Information, Awareness and Substructural Logics

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Abstract. The paper outlines a generalisation of the awareness-based epistemic semantics by Fagin and Halpern. Awareness is construed as a relation between agents and pieces of information instead of formulas. The main motive for introducing the generalisation is that it shows substructural logics to be a natural component of information-based epistemic logic: substructural logics can be seen as describing the logical behaviour of pieces of information. Substructural epistemic logics are introduced and some of their properties are discussed. In addition, extensions of substructural epistemic logics invoking group-epistemic and dynamic modalities are sketched.

Keywords: Awareness, Epistemic Actions, Epistemic Logic, Information, Substructural Logic.

1 Introduction

The present paper provides a generalisation of the awareness-based epistemic framework by Fagin and Halpern [17]: awareness is construed as a relation between agents and *pieces of information*. The main motive for introducing the generalisation is that it connects epistemic logics with *substructural logics* [29,32]. It is shown that the latter are a natural component of information-based epistemic logics: substructural logics can be seen as describing the logical behaviour of pieces of information.

The generalisation provides a framework for studying a large class of *substructural epistemic logics*. The paper is an introductory overview of the framework, focusing on a general discussion of substructural epistemic logics. Consequently, many standard investigations of specific logics (such as completeness proofs) are postponed to a sequel. We note that our approach owes much to justification logics [3,4,5,6] and to the Fitting semantics [19,20] in particular. While being similar in some respects, our approach and Fitting semantics are motivated by different goals: we are aiming solely at explaining the possible applications of substructural logics within epistemic logic.

The paper is organised as follows. Section 2 outlines the awareness-based framework [17]. Section 3 suggests a generalisation of the framework: 'pieces of information' are discussed explicitly and awareness is construed as pertaining to

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these. Building on the relational semantics for distributive substructural logics, Sect. 4 explains that the logics can be seen as natural 'logics of information'. Section 5 expands on the above observations and introduces substructural epistemic logics. Section 6 outlines an information-based generalisation of public announcements and discusses some of its idiosyncrasies. Section 7 concludes the paper and outlines some interesting directions for future work. Proofs of some of the propositions are given in a technical appendix.

2 Logical Omniscience and Awareness

It has been argued since the inception of epistemic logic that the modal-logic-based approach[1] is rather optimistic as to the agents' epistemic abilities. If 'α believes that F' is equivalent to 'F holds in *every α-accessible* alternative' and

$$F_1 \wedge \ldots \wedge F_n \rightarrow G \tag{1}$$

holds in *every* alternative, then if α believes $F_1, \ldots, F_n$, then she is bound to believe G as well.[2] This is rather optimistic indeed. There are no logical reasons why α should 'realise' that (1) holds in every alternative and adjust her beliefs accordingly. The problem is known as the *logical omniscience problem.*[3]

This is a conceptual issue: the notion of belief embodied in the modal-logic-based epistemic logics is rather specific and it does not conform to many intuitions associated with our use of the word 'belief'.[4] The intuitions are numerous, unclear, and perhaps mutually inconsistent. However, one may try to explicate one's intuitions in a little more detail and provide appropriate formalisations. This has been done by many, which led to a number of sophisticated contributions to epistemic logic.[5]

One may argue that the 'pre-theoretical notion of belief' includes a crucial element ignored by the modal-logic-based approach, namely the agent's *active attitudes* towards the believed proposition. Thus, Fagin and Halpern [17] couple the true-in-every-alternative condition with α's *awareness* of the believed formula. Formally, an *awareness model* is a quadruple

$$M = \langle W, R, A, V \rangle \tag{2}$$

where W is the set of alternatives (or 'possible worlds'), R is a binary relation on W (the 'accessibility' relation) and V is a valuation. The crucial element is

[1] This approach originates in Hintikka's classic [21]. For a more recent overview, see [10], for example.

[2] To ensure that or discussion is as general as possible, we shall be using the notion of belief throughout the paper. However, some contexts will allow us to use the stronger 'true belief' and even 'knowledge'.

[3] The term has been coined by Hintikka, see [22]. For more details on the problem, see [18, Ch. 9].

[4] Hintikka [21] concludes his discussion of the problem in a similar vein. For a more recent incarnation of the idea, see [10].

[5] See [18, Ch. 9] for an overview.

A, a function from W to sets of formulas. Informally, $F \in A(w)$ means that α is aware of F at w. 'α believes that F' holds at a world $w \in W$ iff i) Rwv implies that F holds at v (α 'implicitly' believes that F) and, importantly, ii) $F \in A(w)$. The syntactic nature of the awareness function makes it clear that one may avoid the logical omniscience problem for every instance of (1). Since $A(w)$ may be arbitrary, it is always possible to construct a pointed model M, w such that 'α believes that F_i' holds at M, w for every $1 \leq i \leq n$, but $G \notin A(w)$.

However, one can argue that the syntactic flavour of the approach is, in fact, a shortcoming. The syntactical nature of the awareness function can be seen as depriving the approach from the capacity to offer a deeper *internal* justification of the respective failures of omniscience. One may strive for a logic of non-omniscient belief where the properties of belief are arrived at by using a subtler 'inner semantic mechanism'.

3 Awareness and Information

This section introduces the core notions of the paper. First, the information-based generalisation of the awareness framework is discussed on an intuitive level (3.1). After that, information models and related technical notions are defined (3.2).

