Some Remarks on the History of Plasticity – Heinrich Hencky, a Pioneer of the Early Years

Otto T. Bruhns

Abstract. The history of material equations and hence the development of present material theory as a method to describe the behavior of materials is closely related to the development of continuum theory and associated with the beginning of industrialization towards the end of the 19th century. While on the one hand new concepts such as continuum, stresses and strains, deformable body etc. were introduced by Cauchy, Euler, Leibniz and others and mathematical methods were provided to their description, the pressure of industrialization with the need to ever newer, and likewise also reliably secure, developments has led to the fact that, next to the description of elastic behavior of solid bodies, more appropriate models for the description of plastic or elastic-plastic behavior were introduced. Upon this background, this chapter wants to introduce into the history of plasticity and likewise highlight the contributions of the Darmstadt graduate Heinrich Hencky who started his scientific career there 100 years ago.

Keywords: Plasticity, Prandtl-Reuss-theory, deformation theory, hypoelasticity, large deformation.

1 Introduction

Compared to the history of elasticity the plasticity theory is still relatively young, if its birth - perhaps somewhat arbitrary - is identified with the publication of the results of the French engineer Henri Tresca in 1868 [55][1]. Based on the observations of a series of experiments he had published a hypothesis according to which metals begin to flow when the largest shear stress reaches a critical value. We will come

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[1] Reference is also made to a previous publication [54].

E. Stein (ed.), *The History of Theoretical, Material and Computational Mechanics*,
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back to that point later. At that time Henri Tresca was so highly esteemed that his name was engraved as 3rd out of a total of 72 names on the outside of the first platform of the Eiffel Tower. This birth now dates back almost 150 years.

Despite this relatively short period, the theory of plasticity meanwhile has taken a dynamic development. Thus it would be helpful to differentiate between individual development phases:

1. The origins and basics - development until 1945,
2. The expansion of these basics to approximately 1980,
3. The present status and recent developments.

The first of the above phases is characterized by the pioneering efforts of a few researchers in the rapidly developing industry in Europe. Unfortunately, first results produced in France were hardly noticed. It took about 30 years until in the early 20th Century a small group of - primarily German-speaking - engineers and mathematicians adopted this topic. Among these outstanding persons were the two founding fathers of the Society of Applied Mathematics and Mechanics (Gesellschaft für Angewandte Mathematik und Mechanik - GAMM), Richard Edler von Mises and Ludwig Prandtl, but also Willy (or William) Prager and Heinrich Hencky from Darmstadt. With the specific fate of the latter we want to be concerned more in detail in this contribution[2]. Incidentally, this contribution is to remain limited here to the first of the three phases specified above.

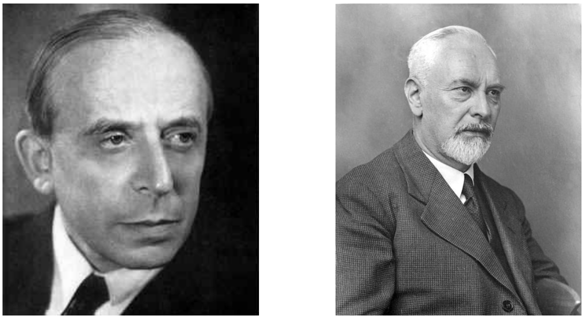

Fig. 1 Richard von Mises (1883-1953) and Ludwig Prandtl (1875-1953)

It is one of the unfortunate facts of the second part of this first phase, that it took place between the two world wars, with all their attendant destruction and distortions. When finally in Germany the Nazi regime came to power, the above mentioned dynamic development soon came to an end; many of the researchers involved were forced to leave their country.

Finally, the following remark appears to be still appropriate: Much of what has been initiated significantly by this – here simply called - "German school" and then,

[2] This text comprises a partly extended version of a lecture held at the GAMM Meeting 2012, March 26-30, Darmstadt, Germany.

Fig. 2 Stepped wheel mechanical calculator, Rheinmetall (Sömmerda)

due to the Nazi regime and its consequences, has been spread all over the world, is today occasionally not appreciated accordingly of its actual achievement. We believe that this attitude does not cope with the partly ground breaking developments of these years. These developments have been made by engineers and applied mathematicians and they could not wait, until the appropriate tools available to us today, were ready. They had to act, i.e.: Find solutions with the tools available at that time. To do this, it is worth remembering: There was no Finite Element Method (FEM), no powerful computers - today assumed as naturally existing - and initially not even simple mechanical calculators, as e.g. the stepped wheel calculator of Rheinmetall, designed in the 1930s (refer to Fig. 2). As a rule, one had to use tables of logarithms[3], as multiplications and divisions were carried out “by hand“.

2 The Origins and Basics – Development Until 1945

Let us mentally go back these 150 years to the beginning of industrialization in Europe. In many plants and constructions steel - or as it was called at that time “iron” - is used. To get an idea of the historical context, we want to recall briefly the following data:

- 1811, in Germany, Friedrich Krupp has founded the first cast steel factory in Essen (Ruhr Area),
- 1825, in England, the first public railway is inaugurated (Stockton & Darlington Railway Company, connecting Witton Park and Stockton-on-Tees in north eastern England, 40 km in length),
- 1835, the first railway is opened in Germany (Bavarian Ludwig Railway from Nuremberg to Fürth, 6 km in length).

An essential prerequisite for the operation of these railways was that from

- 1820, first rails were produced by rolling.

[3] A typical example is e.g. given with the handbook [1]. These tables contain sequences of mantissas of logarithms - preferably on the basis 10 - of natural numbers.

In this way, the rails until then created in casting processes could be replaced by the much smoother rolled profiles. Of course, one knew even at this early stage about the non-linear behavior of the material in use.

Consider a cylindrical specimen subject to a tensile load F, with length L and cross-sectional area A. Also, consider that for small elongations of this specimen the change of this area is negligible[4]. Then stress and strain may be defined as

$$\sigma = \frac{F}{A}, \qquad \varepsilon = \frac{\Delta L}{L_0} = \frac{L}{L_0} - 1, \tag{1}$$

where L_0 is the initial length. From a typical tensile test (refer to Fig. 3) it may be observed that up to a specific (yield) point σ_y the stress increases linearly. Beyond this point we observe a general non-linear monotonic increase during loading, and a linearly decreasing behaviour under reverse loading (unloading). Thus, from this process of loading and subsequent unloading it turns out that the total strain reached at final elongation may be split into a reversible part ε_r and an irreversible so-called plastic part ε_i.

