldsymbol{\varphi}(c+w_*)] < \infty$. For any $0 \le \lambda \le 1$, its clear that

$$\mathbb{E}[\boldsymbol{\varphi}(c+\lambda w_*)] \le \mathbb{E}[\boldsymbol{\varphi}(c+\lambda u_*)] < \infty.$$

Now suppose that $\mathbb{E}[\boldsymbol{\varphi}(c+\lambda_0 w_*)] < \infty$ for some $\lambda_0 > 1$. In view of $1 \le \mathbb{E}[\boldsymbol{\varphi}(c+\lambda_0 w_*)] < \infty$, we can find $\alpha_0 \ge 0$ such that $\mathbb{E}[\boldsymbol{\varphi}(c+\lambda_0 w_* - \alpha_0 u_0)] = 1$. By the definition of u_0, fixed any measurable function $\widetilde{c}$ such that $\mathbb{E}[\boldsymbol{\varphi}(\widetilde{c})] = 1$, we have that $\mathbb{E}[\boldsymbol{\varphi}(\widetilde{c}+\alpha u_0)] < \infty$ for all $\alpha \in \mathbb{R}$. Hence, considering $\widetilde{c} = c + \lambda_0 w_* - \alpha_0 u_0$ and

$$\alpha = \lambda_0 \frac{\mathbb{E}[u_* \boldsymbol{\varphi}'_+(c)]}{\mathbb{E}[u_0 \boldsymbol{\varphi}'_+(c)]} + \alpha_0,$$

we obtain that $\mathbb{E}[\boldsymbol{\varphi}(c+\lambda_0 u_*)] = \mathbb{E}[\boldsymbol{\varphi}(\widetilde{c}+\alpha u_0)] < \infty$, which is a contradiction. Consequently, $\mathbb{E}[\boldsymbol{\varphi}(c+\lambda w_*)] = \infty$ for all $\lambda > 1$, and w_* belongs to $\partial\mathcal{B}_c^\varphi$ and satisfies $\mathbb{E}[\boldsymbol{\varphi}(c+w_*)] < \infty$.

Proceeding as above, we show that $\mathbb{E}[\boldsymbol{\varphi}(c+\lambda w^*)] < \infty$ for all $0 \le \lambda < 1$, and $\mathbb{E}[\boldsymbol{\varphi}(c+\lambda w^*)] = \infty$ for all $\lambda \ge 1$. This result implies that w^* belongs to $\partial\mathcal{B}_c^\varphi$ and is such that $\mathbb{E}[\boldsymbol{\varphi}(c+w^*)] = \infty$.

For a function u in $\partial\mathcal{B}_c^\varphi$, the behavior of the normalizing function $\psi(\lambda u)$ as $\lambda \uparrow 1$ depends on whether $\boldsymbol{\varphi}(c+u)$ is μ-integrable or not. This behavior is partially elucidated by the following proposition:

Proposition 5. *Let u be a function in the boundary of $\mathcal{B}_c^\varphi$. For $\lambda \in [0,1)$, denote $\psi_u(\lambda) := \psi(\lambda u)$, whose right derivative we indicate by $(\psi_u)'_+(\lambda)$. If $\mathbb{E}[\boldsymbol{\varphi}(c+u)] < \infty$ then $\psi_u(\lambda) = \psi(\lambda u)$ converges to some $\alpha \in (0,\infty)$ as $\lambda \uparrow 1$. On the other hand, if $\mathbb{E}[\boldsymbol{\varphi}(c+u)] = \infty$* then *$(\psi_u)'_+(\lambda)$ tends to ∞ as $\lambda \uparrow 1$.*

Proof. Observing that the normalizing function ψ is convex with $\psi(0) = 0$, we conclude that $\psi_u(\lambda) = \psi(\lambda u)$ is non-decreasing and continuous in $[0,1)$. Moreover, $(\psi_u)'_+(\lambda)$ is non-decreasing in $[0,1)$. Fix any function u in the boundary of $\mathcal{B}_c^\varphi$ such that $\mathbb{E}[\boldsymbol{\varphi}(c+u)] < \infty$. Assume that $\psi(\lambda u)$ tends to ∞ as $\lambda \uparrow 1$. In this case, it is clear that

$$\boldsymbol{\varphi}(c+\lambda u - \psi(\lambda u)u_0) \le \boldsymbol{\varphi}(c + u\mathbf{1}_{\{u>0\}} - \psi(\lambda u)u_0) \to 0, \qquad \text{as } \lambda \uparrow 1.$$