3.1 Awareness Generalised

Note that propositions, represented by formulas, can be seen as a special case of *pieces of information*. A piece of information can be tentatively characterised as everything that can *corroborate* (give support to) a proposition.[6]

Example 1. Assume that during a murder trial the jury is shown a video of the defendant entering the victim's home around the established time of death and carrying something that could be the murder weapon. *The video itself* can be seen as a piece of information that corroborates the proposition 'The defendant is guilty'. The prosecution can be said to have made the jury aware of this piece of information by introducing it during the trial. A statement 'The defendant threatened to kill the victim on numerous occasions' of a witness is another possible piece of information that corroborates the same proposition.

[6] Unfortunately, we do not have space in this paper to provide a satisfactory philosophical analysis of the notions of information and corroboration. However, our being vague about these notions can be justified. First, depending on the academic discipline, distinct notions are associated with the term 'information', see [1]. Second, a similar ambiguity pertains to the often used 'agent' as well. However, our use of 'piece of information' is close to the standard use of 'signal', see [16,31]. 'Corroboration' can be seen as a generalisation of 'carrying' information: we do not adopt Dretske's [16] assumption that if a signal carries information that F, then F is the case. 's corroborates F' can be tentatively seen as being close to 'If accepted by an agent, s is likely to cause the agent's belief that F'.

The picture can be complemented by adding general *corroboration conditions*. For example, it may be stipulated that if a piece of information s corroborates a conjunction $F \wedge G$, then it corroborates both conjuncts F, G. In this manner, pieces of information can be endowed with 'logical character'.

Example 2. A perceptual image of my two hands can be seen as a piece of information that corroborates the proposition 'Both of my hands exist now'. The image then obviously corroborates *both* 'My left hand exists now' and 'My right hand exists now'.

Moreover, taking pieces of information into consideration opens door for considering *relations* on pieces of information in addition to the pieces themselves. These may be, in turn, called upon within corroboration conditions. Examples will be provided later on.

3.2 Information Models

Definition 1 (The basic epistemic language). *Let $\Phi = \{p_1, p_2, \ldots\}$ be a denumerable set of propositional variables and let $\mathcal{G}$ be a non-empty set ('agents'). The set of formulas of the basic epistemic language $L_{\mathcal{G}}(\Phi)$ is given by:*

$$F ::= p \mid \neg F \mid F \wedge F \mid F \vee F \mid F \rightarrow F \mid F \leftrightarrow F \mid \Box_\alpha F \tag{3}$$

where $p \in \Phi$ and $\alpha \in \mathcal{G}$. The set of $L_{\mathcal{G}}$-formulas will be denoted as $Form(L_{\mathcal{G}})$. Formulas $\Box_\alpha F$ are read 'α believes that F'. The Boolean fragment of $L_{\mathcal{G}}$ is the subset of $Form(L_{\mathcal{G}})$ consisting of formulas that do not contain occurrences of $\Box_\alpha$, for any $\alpha \in \mathcal{G}$.

Definition 2 (Information structure and L-structure). *An information structure is a couple*

$$\mathcal{I} = \langle I, \triangle \rangle \tag{4}$$

where I is a non-empty set ('pieces of information') and $\triangle$ is a set of relations on I. Let '$X \subseteq \mathcal{I}$' and '$s \in \mathcal{I}$' be short for '$X \subseteq I \in \mathcal{I}$' and '$s \in I \in \mathcal{I}$', respectively.

Let L be a language. An information L-structure is a couple

$$\mathcal{I}(L) = \langle \mathcal{I}, \Vdash \rangle \tag{5}$$

where $\mathcal{I}$ is an information structure and $\Vdash$ is a subset of $I \times Form(L)$ ('corroboration relation'). We shall use '$s \Vdash F$' instead of '$\langle s, F \rangle \in \Vdash$'.

A familiar special case of information structure are sets of atomic programs together with the program operators, known from propositional dynamic logic [23]. A special case of information L-structure are action models, known from dynamic epistemic logics [7,8].

Definition 3 (Information frames and models). *An information frame for $L_{\mathcal{G}}$ is a tuple*

$$\mathcal{F} = \langle W, R, \mathcal{I}, A \rangle \tag{6}$$

where W is a non-empty set ('possible worlds'), $R : \mathcal{G} \to \mathcal{P}(W \times W)$ is a function from the set of agents to binary relations on W, $\mathcal{I}$ is an information structure, $A : (\mathcal{G} \times W) \to (\mathcal{P}(\mathcal{I}) - \emptyset)$ is a function that assigns to every agent α and possible world w a non-empty set of pieces of information $A(\alpha, w) \subseteq \mathcal{I}$. We shall write '$R_\alpha wv$' instead of '$\langle w, v\rangle \in R(\alpha)$' and R_α shall be referred to as the 'α-accessibility' relation.

An information model for $L_\mathcal{G}$ is a tuple

$$\mathcal{M} = \langle \mathcal{F}, V, \Vdash \rangle \tag{7}$$

where $\mathcal{F}$ is an information frame for $L_\mathcal{G}$, $V : \Phi \to \mathcal{P}(W)$ is a valuation and $\Vdash$ is a corroboration relation.