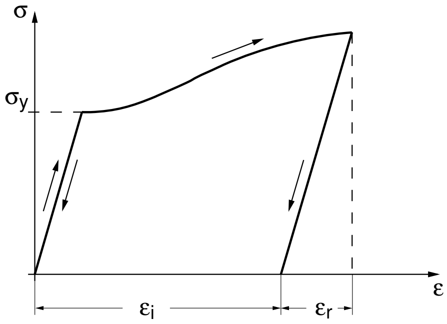

Fig. 3 Stress-strain curve of mild steel during a process of loading and subsequent unloading

What remains is the question how to characterize this behavior. With the above remarks we already have mentioned the two different ways of looking at the behavior of steel in the plastic range:

1. On the one hand, we have for example a vessel of the steam locomotive, which should be tight and safely withstand a given pressure. The necessary strength analyses - eventually carried out – however, were employed on the basis of the elasticity theory developed by Euler (continuum, 1750) and Cauchy (stress concept, 1822). This was admissible because thereby the structure has been assured against the reaching of the yield stress, and even in the case of exceeding this limit a "good-natured" failure - in conjunction with larger in time announcing deformations was observed. We will come back separately to the deviating case of possibly catastrophic failure in stability problems.

   In this first case, the strength properties of the steel are in major focus and accordingly the plastic behavior is interpreted as the behavior of a solid.

[4] This assumption does not hold for larger deformations, where the change of cross-sectional area has to be considered. We will come back to that point later.

2. On the other hand, we consider the rolling of rails as a typical forming process. Here, the material, which also then is often heated, is understood as a viscous fluid. Usually, we are interested in the forces that must be applied in such a rolling process. Similar problems might be given by the extrusion or even - on a very different time scale - the tectonic movements in the collision of two tectonic plates.

The general form of a linear elastic relation as the most simple form of a solid material is due to Cauchy[5] [3]

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}, \qquad \boldsymbol{\varepsilon} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{3\lambda+2\mu}\,\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right), \tag{2}$$

if common today's notations are used. Herein $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are stress and strain tensors, λ and μ are the two Lamé's constants and $\boldsymbol{I}$ is a unit tensor. A fully general expression for a relation combining stress and rate of deformation (strain rate) was first given by Poisson [38]

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu_v\boldsymbol{d} + \lambda_v\mathrm{tr}(\boldsymbol{d})\boldsymbol{I}. \tag{3}$$

Accordingly, λ_v and μ_v are the corresponding viscosities, p is the hydrostatic pressure and $\boldsymbol{d}$ the strain rate tensor (stretching), the symmetrical part of the velocity gradient. Special cases of linear viscous fluids were discovered by Navier, de Saint-Venant and Stokes [36; 47; 52][6]

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu_v\left[\boldsymbol{d} - \frac{1}{3}\mathrm{tr}(\boldsymbol{d})\boldsymbol{I}\right], \qquad \boldsymbol{d} - \frac{1}{3}\mathrm{tr}(\boldsymbol{d})\boldsymbol{I} = \frac{1}{2\mu_v}(\boldsymbol{\sigma} + p\boldsymbol{I}), \tag{4}$$

From the very beginning the development of plasticity has been in this conflict: Does the body under consideration behave more solid-like or like a fluid - and does this possibly also depend on the specific task to be solved?

We now return to Henri Tresca. Tresca wanted to know whether a simple criterion can be specified for achieving the flow state of his material. On the basis of numerous experiments on various metallic materials he concluded[7], that “in the plastic state of the solid, the largest shear stress has a fixed value" (in commonly used today's notations):

$$|\tau|_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = k, \tag{5}$$

with σ_1 the largest and σ_3 the smallest value of the principal stresses, and k the shear yield limit. This was the first yield condition. Using this condition the already 73-year-old Barré de Saint-Venant (1797-1886) [48] 1870 has presented his "five equations of hydro- stereodynamics" for the problem of plane deformations starting from the above described material behavior as a viscous fluid. In addition to balance

[5] Although the anagram ”ut tensio sic vis” of Hooke [22] may be interpreted as a first (uniaxial) step in this direction.

[6] Where in these cases $\mu_v = \text{const.}$ and in addition Navier adopted an incompressible material with $\mathrm{tr}(\boldsymbol{d}) = 0$, whereas the latter two introduced $3\lambda_v + 2\mu_v = 0$. Refer to [56].

[7] He probably may have resorted also to earlier works by Coulomb.

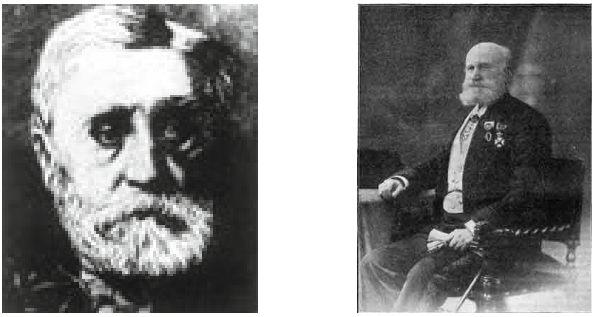

Fig. 4 Barré de Saint-Venant (1797-1886) and Maurice Lévy (1838-1910)

equations, the (assumed) incompressibility, as well as Tresca's yield criterion (5), this was a relationship of the form

$$\frac{d_{xx} - d_{yy}}{d_{xy}} = \frac{\sigma_{xx} - \sigma_{yy}}{\sigma_{xy}} . \tag{6}$$

In the same year, his student Maurice Lévy (1838-1910) [28] has transferred this representation to the general spatial problem. For this we can write

$$\boldsymbol{d} - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{d})\boldsymbol{I} = c\,\boldsymbol{\tau} , \qquad \boldsymbol{\tau} = \boldsymbol{\sigma} - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} , \tag{7}$$

with $\boldsymbol{\tau}$ the deviatoric stresses, and c is a simple proportionality.

Thus, the basics of a simple flow theory were established. Because of the associated mathematical difficulties[8] this theory, however, did not find any application. It took another 30 years before these ideas were taken up again. Some spectacular cases of damage as a result of stability failure may have contributed to this. Namely, unlike the above-mentioned strength analysis after exceeding the yield point, in the case of a stability failure, e.g. the buckling of a simply supported beam (Euler problem), the known elastic solution

$$\sigma_{crit} = EI\frac{\pi^2}{Al^2} , \qquad E = \mu\,\frac{3\lambda + 2\mu}{\lambda + \mu} \tag{8}$$

may lead to an unsafe solution - if plastic deformations occur. Herein the modulus of elasticity E occurs as the slope of the stress-strain curve. Corresponding spectacular failure cases then may have led to the need that already before the turn of the century modifications of the buckling load of elastic-plastic beams were discussed, where - albeit very simplified - the material properties in the plastic range were taken into account (refer to [4; 5; 24]). Incidentally, this discussion about the correct value of

[8] The interested reader will recognize that, formerly, it was almost impossible to solve systems of partial differential equations.

reduction of the modulus of elasticity in the above relationship (8)[9] - as a result of non-linearity of the problem, i.e., due to the fact that here the bifurcation load and the stability limit no longer coincide as in the elastic range - took another 50 years. This solution was finally delivered by Shanley [50; 51].