Since $\boldsymbol{\varphi}(c+\lambda u - \psi(\lambda u)u_0) \le \boldsymbol{\varphi}(c+u\mathbf{1}_{\{u>0\}})$, we can use the Dominated Convergence Theorem to write

$$\mathbb{E}[\boldsymbol{\varphi}(c+\lambda u - \psi(\lambda u)u_0)] \to 0, \qquad \text{as } \lambda \uparrow 1,$$

which is a contradiction to $\mathbb{E}[\boldsymbol{\varphi}(c+\lambda u - \psi(\lambda u)u_0)] = 1$. Thus $\psi(\lambda u)$ is bounded in $[0,1)$, and $\psi(\lambda u)$ converges to some $\alpha \in (0,\infty)$ as $\lambda \uparrow 1$.

Now consider any function u in the boundary of $\mathcal{B}_c^\varphi$ satisfying $\mathbb{E}[\boldsymbol{\varphi}(c+u)] = \infty$. Suppose that $(\psi_u)'_+(\lambda)$ converges to some $\beta \in (0,\infty)$ as $\lambda \uparrow 1$. Then $\psi_u(\lambda) = \psi(\lambda u)$ converges to some $\alpha \in (0,\infty)$ as $\lambda \uparrow 1$. From Fatou's Lemma, it follows that

$$\mathbb{E}[\boldsymbol{\varphi}(c+u-\alpha u_0)] \le \liminf_{\lambda \uparrow 1} \mathbb{E}[\boldsymbol{\varphi}(c+\lambda u - \psi(\lambda u)u_0)] = 1.$$

Since $\varphi(t,\cdot)$ is convex, for any $\lambda\in(0,1)$, we can write

$$\begin{aligned}\boldsymbol{\varphi}(c+\lambda u-\psi(\lambda u)u_0) &= \boldsymbol{\varphi}\Big(\lambda(c+u-\alpha u_0)+(1-\lambda)\Big(c-\alpha u_0+\frac{\alpha-\psi(\lambda u)}{1-\lambda}u_0\Big)\Big)\\ &\le \lambda\boldsymbol{\varphi}(c+u-\alpha u_0)+(1-\lambda)\boldsymbol{\varphi}\Big(c-\alpha u_0+\frac{\alpha-\psi(\lambda u)}{1-\lambda}u_0\Big).\end{aligned}$$

Observing that $\beta=\lim_{\lambda\uparrow 1}(\psi_u)'_+(\lambda)=\lim_{\lambda\uparrow 1}[\alpha-\psi(\lambda u)]/(1-\lambda)$, we can infer that

$$\boldsymbol{\varphi}(c+\lambda u-\psi(\lambda u)u_0)\le\boldsymbol{\varphi}(c+u-\alpha u_0)+\boldsymbol{\varphi}(c-\alpha u_0+\beta u_0),$$

showing that $\boldsymbol{\varphi}(c+\lambda u-\psi(\lambda u)u_0)$ is dominated by an integrable function. Thus, by the Dominated Convergence Theorem, it follows that

$$\mathbb{E}[\boldsymbol{\varphi}(c+u-\alpha u_0)]=\mathbb{E}[\lim_{\lambda\uparrow 1}\boldsymbol{\varphi}(c+\lambda u-\psi(\lambda u)u_0)]=\lim_{\lambda\uparrow 1}\mathbb{E}[\boldsymbol{\varphi}(c+\lambda u-\psi(\lambda u)u_0)]=1.$$

The definition of u_0 tells us that $\mathbb{E}[\boldsymbol{\varphi}(\widetilde{c}+\lambda u_0)]<\infty$ for all $\lambda\in\mathbb{R}$ and any measurable function $\widetilde{c}$ such that $\mathbb{E}[\boldsymbol{\varphi}(\widetilde{c})]=1$. In particular, considering $\widetilde{c}=c+u-\alpha u_0$ and $\lambda=\alpha$, we have that $\mathbb{E}[\boldsymbol{\varphi}(c+u)]<\infty$. This contradicts the assumption that $\mathbb{E}[\boldsymbol{\varphi}(c+u)]=\infty$. Therefore, $\lim_{\lambda\uparrow 1}(\psi_u)'_+(\lambda)=\infty$.

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