The α-accessibility relations are interpreted in the usual way (implicit belief). $A(\alpha, w)$ is to be thought of as the set of pieces of information α is aware of at w. Our target notion of belief can be characterised as follows: α believes that F iff α implicitly believes that F and is aware of a piece of information that corroborates F.[7]

It is clear that awareness models are a special case of information models. Just consider information structures where $I = Form(L_\mathcal{G})$ and $\Vdash$ is the identity relation on I. Specific closure principles may be validated by incorporating extra corroboration conditions. For example, closure under $\wedge$-elimination corresponds to the condition:

– If $s \Vdash F \wedge G$, then $s \Vdash F$ and $s \Vdash G$

Definition 4 (Truth conditions). *The truth conditions of the Boolean formulas are as usual and we state only some:*

– $M, w \models p$ *iff* $w \in V(p)$
– $M, w \models \neg F$ *iff* $M, w \not\models F$
– $M, w \models F \wedge G$ *iff* $M, w \models F$ *and* $M, w \models G$

The target notion of belief is formalised in an obvious way:

– $M, w \models \Box_\alpha F$ *iff i)* $R_\alpha wv$ *implies* $M, v \models F$ *and ii) there is a* $s \in A(\alpha, w)$ *such that* $s \Vdash F$

The usual notions of validity in a model, frame and class of frames are assumed.

Note that, to ensure maximal generality, we have not introduced specific corroboration conditions yet. However, Sect. 4 points out that truth conditions in substructural models are natural candidates.

[7] The requirement that the awareness sets be non-empty is a useful idealisation, see the proof of Prop. 3.

4 Information and Substructural Logics

'Pieces of information' have been invoked within informal interpretations of the semantics of many substructural logics. For example, Kripke [24] describes the points of intuitionistic models as 'points in time (or "evidential situations"), at which we may have various pieces of information' [24, p. 98]. Urquhart's interpretation of his semi-lattice semantics for relevant implication [33] invokes 'pieces of information' together with specific operations on them. More recent interpretations [26,28] invoke a related notion of situation. Hence, there is hope that epistemic models where pieces of information are considered explicitly will be a natural 'meeting point' of epistemic and substructural logics.

Definition 5 (Substructural frames and models). *We shall use a slight modification of the standard definitions [29, Ch. 11]. A substructural frame is a tuple*

$$\mathfrak{F} = \langle P, \sqsubseteq, \bullet, C \rangle \tag{8}$$

where P is a non-empty set ('points'), $\sqsubseteq$ is a partial order on P ('informational containment'), $\bullet$ is a binary operation on P ('application') and C is a symmetric binary relation on P ('compatibility'). It is assumed that

- *If Cxy, $x' \sqsubseteq x$ and $y' \sqsubseteq y$, then $Cx'y'$*
- *if $x \bullet y \sqsubseteq z$, $x' \sqsubseteq x$, $y' \sqsubseteq y$ and $z \sqsubseteq z'$, then $x' \bullet y' \sqsubseteq z'$*

A substructural model for $L_{\mathcal{G}}$ is a couple

$$\mathfrak{M} = \langle \mathfrak{F}, \Vdash \rangle \tag{9}$$

where $\mathfrak{F}$ is a frame and $\Vdash$ is a relation between points and members of $Form(L_{\mathcal{G}})$ such that:

- *$x \sqsubseteq y$ and $x \Vdash p$ implies $y \Vdash p$*
- *$x \Vdash \neg F$ iff Cxy implies $y \nVdash F$ for all y*
- *$x \Vdash F \wedge G$ iff $x \Vdash F$ and $x \Vdash G$*
- *$x \Vdash F \vee G$ iff $x \Vdash F$ or $x \Vdash G$*
- *$x \Vdash F \to G$ iff $y \Vdash F$ and $x \bullet y \sqsubseteq z$ imply $z \Vdash G$, for all y, z*
- *$x \Vdash F \leftrightarrow G$ iff $x \Vdash F \to G$ and $x \Vdash G \to F$*

F entails G in $\mathfrak{M}$ iff $x \Vdash F$ implies $x \Vdash G$ for every $x \in \mathfrak{M}$. Entailment in frames and classes of frames is then defined in the usual way.

Substructural frames (models) clearly are a special case of information structures ($L_{\mathcal{G}}$-structures). Substructural frames correspond to a set of pieces of information together with a binary relation $\sqsubseteq$ of informational containment, a relation of compatibility C and an operation of application $\bullet$. Let us discuss these in reverse order.

Example 3. Application corresponds to 'taking two pieces of information together': the two pieces of information taken together can be seen as a 'new' piece of information. For example, two consecutive statements s, t by witnesses during a trial can be seen as a 'complex' piece of information $s \bullet t$, considered by the jury.

Example 4. As an example of two compatible pieces of information, consider a sworn statement of a witness to the effect that the defendant's car was parked somewhere far away from the crime scene around the time of the victim's death (s) and the video from Exam. 1 (t). The two are obviously compatible. Consequently, s does not corroborate the proposition 'The defendant is not guilty', $\neg F$, since it is consistent with t, a piece of information that corroborates F.

Proposition 1. *Let F be a member of the Boolean fragment of $L_{\mathcal{G}}$, $\mathfrak{M}$ a substructural model and x, y points of $\mathfrak{M}$. If $x \Vdash F$ and $x \sqsubseteq y$, then $y \Vdash F$.*

Hence, if x is informationally contained in y, then every Boolean formula that holds at x holds at y as well. The Proposition is a standard result in substructural logic and a simple consequence of Def. 5.

Example 5. Informational containment can be seen as a complex relation: $s \sqsubseteq t$ iff i) s is contained in t *and* ii) every F corroborated by s is corroborated by t as well. For an example of a piece of information contained in a 'larger' piece, consider two fingerprints found on a crime scene. The couple of prints can be seen as a piece of information containing the two single prints. It is plausible to assume automatically that if one of the prints corroborates a proposition (e.g. that a given suspect is guilty), then the couple does so as well. However, this is not plausible in general. For example, consider a sworn statement that the defendant was playing poker at a local casino at the estimated time of the murder (s) and, again, the video from Exam. 1 (t). The jury can take these together, i.e. consider $s \bullet t$. In a sense, both s, t are contained in $s \bullet t$. However, they are not so in the *informational sense*: while s alone can be said to corroborate $\neg F$, $s \bullet t$ cannot. The new piece of information t 'neutralised' the force of s.