To simplify the flow theory outlined above, in 1913 v. Mises [31] replaced Tresca's yield condition (5) by

$$(\tau_1)^2 + (\tau_2)^2 + (\tau_3)^2 = 2k^2 \tag{9}$$

with τ_i the principal deviator stresses. The meaning of this condition and its relationship with a strength hypothesis already indicated by M.T. Huber[10] in 1904 [23] has been explained by Heinrich Hencky first 1924 [14]. The deformation law

$$\boldsymbol{d} - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{d})\boldsymbol{I} = c\,\boldsymbol{\tau}$$

of v. Mises has indeed taken over from de Saint-Venant and Lévy $(7)_1$. In the addressed work Hencky also specifies the until today used geometrical interpretation of the v. Mises and the Tresca yield conditions as surfaces in the 9-dimensional space of stresses (refer to Fig. 5)[11].

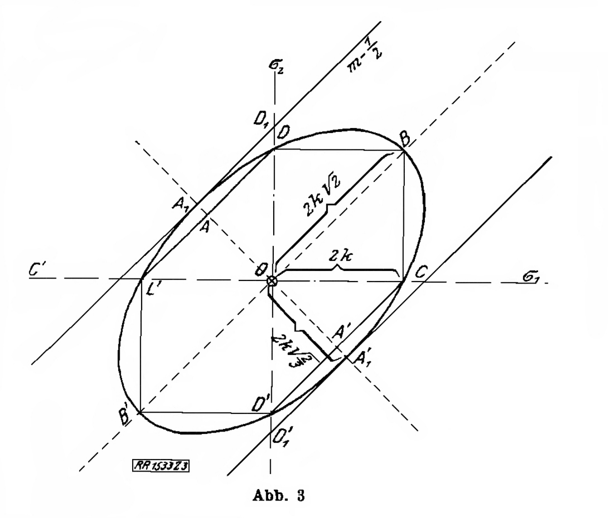

Fig. 5 Yield conditions according to Tresca and v. Mises, refer to Hencky, 1924 [14]. Copyright Wiley-VCH Verlag GmbH & Co. KGaA. Reproduced with permission.

[9] One might only contemplate the classical confusion regarding whether the tangent-modulus or the reduced- modulus theory gives an adequate description of the critical load, see e.g. [25; 6; 27]

[10] Refer to footnote 3 on page 327 of the Hencky paper [14].

[11] Figure 5 with a slight modification has been taken from the Hencky paper [14].

This system of partial differential equations at the time has been considered unsolvable – with a few pathological exceptions. In contrast, however, was the wish to develop practically manageable procedures that allowed to regain the progress of the material description in the calculation procedure and design rules. Thus, meaningful and yet reasonable simplifications had to be made. Today, we would say: There was a need for developing simplifying "models". In this sense, also the simplifications introduced by Prandtl [43] to consider the continuum as a) ideal-plastic or b) elastic-plastic body are to be understood. A further simplification is pursued in 1921 and 1923 by Prandtl [44] and Hencky [13], emerging from the following question: Do, possibly, exist special cases such that with the help of balance equations (equilibrium conditions) and the additional yield condition alone, solutions to the given problem can be found?

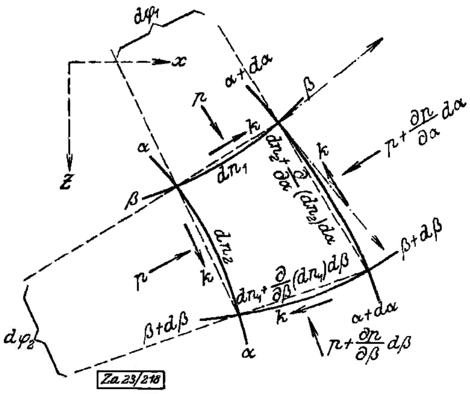

Abb. 2

Fig. 6 Orthogonal families of α and β-slip lines, refer to Hencky, 1923 [13]. Copyright Wiley-VCH Verlag GmbH & Co. KGaA. Reproduced with permission.

For the general spatial problem with its 4 conditions and 6 unknown stress components this obviously would be not the case. If we restrict our considerations, however, to axially symmetric problems, we note that only 3 unknown stress components are facing 3 conditions. The focus of the forthcoming work is therefore on the solution of such "statically determinate" cases. For problems of plane deformations, the yield conditions of Tresca and von Mises (5) and (9) coincide[12]. Based on a system of hyperbolic differential equations then two orthogonal families of characteristics can be introduced as geometrical places of directions of the principal shear stresses. Following suggestions from Prandtl [44] and Nádai [34] these curves according to their meaning in plastic behavior will be interpreted by Hencky [13] in 1923 as "slip-lines" (α– and β-lines, see Fig. 6). Inclined at 45° to this family, we

[12] We note that this is not the case for problems of plane stress.

find the directions of the largest tensile and compressive stresses[13]. Between these families, there exist certain phenomena which Hencky summarizes in 3 theorems. That same year, Prandtl [45] as well as Carathéodory and Schmidt [2] adopt these thoughts and complement them by graphical solution methods and numerous additional statements. 1925 v. Mises [32] finally summarizes this development: ". . . from (.) immediately follow the beautiful differential geometrical properties of the slip-lines discovered by Mr. Hencky."

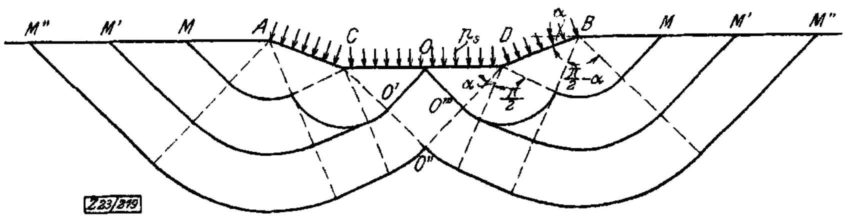

Abb. 3

Fig. 7 Half plane with prismatic indentation, refer to Hencky, 1923 [13]. Copyright Wiley-VCH Verlag GmbH & Co. KGaA. Reproduced with permission.