Consequently, the corroboration condition for $F \to G$ makes sense: s can be said to corroborate $F \to G$ iff taking s together with any possible piece of information t that corroborates F results in $s \bullet t$ that corroborates G, and so does every u such that $s \bullet t \sqsubseteq u$.

Observe that the 'boxed' formulas $\Box_\alpha F$ have not received attention yet. To retain generality, we shall not provide truth conditions, but we will focus on a specific class of substructural models.

Definition 6 (Intended substructural models). *An intended substructural model for $L_{\mathcal{G}}$ is a substructural model for $L_{\mathcal{G}}$ such that*

$$s \Vdash F \textit{ only if } s \Vdash \Box_\alpha F \textit{ for every } \alpha \in \mathcal{G} \tag{10}$$

In intended models, boxed formulas behave somewhat like propositional atoms. Extension of Prop. 1 to the whole $L_{\mathcal{G}}$ within intended models is a trivial consequence of Def. 6. The clause (10) is included also for technical reasons that will become clear in Sect. 5.2. But it can be motivated independently as well: if s corroborates F, then s should be a sufficient reason to believe F. Of course, (10) can be dropped in case it is not considered intuitive enough, but we choose to keep it.[8] Another reason to keep the clause is that it allows for a natural way of dealing with common belief (Sect. 5.3).

5 Substructural Epistemic Logics

Substructural epistemic logics emerge as soon as we use substructural models as the information structures in information models.

5.1 Substructural Information Models

Definition 7 (Substructural information $\mathcal{C}$-frames and $\mathcal{C}$-models). *Let $\mathcal{C}$ be a class of substructural frames. A substructural information $\mathcal{C}$-frame is a tuple*

$$\mathbf{F} = \langle W, R, \mathfrak{F}, A \rangle \tag{11}$$

where W, R, A are as in Def. 3 and $\mathfrak{F}$ is a substructural frame such that $\mathfrak{F} \in \mathcal{C}$. Moreover, let us assume for technical convenience[9] that

$$s \in A(\alpha, w) \text{ and } s' \sqsubseteq s \text{ only if } s' \in A(\alpha, w) \tag{12}$$

A substructural information $\mathcal{C}$-model built on $\mathbf{F} = \langle W, R, \mathfrak{F}, A \rangle$ is a tuple

$$\mathbf{M} = \langle W, R, \mathfrak{M}, A, V \rangle \tag{13}$$

where W, R, A are as in Def. 3, $\mathfrak{M} = \langle \mathfrak{F}, \Vdash \rangle$ is an intended substructural model and V is a valuation.

The truth conditions of $L_{\mathcal{G}}$-formulas are those of Def. 4. The usual notions of validity in a model and a frame are assumed. F is $\mathcal{C}$-valid iff it is valid in every substructural information $\mathcal{C}$-frame. The set of $\mathcal{C}$-valid formulas shall be denoted as $\mathsf{K}(\mathcal{C})$.

The sets $\mathsf{K}(\mathcal{C})$ can be seen as basic information-based epistemic logics where the pieces of information are 'described' by the logic of $\mathcal{C}$. The actual choice of $\mathcal{C}$ will depend on the application.[10] However, since this paper is focused on the general framework, we shall not discuss such special cases here. We shall limit our discussion to a rather general observation instead.

Proposition 2. *F is $\mathcal{C}$-valid if (but not only if) i) F is a propositional tautology or ii) $F = \Box_\alpha G \to \Box_\alpha G'$, where G entails G' in $\mathcal{C}$.*

[8] There is a standard way of dealing with 'boxes' that invokes additional relations, see [29]. A different evidence-based approach that builds only on C and $\sqsubseteq$ is discussed in [12].

[9] See the proof of Prop. 4 in the appendix.

[10] For example, Sequoiah-Grayson [30] argues that, when modelling the flow of information in inference, associativity $s \bullet (t \bullet u) = (s \bullet t) \bullet u$, contraction $s \bullet s = s$ and other assumptions have to be rejected, leaving only weak commutativity $s \bullet t = t \bullet s$.

Hence, substructural epistemic logics in general respect propositional validity and belief is closed under $\mathcal{C}$-entailment.

Example 6. An example of such a closure principle would be closure under conjunction elimination:

$$\Box_\alpha(F \wedge G) \rightarrow (\Box_\alpha F \wedge \Box_\alpha G) \tag{14}$$

On the other hand, some of the more problematic closure principles are not valid.

Example 7. Examples of invalid closure principles include closure under conjunction introduction and Modus Ponens:

$$(\Box_\alpha F \wedge \Box_\alpha G) \rightarrow \Box_\alpha(F \wedge G) \tag{15}$$

$$\Box_\alpha(F \rightarrow G) \rightarrow (\Box_\alpha F \rightarrow \Box_\alpha G) \tag{16}$$

The construction of counterexamples is easy and the reader may try it as an exercise.

Closure under logical equivalence does not hold either, i.e. it is not the case that if $F \leftrightarrow G$ is $\mathcal{C}$-valid, then $\Box_\alpha F \leftrightarrow \Box_\alpha G$ is $\mathcal{C}$-valid as well. For example, counterexamples concerning the classical tautology $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ are easily constructed for most classes $\mathcal{C}$.

Consequently, substructural epistemic logics achieve the goal mentioned in Sect. 2: the non-omniscient properties of belief are explained by reference to the 'logical character' of pieces of information.

5.2 Factive and Introspective Models

Definition 8 (Factive frames and models). *A substructural information frame*

$$\mathbf{F} = \langle W, R, \mathfrak{F}, A \rangle$$

is factive iff every $R(\alpha)$ is reflexive on W. A substructural information model $\mathbf{M}$ built on $\mathbf{F}$ is factive iff $\mathbf{F}$ is factive. The set of formulas valid in every factive $\mathcal{C}$-frame will be denoted $\mathsf{T}(\mathcal{C})$.

Proposition 3. *$\Box_\alpha F \rightarrow F$ is $\mathbf{F}$-valid iff $\mathbf{F}$ is a factive frame. Consequently, $\Box_\alpha F \rightarrow F \in \mathsf{T}(\mathcal{C})$ for every $\mathcal{C}$.*

Hence, in the context of factive models and frames, $\Box_\alpha$ may be read in terms of 'true belief' or even 'knowledge'.

Definition 9 (Introspective frames and models). *A substructural information frame $\mathbf{F} = \langle W, R, \mathfrak{F}, A \rangle$ is introspective iff every $R(\alpha)$ is transitive on W and $R_\alpha wv$ implies $A(\alpha, w) \subseteq A(\alpha, v)$ for every α and w. A substructural information model $\mathbf{M}$ built on $\mathbf{F}$ is introspective iff $\mathbf{F}$ is introspective. The set of formulas valid in every introspective $\mathcal{C}$-frame will be denoted $\mathsf{K4}(\mathcal{C})$ and the set of formulas valid in every factive and introspective $\mathcal{C}$-frame will be denoted $\mathsf{S4}(\mathcal{C})$.*

Proposition 4. *$\Box_\alpha F \rightarrow \Box_\alpha \Box_\alpha F$ is $\mathbf{F}$-valid iff $\mathbf{F}$ is an introspective frame. Consequently, $\Box_\alpha F \rightarrow \Box_\alpha \Box_\alpha F \in \mathsf{K4}(\mathcal{C})$ for every $\mathcal{C}$.*

5.3 Common Belief

This section outlines a way to deal with common belief in the substructural epistemic framework. We will work with the standard construal of common belief as an infinite iteration of the 'everybody-believes-operator'. It is noted that common belief lacks some of the standard properties.

Definition 10 (Language with common belief). *The language with common belief $L^*_{\mathcal{G}}$ extends the basic epistemic language with a family of operators $\circledast_B$ for every $B \subseteq \mathcal{G}$. Formulas $\circledast_B F$ are read 'It is common belief in B that F'. Moreover, $\Box_B F$ is a shorthand for $\bigwedge_{\alpha \in B} \Box_\alpha F$, read 'every agent in B believes that F'.*

Definition 11 (Common belief information structures). *A B-sequence σ_B is a sequence of belief-operators $\Box_{\alpha_1} \ldots \Box_{\alpha_n}$ where $n \geq 1$ and every $\alpha_i \in B$. $\mathcal{I}(L^*_{\mathcal{G}})$ is a common belief information $L^*_{\mathcal{G}}$-structure iff it is the case that*

- *$s \Vdash \circledast_B F$ iff $s \Vdash \sigma_B F$ for every B-sequence σ_B*

A common belief $\mathcal{C}$-model is a substructural information $\mathcal{C}$-model where the information $L^*_{\mathcal{G}}$-structure is a common belief information $L^*_{\mathcal{G}}$-structure. $\mathsf{K}^*(\mathcal{C})$, $\mathsf{T}^*(\mathcal{C})$, $\mathsf{K4}^*(\mathcal{C})$ and $\mathsf{S4}^*(\mathcal{C})$ are sets of $L^*_{\mathcal{G}}$-formulas valid in every common belief $\mathcal{C}$-model, every factive, introspective and factive and introspective common belief $\mathcal{C}$-model, respectively.

Lemma 1. *If $s \Vdash F$, then $s \Vdash \sigma_B F$ for every B-sequence σ_B and every $B \subseteq \mathcal{G}$.*

Definition 12 (Group accessibility). *Let $B \subseteq \mathcal{G}$. Let a B-path from w to v be a sequence of couples $\langle w_1, w_2 \rangle, \ldots, \langle w_{n-1}, w_n \rangle$ such that $w_1 = w$, $w_n = v$ and every $\langle w_i, w_{i+1} \rangle \in R(\alpha)$ for some $\alpha \in B$. Let $R(B)$ ('B-accessibility') be a binary relation on W such that $\langle w, v \rangle \in R(B)$ iff there is a B-path from w to v.*

Definition 13 (Truth conditions for common belief). *The truth conditions for every $L^*_{\mathcal{G}}$-formula are specified by adding the following clause to Def. 4:*

- *$\mathbf{M}, w \models \circledast_B F$ iff i) $\mathbf{M}, w \models \Box_B F$ and ii) $R_B wv$ implies $\mathbf{M}, v \models \Box_B F$.*

Let us close the section by pointing out that two of the well known axioms for 'common knowledge' hold also for common belief in the substructural epistemic setting, if we limit our attention to factive frames (i.e. if we are studying 'common true belief' or 'common knowledge').