In the above work Hencky, as v. Mises elsewhere, indicates the limits of this approach which are given by the respective boundary conditions: Ideally here only stresses should be prescribed. However, this is generally not the case.

Thus, the slip line theory is developed. Besides v. Mises, Prandtl and Hencky especially Hilda Geiringer and Willy Prager [8] contribute to the spreading of this theory.

Who was actually this Heinrich Hencky[14] from (at the time) Delft, who together with v. Mises and Prandtl has published numerous contributions to the development of the theory of plasticity in the ZaMM[15] (Zeitschrift für angewandte Mathematik und Mechanik), at that time the leading journal of mechanics?

Heinrich Hencky was born in Ansbach (Bavaria), Germany on the 2nd of November 1885 as elder son of a Bavarian school administrator. After the early death of his father and after finishing the senior high school in Speyer, he studies civil engineering at the TH (Technische Hochschule) in Munich from 1904-1908; at the same college his 4 years younger brother Karl Georg follows him, studying mechanical engineering (1908-1912). After his military service in 1908/09 with the 3rd Pioneer Battalion in Munich and a short activity as engineer for the Alsacian Railway in Strasbourg (1910-1912), in 1912 he moves to the Grand Ducal TH (Großherzogliche Technische Hochschule) in Darmstadt as "design engineer and assistant for engineering sciences". In 1913 he receives his doctorate of engineering from the

[13] The construction of a typical field of slip-lines is shown in Fig. 7 for a prismatic indentation into a half-plane.

[14] We here also refer to the sensitive article of Tanner and Tanner [53] where especially Hencky's pioneering work on rheology is underlined.

[15] Now ZAMM.

Fig. 8 Heinrich Hencky (1885-1951) and Boris G. Galerkin (1871-1945)

TH Darmstadt with a thesis on the numerical calculation of stresses in thin plates [10].

After completing his doctorate he seeks for a new position in the field of railway engineering[16] and in 1914 moves to Kharkov (Kharkiv) in the Ukraine, an emerging industrial and commercial center. The beginning of World War I and the revolutionary upheavals in Russia procure him his first unpleasant experience. His career is abruptly terminated. Like many others, he is interned 1915-1918 in the Urals region. Although during this time he met his later wife Alexandra Yuditskaya, this period must have had a decisive impact. After the war ended the Hencky family is sent back to Germany. His attempt to restart as test engineer in Warnemünde (on a project to develop a seaplane) fails. As a result of the general demobilization, the project had to be stopped.

1919 he is again ”assistant for engineering sciences” at the TH Darmstadt. During this time he turns to the treatment of stability problems of elastic structures. Within this new subject he also wrote his habilitation treatise [11]. In 1920 he moves to the mechanical department of the TH of Dresden, first as assistant, and after acceptance of his habilitation in 1921 as lecturer. He is striving for a professorship. If one may believe the rumors, this, however, failed due to "not convincing achievements in teaching". Perhaps, his occasional reticence or shyness may have been an obstacle on this way. In this Dresden period he also reflects on possible relationships between philosophy and the description of nature [12]. Anyway, in 1922 he moves to the Technical University of Delft (at the chair of Cornelis B. Biezeno) - hoping to get a permanent professorship there. After all he is at the age of 37 and has a wife and two children.

In Delft Hencky probably wrote his most significant works - we come back to that in a moment. This, however, seems to have no influence on his position. In Delft, he all the time remains lecturer. It seems only too understandable that this situation must have affected his relationship with Biezeno. Therefore, in 1929 he leaves Delft with his family towards the USA. Biezeno indeed believes that it was merely a temporary research period; Hencky, however, is looking for a permanent job (he is now at the age of 44). In the summer of 1930, he then becomes associate professor at

[16] In those days one of the most attractive branches in civil engineering.

MIT in Cambridge, Massachusetts. This position, again, is not permanent. In 1932 the MIT is reorganized, his former advocate has died and Hencky is no longer employed. Here again, if we can trust the sources, he was regarded as ”too theoretical” in a department of mechanical engineering, mainly interested in practical problems. In the following years he tried to survive as a consultant - interrupted by a short job at Lafayette College in Easton, Penn., with E.C. Bingham.

What should he do? In Germany, he could not find any kind of work, as in 1935 an offer arrived from Boris Galerkin, whom he knows from his Delft period. After careful consideration – in fact he probably had no choice – 1936 he accepts this proposition as a professor of engineering mechanics at Kharkov Polytechnic Institute and later at the Institute of Mechanics of the Lomonosov University of Moscow with A.A. Ilyushin[17]. Although he now has an adequate position, in his heart he must have felt as a prisoner. Whether he expressed this accordingly and therefore is "fallen from grace" can not be ascertained anymore. In any case, 1938, he has to leave the Soviet Union with his family within 24 hours.

Back in Germany, he gets help from his brother Karl Georg, who meanwhile was extraordinary professor (apl. Professor) at RWTH Aachen and now holds a senior position at the IG Farben in Ludwigshafen, to find a position at MAN company in Gustavsburg. Here he remains, suspiciously observed by the ”security service” (SD), the intelligence agency of the SS, but under the protection of his supervisor at MAN (Dr. Richard Reinhardt) until the end of the war and then until his retirement in 1950. On the 6.07.1951, he died in a climbing accident at the age of 65.

Back to historical development. In the above mentioned paper from 1924 [14], where Hencky explained the v. Mises yield condition and interpreted it in a today still usual way as a hypothesis, according to which the boundary to plasticity is described by the elastic shear strain energy, he also developed a simple constitutive law for elastic as well as elastoplastic behavior, specifying a relationship between stresses and strains. As he mentions in his introduction, he hereby revisits an approach by Haar and v. Kármán [9], whose meaning seems to be not properly recognized.

For this purpose he assumes the specific complementary energy

$$A = \frac{1}{4\mu}\left(\sigma_{ik}\sigma_{ik} - \frac{9\lambda}{2+\lambda}\,\sigma_m^2\right) = \frac{1}{4\mu}\left(S_2 - \frac{\lambda}{2+\lambda}\,S_1^2\right), \qquad (10)$$

with $S_1 = \mathrm{tr}(\boldsymbol{\sigma})$, $S_2 = \mathrm{tr}(\boldsymbol{\sigma}^2)$ and the limit of elastic behavior, which here is specified with the condition

$$\Phi = T_2 - 2k^2\,, \qquad T_2 = \mathrm{tr}(\boldsymbol{\tau}^2) = S_2 - \frac{1}{3}\,S_1^2. \qquad (11)$$

[17] Unfortunately, informations about Hencky’s second stay in the Soviet Union are very rare. This is certainly due to the secrecy of those days. It is, however, believed that with the help of Galerkin Hencky was hired to improve the Soviet Union lightweight (airplane) construction. His deformation theory could contribute a lot to this matter, and it is known that Ilyushin later was very much impressed by this theory.

Hencky now solves the variational problem by seeking with

$$W = A + Lu + Mv + Nw + \varphi\Phi \tag{12}$$

for an extremum of $\int W \,\mathrm{d}V$. L, M and N herein are the balance equations as auxiliary conditions. Thus, the bounding (yielding) condition $\Phi = 0$ is multiplied by a local function φ and then added to the elastic complementary energy. As a result, he receives

$$\boldsymbol{\varepsilon} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} = \frac{1+\varphi}{2\mu}\boldsymbol{\tau}, \qquad \mathrm{tr}(\boldsymbol{\varepsilon}) = \frac{1}{3K}\mathrm{tr}(\boldsymbol{\sigma}), \quad 3K = 3\lambda + 2\mu \tag{13}$$

a model very similar to the elasticity law $(2)_2$, where φ is a still undetermined Lagrange parameter, and K is the bulk modulus. For $\varphi = 0$, this law changes into that of the elastic material. The compression is purely elastic and at plastic behavior the shear modulus μ is reduced by $(1+\varphi)$, the material thus becomes ”softer”.

Thus, for the first time it is possible to formulate a constitutive law to describe elastoplastic behavior. This formulation later referred to as “deformation theory” is rapidly accepted, even if it soon meets its limits: A neutral change of stresses, as it for instance occurs in non-proportional loading, cannot be reflected. For many so-called “proportional” problems, however, it represents not only the first but also a very simple method. It should be noted that Hencky in this development has assumed that a body under increasing load will be deformed first elastically and then plastically after having reached the yield point. In the interior of the structure, however, still remains a so-called “elastic core”. This corresponds to the above mentioned concept of plastic behavior as that of a solid material.

In the case that the considered body, under further load increase, merges into a “free flow”, Hencky in a paper [16] of 1925 reverts to Lévy’s approach (7), relating the stresses with strain rates, thus describing plastic flow as a behavior of a fluid.

Apparently independent[18] from a work by L. Prandtl [43], where already elastic deformations have been considered in a plane problem, the Hungarian András (Endre) Reuss in 1930 connects this Saint-Venant/Lévy approach (7) with the description of elastic behavior. For this purpose, like Hencky, he emanates from the v. Mises yield condition and obtains on a comparable way a constitutive law

$$\boldsymbol{d} - \frac{1}{3}\mathrm{tr}(\boldsymbol{d})\boldsymbol{I} = \frac{1}{2\mu}\dot{\boldsymbol{\tau}} + \lambda\boldsymbol{\tau}, \tag{14}$$

that with λ still contains a yet undetermined function, and

$$\mathrm{tr}(\boldsymbol{d}) = \frac{1}{3K}\mathrm{tr}(\dot{\boldsymbol{\sigma}}) \tag{15}$$

for its (purely elastic) compressible part. Thus, the basic version of the nowadays commonly used Prandtl-Reuss theory is introduced.

[18] We refer to a footnote of A. Reuss in [46] where it is stated that while elaborating his 1930 paper (Berücksichtigung der elastischen Formänderung in der Plastizitätstheorie, Z. angew. Math. Mech. 10, 266–274, 1930), the lecture of Prandtl was not known to him.

Fig. 9 Hilda Geiringer-v. Mises (1893-1973) and Willy Prager (1903-1980)

Many generalizations of this theory are given shortly later. So H. Geiringer and W. Prager [8] in addition to the yield condition $\Phi = 0$ introduce a second condition $H = 0$ as flow potential. The self-evident case $\Phi = H$ then is called an associated theory and as with Hencky's deformation theory a normality rule can be derived from the potential property

$$\boldsymbol{d}_p = \lambda \,\frac{1}{2}\,\frac{\partial \Phi}{\partial \boldsymbol{\tau}}\,. \tag{16}$$

For the general case of elastoplastic deformations, the yet undetermined parameter λ can be eliminated with the aid of the so-called condition of consistency, e.g. $\dot{\Phi} = 0$. This idea stems from H. Geiringer [7].

Also, the hardening of the material is taken into account by modifying the yield condition. First steps towards the "isotropic" hardening adapt the function k of the uniaxial tensile test for corresponding experiments and arrive at, e.g. the following proposals:

$$k = k_0 + F(J_1), \quad J_1 = \sqrt{\int \boldsymbol{\sigma} : \boldsymbol{d}_p \, dt}\,, \tag{17}$$

$$k = k_0 + F(J_2), \quad J_2 = \int \sqrt{\mathrm{tr}(\boldsymbol{d}_p^2)\, dt}\,, \tag{18}$$

where eq. (17) is introduced by Schmidt [49], and eq. (18) by Odqvist [37].

Approaches to the description of the "kinematic" hardening are derived from Reuss and Prager [39][19] in the form

$$\Phi = (\boldsymbol{\tau} - c\,\boldsymbol{\varepsilon}_p) : (\boldsymbol{\tau} - c\,\boldsymbol{\varepsilon}_p) - 2k^2 \tag{19}$$

and allow to take into account the Bauschinger effect. Then, the first evolution equation for this kinematic hardening originates from Melan [30]

$$\Phi = (\boldsymbol{\tau} - \boldsymbol{\alpha}) : (\boldsymbol{\tau} - \boldsymbol{\alpha}) - 2k^2\,, \quad \dot{\boldsymbol{\alpha}} = C\boldsymbol{d}_p\,. \tag{20}$$

[19] We refer to footnote 4 on page 79 of [39].

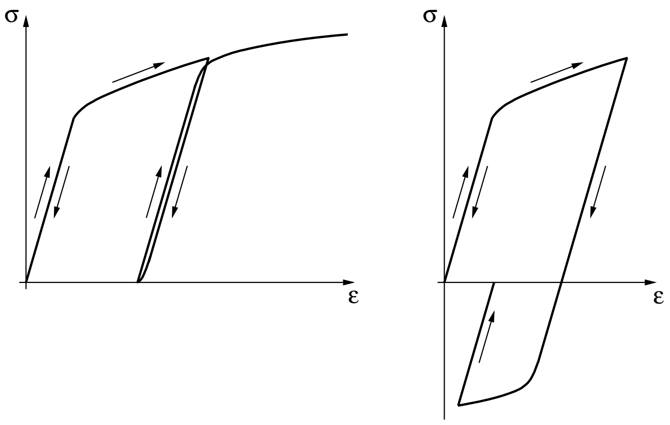

Fig. 10 Tensile test, cyclic loading with reloading in different directions: same direction (left Figure) and reverse loading expressing the Bauschinger effect (right Figure)

The debate as to what might be the “better” theory, starts of course immediately and drags on until the 1950’s, 1960’s[20]. At the beginning, it even seems as if the deformation theory of H. Hencky and A. Nádai[21] has advantages compared to the Prandtl-Reuss theory. K. Hohenemser [21] and W. Prager [40] for example indicate that in comparison with numerous experiments the deformation theory can reflect these results better. Moreover, in the aforementioned dispute about the “correct” buckling load in the range of plastic deformations, this deformation theory at first provides the “better” results. Finally, one also must recognize that in these pre-war years, e.g. in aircraft construction, the deformation theory simply prevails in practical applications because of its easy handling.