Proposition 5. *The following belong to $\mathsf{T}^*(\mathcal{C})$ for every $\mathcal{C}$:*

1. *$\circledast_B F \to (F \wedge \Box_B \circledast_B F)$ ('Mix')*
2. *$\circledast_B (F \to \Box_B F) \to (F \to \circledast_B F)$ ('Induction')*

However, other standard axioms, such as $\circledast_B$-closure under Modus Ponens and $\circledast_B$-necessitation, do not hold due to the specifics of the simple $\Box_\alpha$-belief.

6 Public Information Introduction

This section investigates into a generalisation of public announcements (not necessarily truthful). If we see formulas as special cases of pieces of information, then the action of publicly announcing a formula is a special case of publicly introducing a piece of information. Hence, it is interesting to look at the more general case.

Definition 14 (The announcement language). *Let* AI *('active pieces of information') and* $\mathcal{G}$ *('agents') be non-empty sets of labels. Formulas of the announcement language* $L^+_{\mathcal{G}}(\Phi, \mathsf{AI})$ *are constructed as follows:*

$$F ::= p \mid \neg F \mid F \wedge F \mid F \vee F \mid F \to F \mid F \leftrightarrow F \mid \Box_\alpha F \mid s : F \mid [+s]F \tag{17}$$

where $p \in \Phi$*,* $\alpha \in \mathcal{G}$ *and* $s \in \mathsf{AI}$*.*

Formulas $s : F$ are read 's corroborates F' and $[+s]F$ is read 'F is the case after the public introduction of s'. We shall not assume special corroboration conditions for formulas $s : F$ and $[+s]F$.[11]

Definition 15 (Information models for the announcement language). *We shall use the models of Def. 7 with the proviso that* $I \subseteq \mathsf{AI}$*. Validity of formulas in models, frames and classes of frames is defined in the usual way. Truth conditions for the 'basic epistemic fragment' of the announcement language are as before (Def. 4). Moreover:*

- $\mathbf{M}, w \models s : F$ *iff* $s \in \mathcal{I}$ *and* $s \Vdash F$
- $\mathbf{M}, w \models [+s]F$ *iff* $s \in \mathcal{I}$ *implies* $M^{+s}, w \models F$

where

$$\mathbf{M}^{+s} = \langle W^{+s}, R^{+s}, \mathcal{I}(L^+_{\mathcal{G}})^{+s}, A^{+s}, V^{+s} \rangle \tag{18}$$

such that

- $W^{+s} = W$, $\mathcal{I}(L^+_{\mathcal{G}})^{+s} = \mathcal{I}(L^+_{\mathcal{G}})$ *and* $V^{+s}(p) = V(p)$ *for all* $p \in \Phi$
- $R^{+s}_\alpha(w) = R_\alpha(w) - [\![\overline{s}]\!]_M$ *for every* α, w
- $A^{+s}(\alpha, w) = A(\alpha, w) \cup \{s\}$ *for every* α, w

where $R_\alpha(w) = \{v \mid R_\alpha wv\}$ *and* $[\![\overline{s}]\!]_{\mathbf{M}} = \{w \mid \mathbf{M}, w \models \neg F$ *for some* F *such that* $s \Vdash F\}$*.*

$\mathsf{K}^+(\mathcal{C})$ *is the class of* $L^+_{\mathcal{G}}$*-formulas valid in every information* $\mathcal{C}$*-model for* $L^+_{\mathcal{G}}$*.* $\mathsf{T}^+(\mathcal{C})$*,* $\mathsf{K4}^+(\mathcal{C})$ *and* $\mathsf{S4}^+(\mathcal{C})$ *are the classes of* $L^+_{\mathcal{G}}$*-formulas valid in every factive, introspective, and factive and introspective* $\mathcal{C}$*-model for* $L^+_{\mathcal{G}}$*.*

's corroborates F' holds in a pointed model only if s is 'active' in the model, i.e. if $s \in \mathcal{I} \subseteq \mathsf{AI}$. A public introduction of s inserts s into every $A(\alpha, w)$ and 'cuts off' the accessibility arrows leading to points where the negation of a formula corroborated by s holds. Such an introduction is 'persuasive' and 'monotonic':

[11] However, it might be plausible to assume that $s \Vdash [+t]F$ iff $s \bullet t \Vdash F$.

Lemma 2 (Persuasiveness and monotonicity). *The following are contained in* $\mathsf{K}^+(\mathcal{C})$*, for every* $\mathcal{C}$*:*

1. $s : F \to [+s]\Box_\alpha F$
2. $\Box_\alpha F \to [+s]\Box_\alpha F$

There is a stronger version of information introduction for which these properties do not hold. It is possible to add the assumption that Cst holds for every $t \in A^{+s}(\alpha, w)$. In other words, we could assume that the introduction of s results in 'removing' every t that is not consistent with s from the awareness set. For sake of simplicity, we shall not discuss this version in more detail here.[12]

Note that application of the standard 'reduction-axioms-technique' is seriously limited in the substructural epistemic framework. Importantly, there is no hope of being able to find an equivalent $L_\mathcal{G}$-formula for every $L_\mathcal{G}^+$-formula. The reason is explained by Exam. 7: it is possible that there are formulas $[+s]F$ and G such that $[+s]F \leftrightarrow G$ is valid, but $\Box_\alpha[+s]F \leftrightarrow \Box_\alpha G$ is not. However, variants of some of the well-known reduction axioms are still valid.