It is noticeable that until the 1930’s in the mathematical representation of the so far considered contributions, we find Cartesian coordinates, small deformations, and all relations written in detailed component notation. In his 1925 paper [15], Hencky is one of the first to introduce the tensor analysis into mechanics, “to avoid the maze of formulas, that so far has prevented from calculating finite deformations...” as he says. To this end, in this work he also for the first time introduces convective coordinates. Maybe he is thus too far ahead of the times.

Hencky now turns increasingly to the issues of finite deformations and thus in [17; 18; 19] introduces first of all a logarithmic strain measure[22], which is the only one to allow for a correct superposition and, moreover, the only one capable to allow for a physically meaningful description of a total compression[23],

[20] We e.g. refer to Prager [41] and the discussion between B. Budiansky and W. Prager in [42].

[21] A. Nádai with his book [35] has very much contributed to its publicity.

[22] This strain is sometimes called natural strain or - simply - logarithmic strain. Here, we will use the term Hencky strain $\boldsymbol{h}$.

[23] Such a compression to ”zero length” can be accompanied - think of an elastic body - only with an infinitely large force.

$$\boldsymbol{h} = \frac{1}{2} \ln \boldsymbol{b}, \quad \boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}. \tag{21}$$

$$\boldsymbol{h} = \boldsymbol{e}^{(0)} = \frac{1}{2} \ln \boldsymbol{b} = \frac{1}{2} \sum_{\sigma=1}^{n} (\ln \chi_\sigma) \boldsymbol{b}_\sigma. \tag{22}$$

where $\boldsymbol{F}$ is the deformation gradient and χ_σ and $\boldsymbol{b}_\sigma$ are n distinct eigenvalues and eigenprojections[24], respectively, of the left Cauchy-Green tensor $\boldsymbol{b}$[25]. Moreover, $\boldsymbol{e}^{(0)}$ is an Eulerian strain of the family of the Seth-Hill-Doyle-Ericksen strains ($m = 0$)

$$\boldsymbol{e}^{(m)} = \frac{1}{2m} (\boldsymbol{b}^m - \boldsymbol{I}). \tag{23}$$

Hencky introduces the Lagrangian and the Eulerian descriptions and discusses in this context the importance of time derivatives occurring in the relevant constitutive laws. For the Lagrangian analysis these will be understood as material time derivatives. In an Eulerian description, which he prefers for physical reasons, he notes that the time derivatives must be independent of the respective rigid-body rotation - or simply objective in today's notation. In [18] he therefore replaces the specified time derivative of the stress tensor by

$$\dot{\boldsymbol{\sigma}} \quad \Rightarrow \quad \overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{w} - \boldsymbol{w}\boldsymbol{\sigma}. \tag{24}$$

Herein, the spin tensor (vorticity) $\boldsymbol{w}$ has been introduced as skew-symmetrical part of the velocity gradient $\boldsymbol{l}$

$$\boldsymbol{w} = \frac{1}{2}\left(\boldsymbol{l} - \boldsymbol{l}^{\mathrm{T}}\right), \quad \boldsymbol{d} = \frac{1}{2}\left(\boldsymbol{l} + \boldsymbol{l}^{\mathrm{T}}\right), \quad \boldsymbol{l} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}. \tag{25}$$

Unfortunately, the original work by Hencky [18] contains a small error. Instead of the above spin tensor he used an alternative definition, which differs by a minus sign. Due to this deviation, however, his derivative (24) looses its objectivity[26].

We immediately recognize the so-called Jaumann derivative, previously already discussed in this context by M.S. Zaremba [60] and G. Jaumann [26].[27]

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[24] In numerous discussions Hencky is confronted with the difficulties of this measure for problems of non-principal axes. He then always refers to a possible transformation into principal axes. He also notes that this approach is quite common among technologists (refer e.g. to P. Ludwik, [29]). Nevertheless, this measure does not prevail. Knowing the today's computational possibilities, however, we should overcome this position and always use the logarithmic Hencky strain measure in the description of finite deformations.

[25] In today's notations.

[26] This may explain why proposition (24) is criticized by Truesdell in [56; 57], refer also to [53].

[27] It is not delivered whether Hencky knew these works. Rather not, we suspect. Otherwise he certainly would have quoted them. The today occasionally widespread habit, to quote only the works of the own school, has come up later.

But even this - from today's perspective - important statement is ignored. Until much later–after the end of World War II - these ideas are taken up again, e.g. by Oldroyd, Hill, Rivlin, Truesdell, and others. W. Prager e.g. in a work of 1960 refers to Jaumann: "Jaumann's work does not seem to be well known: the definition ... is frequently used in the recent literature without reference to Jaumann".

The perception that even in this accordingly corrected Prandtl-Reuss law still errors can occur, is generally attributed to C.A. Truesdell. For purely elastic processes, e.g. during unloading of a plastically deformed material, the constitutive law formulated in rates of stresses and deformation (which not until later is denoted hypoelastic by Truesdell [57]) does not have the properties of an elastic body. This can be shown easily if a circular process is calculated and thereby at the end an accumulated dissipation is observed. But even this can already be read in the work of G. Jaumann, who then denotes his rate material a material with fluxes (Fluxionen). He states that the behavior of this rate material is indeed not elastic and only in the limit of infinitesimal deformations changes to that of an elastic body. The full solution of this ”defect” is given only in recent times by introducing the so-called logarithmic rate [58] to replace the rate of stress in the Prandtl-Reuss law. This allows for an exact integration of these rate equations to reflect elastic behavior [59].

Of course, one could ask where actually the contributions of materials science and materials physics remain, because naturally plastic behavior results primarily from gliding as a consequence of dislocation movement. In the 1920's, Bridgman had managed to produce single crystals. Subsequently, Taylor and Elam, Schmid and Sachs conducted experiments on such mono-crystalline metals. Sachs (1928) and later Taylor (1938) also have determined yield stresses for polycrystalline material by averaging the values of single crystals. As a result of the works of Egon Orowan, Michael Polanyi and Geoffrey Ingram Taylor, 1934 may be considered the birth year of the dislocation theory. The fundamental work of Johannes Martinus Burgers followed in 1939.

And there is also a work by v. Mises [33] on the yield behaviour of single crystals and a work by Hencky from his time at MIT [20], which begins as follows: ”The behavior of metals in the inelastic states cannot be explained by theories which do not assume a microstructure.” These pioneering works contain a proposal for the description of polycrystalline metals based on a statistical method.

In this first phase, however, such considerations had no further influence on the development of the plasticity theory. Yet the numerical methods and the associated high-performance computers were missing. The plasticity theory was a purely phenomenological one and remained so until the 70's and 80's of the last century.

At the beginning, we stated that research in the theory of plasticity theory took place primarily in German-speaking countries, which among other things can be detected from the fact that the Journal of applied Mathematics and Mechanics ZaMM was the leading journal of mechanics at the time. Certainly, the question comes to our mind, how this situation has changed due to political changes in Germany. Consider the year 1938: Most of the aforementioned persons were Jewish or simply of Jewish descents and as such no longer safe in Germany. Von Mises, 1933 has emigrated to Turkey - although he initially as a highly decorated veteran of World War

I and respected test pilot thought to be safe from reprisals. Via Istanbul, in 1939, he then attended Harvard University, Cambridge.

Prager has studied at the TH in Darmstadt, made his PhD in 1926, and remained there as lecturer (1927-1929). In 1929 he became lecturer in Göttingen, and then in 1932 – as at the time youngest professor in Germany – he went to Karlsruhe. 1933, he also emigrated to Turkey as a professor of theoretical mechanics in Istanbul. In 1940, he left Istanbul for Brown University, Providence.

Hilda Geiringer was assistant to von Mises. In 1927, she habilitated in Berlin where she was a lecturer - incidentally after a short marriage as a single child's mother. After Hitler came to power, however, she was dismissed and 1934 followed von Mises to Turkey - with a small detour over Brussels - and later in the United States, where in 1944 she became a professor at Wheaton College, Norton. At that time Hencky was already in the Soviet Union and was close to his deportation.

The Hungarian Arpád Nádai (1883-1963) and Theodore v. Kármán (1881-1963) both have studied first at the Technical University of Budapest. Nádai then went to the Technical University of Berlin and received his PhD in 1911; in 1918 he went to Göttingen to Prandtl and became a professor there. In 1927 he finally became the successor of Stepan Timoshenko at Westinghouse, Philadelphia. Von Kármán 1906 went to Göttingen to Ludwig Prandtl and Felix Klein, habilitated there, and in 1913 became professor at the RWTH Aachen. He also should be dismissed from his office, however, anticipating his release, 1934 he emigrated to the United States at Caltech, Pasadena.

The third Hungarian was by the way András Reuss (1900-1968), but also Orowan (1902-1989) and Polanyi (1891-1976) were Hungarian, who made their PhD in Berlin and Karlsruhe. Reuss never had left his home country, Orowan and Polanyi went in time via Birmingham at the MIT and to Manchester, respectively.

The hitherto fruitful research in the surrounding of the two nuclei Richard von Mises and Ludwig Prandtl had so definitively lost some of its most important figures; on the horizon, World War II announced itself in the course of which large parts of Europe and Asia should sink into wrack and ruin.

References

[1] Bruhns, C.: Neues logarithmisch-trigonometrisches Handbuch auf sieben Decimalen, 13 edn. Bernhard Tauchnitz, Leipzig (1920)

[2] Carathéodory, C., Schmidt, E.: Über die Hencky-Prandtlschen Kurven. Z. angew. Math. Mech. 3, 468–475 (1923)

[3] Cauchy, A.L.: Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques. Bull. Soc. Philomath., 9–13 (1823)

[4] Considère, A.: Résistance des pièces comprimées. Congr. Int. des Procédés de Constr. 3, 371 (1889)

[5] Engesser, F.: über die Knickfestigkeit gerader Stäbe. Z. Arch. Ing-Ver. 35, 455 (1889)

[6] Engesser, F.: Über Knickfragen. Schweiz. Bauzeitung 26, 24–26 (1895)

[7] Geiringer, H.: Fondements mathématiques de la théorie des corps plastiques isotropes. Mém. Sci. Math., Gauthier-Villars, Paris 86, 19–22 (1937)

[8] Geiringer, H., Prager, W.: Mechanik isotroper Körper im plastischen Zustand. Ergebnisse der Exakten Naturwissenschaften 13, 310–363 (1934)
[9] Haar, A., von Kármán, T.: Zur Theorie der Spannungszustände in plastischen und sandartigen Medien. Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl, 204–218 (1909)
[10] Hencky, H.: Der Spannungszustand in rechteckigen Platten. Oldenbourg, München und Berlin (1913)
[11] Hencky, H.: Über die angenäherte Lösung von Stabilitätsproblemen im Raum mittels der elastischen Gelenkkette. Der Eisenbau 11, 437–452 (1921)
[12] Hencky, H.: Über die Beziehungen der Philosophie des 'Als Ob' zur mathematischen Naturbeschreibung. Annalen der Philosophie 3(2), 236–245 (1923)
[13] Hencky, H.: Über einige statisch bestimmte Fälle des Gleichgewichts in plastischen Körpern. Z. Angew. Math. Mech. 3, 241–251 (1923)
[14] Hencky, H.: Zur Theorie plastischer Deformationen und der hierdurch im Material hervorgerufenen Nachspannungen. Z. Angew. Math. Mech. 4, 323–334 (1924)
[15] Hencky, H.: Die Bewegungsgleichungen beim nichtstationären Fließen plastischer Massen. Z. Angew. Math. Mech. 5, 144–146 (1925)
[16] Hencky, H.: Über langsame stationäre Strömungen in plastischen Massen mit Rücksicht auf die Vorgänge beim Walzen, Pressen und Ziehen von Metallen. Z. Angew. Math. Mech. 5, 115–124 (1925)
[17] Hencky, H.: Über die Form des Elastizitätsgesetzes bei ideal elastischen Stoffen. Z. Techn. Phys. 9, 215–220, 457 (1928)
[18] Hencky, H.: Das Superpositionsgesetz eines endlich deformierten relaxationsfähigen elastischen Kontinuums und seine Bedeutung für eine exakte Ableitung der Gleichungen für die zähe Flüssigkeit in der Eulerschen Form. Annalen der Physik. 5, 617–630 (1929)
[19] Hencky, H.: The law of elasticity for isotropic and quasi-isotropic substances by finite deformations. J. Rheol. 2, 169–176 (1931)
[20] Hencky, H.: On a simple model explaining the hardening effect in poly-crystalline metals. J. Rheol. 3, 30–36 (1932)
[21] Hohenemser, K.: Elastisch-bildsame Verformungen statisch unbestimmter Stabwerke. Ing.-Archiv. 2, 472–482 (1931)
[22] Hooke, R.: Lectures de potentia restitutiva, or of spring explaining the power of springing bodies, London (1678)
[23] Huber, M.T.: Właściwa praca odkształcenia jako miara wytężenia materiału. Czasopismo Techniczne 22, 34–40, 49–50, 61–62, 80–81 (1904)
[24] Jasinski, F.: Recherches sur la flexion des pièces comprimées. Ann. Ponts Chaussées 8, 233, 654 (1894)
[25] Jasinski, F.: Noch ein Wort zu den "Knickfragen". Schweiz. Bauzeitung 25, 172–175 (1895)
[26] Jaumann, G.: Geschlossenes System physikalischer und chemischer Differentialgesetze. Sitzber. Akad. Wiss. Wien, Abt. Iia. 120, 385–530 (1911)
[27] von Kármán, T.: Untersuchungen über Knickfestigkeit. Mitt. VDI, p. 81 (1910)
[28] Lévy, M.: Mémoire sur les équations générales des mouvements intérieurs des corps solides ductiles au delà des limites où l'élasticité pourrait les ramener à leur premier état. C. R. Acad. Sci., Paris 70, 1323–1325 (1870)
[29] Ludwik, P.: Elemente der technologischen Mechanik. Springer, Berlin (1909)
[30] Melan, E.: Zur Plastizität des räumlichen Kontinuums. Ing.-Archiv 9, 116–126 (1938)