Proposition 6. *The following belong to* $\mathsf{K}^+(\mathcal{C})$*, for every* $\mathcal{C}$*:*

1. $s : G \to ([+s]p \leftrightarrow p)$
2. $s : G \to ([+s]\neg F \leftrightarrow \neg[+s]F)$
3. $[+s](F \wedge G) \leftrightarrow ([+s]F \wedge [+s]G)$
4. $[+s](F \vee G) \leftrightarrow ([+s]F \vee [+s]G)$
5. $[+s](F \to G) \leftrightarrow ([+s]F \to [+s]G)$
6. $[+s](F \leftrightarrow G) \leftrightarrow ([+s]F \leftrightarrow [+s]G)$
7. $s : G \to ([+s]\Box_\alpha F \leftrightarrow (s : F \vee \Box_\alpha F))$
8. $s : G \to ([+s]t : F \leftrightarrow t : F)$

Notice items 1., 2., 7. and 8.: the antecedent $s : G$ is necessary, since not every s is 'active' in every model.[13] To sidestep this, we could narrow our attention down to models where every piece of information expressible in the language is active.

Definition 16 (Full frames). *An information frame is* AI*-full iff* $\mathcal{I} = \langle \mathsf{AI}, \triangle \rangle$.

Corollary 1. *The following are valid in every* AI*-full information frame:*

1. $[+s]p \leftrightarrow p$
2. $[+s]\neg F \leftrightarrow \neg[+s]F$
3. $[+s]\Box_\alpha F \leftrightarrow (s : F \vee \Box_\alpha F)$
4. $[+s]t : F \leftrightarrow t : F$

A note on related work. Combinations of public announcements and information-based epistemic logics are widely studied within dynamic justification logics. However, there are notable differences between the justification-logic-based

[12] However, there is hope that working with both of these versions will yield interesting results concerning the relation of the present framework to the AGM belief revision theory, see [2].

[13] This fact partly justifies our inclusion of formulas $s : F$ into the announcement language. The other part of the justification is the fact that without such formulas, no interesting 'recursion implication' for formulas $[+s]\Box_\alpha F$ would be provable.

approaches and the approach of the present paper. Bucheli et al. [13,14] and Renne [27] combine justification logic with announcements, but the latter are classical formula announcements. Kuznets and Studer [25] combine formula announcements with evidence introduction, in that the announcement itself is considered as a new piece of evidence. The rich framework of Baltag et al. [9] deals with various versions of evidence dynamics, but does so only for the singe-agent case and the 'pieces of information' are considered from the viewpoint of justification logic, not substructural logic. Apart from the justification-logic-based approaches, an interesting contribution has been made by van Benthem and Pacuit [11], but they construe evidence in terms of sets of possible worlds and their announcements are formula-based as well.

7 Conclusion

Our primary goal in this paper was to explain that substructural logics are a natural part of information-based epistemic logic. This observation may stimulate productive collaborations between sub-fields of logic that have perhaps been thought of as rather remote from one another. The paper is an introductory outline and, consequently, there are many interesting directions for future work. First, we plan to concentrate on specific substructural epistemic logics: to explain their respective philosophical motivations in more detail and to provide axiomatisations. Second, as the present framework is rather general, its will be interesting to expound connections to the established formalisms. Third, the information-introduction-extensions of substructural epistemic logics deserve systematic attention: sound and complete axiomatisations are a natural goal, as is establishing connections with the well-known version of dynamic-epistemic logics. Moreover, there are many other dynamic extensions that have been left out of the present outline. In addition, it is interesting to dwell upon the 'philosophical background' of the present framework: one could formulate different readings of 'corroboration' and 'piece of information' and provide various versions of information-based logics built to fit the different readings. There is hope that this will result in non-trivial applications of the present framework in epistemology.

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Appendix: Proofs of Propositions

This technical appendix contains proofs of some of the Propositions stated in the paper.

Proof of Prop. 3. The implication from right to left is trivial. The converse implication is easily demonstrated in the usual way. Assume that $\mathbf{F}$ is not a factive frame. Hence, there are α and w such that $\neg R_\alpha ww$. Since $A(\alpha, w)$ is non-empty by Def. 3, we may choose an arbitrary $s \in A(\alpha, w)$. Now define a model $\mathbf{M}$ built on $\mathbf{F}$ such that $s \Vdash p$, $\mathbf{M}, w \not\models p$ and $R_\alpha wv$ implies $\mathbf{M}, v \models p$ for every v. It is obvious that $\mathbf{M} \not\models \Box_\alpha p \to p$.

Proof of Prop. 4. The implication from right to left is trivial. Again, the converse implication is easily demostrated in the usual way. Assume that $\mathbf{F}$ is not introspective. The assumption entails that 1) there are α, w, v, v' such that $R_\alpha wv$, $R_\alpha vv'$ and $\neg R_\alpha wv'$, *or* 2) there are α, w, v such that $R_\alpha wv$ and $A(\alpha, w) \nsubseteq A(\alpha, v)$.

Assume 1). Build a model $\mathbf{M}$ as follows. Choose an arbitrary $s \in A(\alpha, w)$ and set $s \Vdash p$. In addition, set $\mathbf{M}, u \models p$ for every u such that $R_\alpha wu$ and $\mathbf{M}, u' \not\models p$ for every u' such that $R_\alpha vu'$. It is plain that $\mathbf{M}, w \not\models \Box_\alpha p \to \Box_\alpha \Box_\alpha p$.

Now assume 2). There is a $s \in A(\alpha, w)$ such that $s \notin A(\alpha, v)$. Build a model $\mathbf{M}$ as follows. Let $t \Vdash p$ iff $t \in \{s' \mid s \sqsubseteq s'\}$ for all t. Moreover, let $t \nVdash p$ for every $t \in A(\alpha, v)$. This choice is possible due to (12) of Def. 7. Moreover, set $\mathbf{M}, u \models p$ for every u such that $R_\alpha wu$. It is plain that $\mathbf{M}, w \not\models \Box_\alpha p \to \Box_\alpha \Box_\alpha p$.