[31] von Mises, R.: Mechanik der festen Körper im plastisch-deformablen Zustand. Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl, 582–592 (1913)
[32] von Mises, R.: Bemerkungen zur Formulierung des mathematischen Problems der Plastizitätstheorie. Z. Angew. Math. Mech. 5, 147–149 (1925)
[33] von Mises, R.: Mechanik der plastischen Formänderung von Kristallen. Z. Angew. Math. Mech. 8, 161–185 (1928)
[34] Nádai, A.: Versuche über die plastischen Formänderungen von keilförmigen Körpern aus Flußeisen. Z. Angew. Math. Mech. 1, 20–28 (1921)
[35] Nádai, A.: Plasticity, a Mechanics of the Plastic State of Matter. McGraw-Hill, New York (1931)
[36] Navier, L.: Sur les lois des mouvements des fluides, en ayant égard à l'adhésion des molécules. Ann. de Chimie 19, 244–260 (1821)
[37] Odqvist, F.K.G.: Die Verfestigung von flußeisenähnlichen Körpern. Z. Angew. Math. Mech. 13, 360–363 (1933)
[38] Poisson, S.D.: Mémoire sur les équations générales de l'équilibre et du mouvement des corps solides élastiques et des fluides. J. École Poly. 13 (1831)
[39] Prager, W.: Der Einfluß der Verformung auf die Fließbedingung zähplastischer Körper. Z. Angew. Math. Mech. 15, 76–80 (1935)
[40] Prager, W.: Strain hardening under combined stresses. J. Appl. Phys. 16, 837–840 (1945)
[41] Prager, W.: Theory of plastic flow versus theory of plastic deformation. J. Appl. Phys. 19, 540–543 (1948)
[42] Prager, W.: A new method of analyzing stresses and strains in work-hardening plastic solids. ASME J. Appl. Mech. 78, 493–496, 79, 481–484 (1956)
[43] Prandtl, L.: Spannungsverteilung in plastischen Körpern. In: Proc. 1st Int. Congr. Appl. Mech., Delft, pp. 43–46 (1924)
[44] Prandtl, L.: Über die Eindringungsfestigkeit (Härte) plastischer Baustoffe und die Festigkeit von Schneiden. Z. Angew. Math. Mech. 1, 15–20 (1921)
[45] Prandtl, L.: Anwendungsbeispiele zu einem Henckyschen Satz "uber das plastische Gleichgewicht. Z. Angew. Math. Mech. 3, 401–406 (1923)
[46] Reuss, A.: Fließpotential oder Gleitebenen? Z. Angew. Math. Mech. 12, 15–24 (1932)
[47] de Saint-Venant, B.: Note à joindre au mémoire sur la dynamique des fluides, présenté le 14 avril 1834. C. R. Acad. Sci. 17, 1240–1243 (1843)
[48] de Saint-Venant, B.: Sur l'établissement des équations des mouvements intérieurs operes dans les corps solides ductiles au delà des limites où l'élasticité pourrait les ramener à leur premier état. C. R. Acad. Sci. 70, 473–480 (1870)
[49] Schmidt, R.: über den Zusammenhang von Spannungen und Formänderungen im Verfestigungsgebiet. Ing.-Archiv 3, 216–235 (1932)
[50] Shanley, F.R.: The column paradox. J. Aeronaut. Sci. 13, 678 (1946)
[51] Shanley, F.R.: Inelastic column theory. J. Aeronaut. Sci. 14, 261–268 (1947)
[52] Stokes, G.G.: On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids. Trans. Cambr. Phil. Soc. 8, 287–319 (1845)
[53] Tanner, R.I., Tanner, E.: Heinrich Hencky: a rheological pioneer. Rheologica Acta 42, 93–101 (2003)
[54] Tresca, H.: Mémoire sur lécoulement des corps solides soumis à des fortes pressions. C. R. Acad. Sci. 59, 754–758 (1864)

[55] Tresca, H.: Mémoire sur lécoulement des corps solides. Mém. Pres. Par. Div. Sav. 18, 733–799 (1868)
[56] Truesdell, C.: The mechanical foundations of elasticity and fluid dynamics. J. Rational Mech. Anal. 1, 125–300, 2, 595–616, 3, 801 (1952)
[57] Truesdell, C.: Hypo-elasticity. J. Rational Mech. Anal. 4, 83–133 (1955)
[58] Xiao, H., Bruhns, O.T., Meyers, A.: Logarithmic strain, logarithmic spin and logarithmic rate. Acta Mechanica 124, 89–105 (1997)
[59] Xiao, H., Bruhns, O.T., Meyers, A.: Hypo-elasticity model based upon the logarithmic stress rate. J. Elasticity 47, 51–68 (1997)
[60] Zaremba, S.: Sur une forme perfectionnée de la théorie de la relaxation. Bull. Intern. Acad. Sci. Cracovie, 595–614 (1903)