Proof of Prop. 5. Item 1. $\boxast_B F$ obviously entails F (Def. 8, 13 and Prop. 3). It remains to prove that $\boxast_B F$ entails $\Box_B \boxast_B F$. Now $\mathbf{M}, w \models \boxast_B F$ entails $\mathbf{M}, w \models \Box_\alpha F$ for every $\alpha \in B$ (Def. 13). The latter entails that there is a $t \in A(\alpha, w)$ such that $t \Vdash F$. Consequently, $t \Vdash \sigma_B F$ for every B-sequence σ_B (Lem. 1) and $t \Vdash \boxast_B F$.

Hence, it remains to prove that $R_\alpha wv$ and $\mathbf{M}, w \models \boxast_B F$ together imply $\mathbf{M}, v \models \boxast_B F$ for every v and $\alpha \in B$. Assume to the contrary. The assumption entails that a) $\mathbf{M}, v \not\models \Box_B F$ or b) $R_B vu$ and $\mathbf{M}, u \not\models \Box_B F$ for some u. However, both are impossible, since $R_B wv$ and $R_B wu$: consequently, the assumption entails that $\mathbf{M}, v \models \Box_B F$ and $\mathbf{M}, u \models \Box_B F$.

Item 2. The proof is virtually identical to the standard inductive proof of a similar claim in modal-logic-based epistemic logic [15, p. 37]. Assume that $\mathbf{M}, w \models \boxast_B (F \to \Box_B F) \land F$. We have to show that $\mathbf{M}, w \models \boxast_B F$, i.e. that $R^n_B wv$ entails $\mathbf{M}, v \models \Box_B F$ for every $n \geq 0$, where $R^0_B wv$ iff $w = v$ and $R^m_B wv$ iff v is reachable from w by a B-path of length m. The base case for $n = 0$ is trivial. Now assume that the claim holds for a specific m: there is a v such that $R^m_B wv$ and $\mathbf{M}, v \models \Box_B F$. To prove the claim for $m + 1$, pick an $\alpha \in B$ and a u such that $R_\alpha vu$. Now $\mathbf{M}, v \models \Box_\alpha F$ obviously entails $\mathbf{M}, u \models F$. But $R_B wu$ and, consequently, $\mathbf{M}, u \models F \to \Box_B F$. Thus, $\mathbf{M}, u \models \Box_B F$ as desired.

Proof of Prop. 6. Item 1. For every $\mathbf{M}, w$: $\mathbf{M}, w \models s : G \to [+s]p$ iff ($s \in \mathcal{I}$ and $s \Vdash G$) implies ($s \in \mathcal{I}$ and $\mathbf{M}^{+s}, w \models p$) iff ($s \in \mathcal{I}$ and $s \Vdash G$) implies ($s \in \mathcal{I}$ and $\mathbf{M}, w \models p$) iff $\mathbf{M}, w \models s : G \to p$. By propositional logic, $\mathbf{M}, w \models s : G \to ([+s]p \leftrightarrow p)$, for every $\mathbf{M}, w$.

Item 2. First, let us prove that $s : G \wedge \neg[+s]F$ implies $[+s]\neg F$. By Def. 15, $\neg[+s]F$ is equivalent to the conjunction of $s \in \mathcal{I}$ and $\mathbf{M}^{+s}, w \models \neg F$. The conjunction implies that $s \in \mathcal{I} \Rightarrow \mathbf{M}^{+s}, w \models \neg F$, i.e. that $\mathbf{M}, w \models [+s]\neg F$. The desired result follows by propositional logic. Second, let us prove that $s : G \wedge [+s]\neg F$ implies $\neg[+s]F$. The assumption $\mathbf{M}, w \models s : G \wedge [+s]\neg F$ is equivalent to the conjunction of $s \in \mathcal{I}$, $s \Vdash G$ and ($s \in \mathcal{I} \Rightarrow \mathbf{M}^{+s}, w \models \neg F$). The conjunction obviously entails $s \in \mathcal{I}$ and $\mathbf{M}^{+s}, w \not\models F$, i.e. $\mathbf{M}, w \models \neg[+s]F$.

Items 3. – 6. can be demonstrated by simple propositional reasoning. Item 7. One half of the result follows from Lemma 2. To prove the second half, assume that $\mathbf{M}, w \models s : G \wedge \neg s : F \wedge \neg\Box_\alpha F$. The first two conjuncts entail that $s \nVdash F$. The third conjunct entails that i) there is a v such that $R_\alpha wv$ and $\mathbf{M}, v \models \neg F$, or ii) there is no $t \in A(\alpha, w)$ such that $t \Vdash F$. Assume i). Since $s \nVdash F$, $R_\alpha^{+s}wv$ for the $\neg F$-world v. Consequently, $\mathbf{M}, w \not\models [+s]\Box_\alpha F$. Assume ii). Since, $s \nVdash F$, there is no $t' \in A^{+s}(\alpha, w)$ such that $t' \Vdash F$ and, consequently, $\mathbf{M}, w \not\models [+s]\Box_\alpha F$.

Item 8. can be proved easily by propositional reasoning and by using the fact that $\mathcal{I}(L_\mathcal{G}^+)^{+s} = \mathcal{I}(L_\mathcal{G}^+)$ (Def. 15).

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