A Historical View on Shakedown Theory

Dieter Weichert and Alan Ponter

Abstract. Plastic design started in the early 20^{th} century with the arrival of steel constructions in civil engineering. The objective was to determine the load carrying capacity in particular of steel bridges and steel skeleton buildings beyond the elastic limit. The related studies were first focused on monotonically increasing, "dead" loading. From this point of view they were directly related to the ancient question of determining the load carrying capacity of masonry construction like domes of churches.

It was in the extension of these studies that the problem of plastic design under variable loads came into the picture. Martin Grüning was the first to be attracted by the beneficial effect of limited plastic deformation in redundant elements in hyperstatic structures and opened the door to the fascinating theory of shakedown.

Keywords: Shakedown Theory, Plasticity.

1 From the Beginnings to the Formulation of the Classical Theorems of Shakedown Theory

The usual path to solve problems in solid mechanics today starts with an appropriate formulation of the set of differential equations governing kinematics, balance laws and material behaviour. In particular in the non-linear case, generally the evolutionary or rate formulation of the problem is chosen, which is then reformulated in weak form. This opens the way to approximation methods by introducing appropriate test functions. Solutions are then constructed in discrete form by a

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cascade of systems of linear algebraic equations.-The finite element method most prominently represents this strategy, which we may call “classical” in what follows.

This highly successful methodology relies on several important assumptions: The loading history is deterministically known, test functions and material law formulation are sufficient smooth and, above all, means for solving a cascade of very large systems of linear algebraic equations are available. - Non-linear material behaviour such as plasticity renders the problem within this strategy more complicated than linear elasticity which is a simple linear mapping of the space of stresses onto the space of strains. As well-known example, the elastic-perfectly plastic material model rather badly fits into this scheme.

Going back a century, there was no question of adequate computational tools nor did there even exist a general mathematical framework for non-linear solid mechanics. Nevertheless, the question whether a mechanical structure may fail or not under variable loads is as old as engineering activity itself. It is therefore very difficult to pinpoint the instant when mathematical models were first used to predict with rational arguments if a structure will resist the acting forces or collapse. Edoardo Benvenuto [1] gives an excellent view on early developments, going back to antiquity. – We choose more or less arbitrarily the prominent and representative “Poleni’s problem” as the starting point for the rational treatment of limit states of structures. By his study published in 1748 [2] G. Poleni answered the question, whether the dome of St. Peter’s would collapse due to cracks that had formed on its top? He solved this problem, that would today be classified as a problem of limit analysis with unilateral material behaviour, semi-empirically: he used Hooke’s analogy between the “hanging chain problem” and the “arch-problem” (“*as hangs the flexible line, so but inverted will stand the rigid arch*” cited from J. Heyman [3]) which needed to translate the plane problem of the arch to a shell problem (for details see J. Heyman’s paper on Poleni’s problem already quoted before [3]).

This typical limit-analysis problem does not involve the characteristic features of a shakedown problem which are variable loads, material ductility and residual stresses. However, it illustrates several important historical aspects:

(i) Long before mechanical engineering had gained significant importance, in civil engineering safe design, that means avoiding collapse of masonry structures, was of crucial importance.
(ii) Irreversible material behaviour is present through cracks and pseudo-hinges.
(iii) Elastic behaviour is neglected.

In the beginning of the 20th century, steel was increasingly used to construct bridges and buildings. Very soon engineers found that elastic design, commonly used at that time, was excessively conservative and plastic design became an issue for this type of structures in the 2nd decade. In particular methods to determine the load carrying capacity of beams and trusses under monotonically increasing loads were investigated both from experimental and theoretical point of view (Maier-Leibnitz in 1928 [4], Schaim in 1930 [5], Fritzsche 1931 [6]). But as early as 1926, Martin Grüning (Fig.1) discussed the influence of plasticity in hyperstatic truss systems under repeated loading: He observed the beneficial effect of redundant elements in case of repeated loading: Redistribution of stresses caused by plastic deformation occurring in first loading cycles may be such that in subsequent loading, the members of the structure may be operating solely in the elastic domain for higher load levels than determined if the initial elastic limit would be taken as reference. This might be considered as the first appearance, yet in embryonic form, of the idea of shakedown. Grüning published his results shortly after his famous work "Die Statik des ebenen Tragwerks" [7] in 1925 (still available today) in a 30-pages booklet entitled "Die Tragfähigkeit statisch unbestimmter Tragwerke aus Stahl bei beliebig häufig wiederholter Belastung" [8]: The decisive sentence in German:

"Überschreiten die Spannungen in n Stäben eines n-fach statisch unbestimmten Fachwerks, die als Überzählige eines stabilen (statisch bestimmten) Systems aufgefaßt werden können, infolge einer Belastung die Elastizitätsgrenze, so gehen sie unter hinreichend häufig wiederholter Be-und Entlastung in und unter Umständen unter die Elastizitätsgrenze zurück, sofern die Spannung in keinem Stabe des stabilen Systems sich über die Elastizitätsgrenze hebt".
("Do the stresses in n bars of a n-times over-determined bar-system, which can be considered as redundant elements of a stable (statically determined) system, exceed the elastic limit due to some loading, so they go, under sufficiently often repeated loading and unloading, back to, or under circumstances below the elastic limit, in case that the stress in none of the bars of the stable system goes beyond the elastic limit")

Fig. 1 (Bernhard) Martin Grüning (10.12.1869 – 30.6.1932) was a civil engineer, had several positions in German administration before he became in 1918 professor in Statics and Steel Construction (Eisenbau) at TH Hannover and in 1923 professor in Statics (Baustatik) at TH Wien. The above cited papers were published under his affiliation to TH Hannover.

We see the vicinity to the limit-load problem of civil engineering on one hand, but also the differences that are decisive for the different strands that take shakedown-theory and limit-analysis: Metal ductility combined with elasticity, at the origin of residual stresses, which are introduced as key element and can only play their role in repeated loading, in contrast to monotonically increasing loading in limit analysis that does not need to introduce elasticity and residual stresses in order to determine limit loads.

It was Hans Bleich (Fig.2) from TH Wien who picked up and generalised in 1932 Grünings findings [9], who's studies concerned only locally stationary repeated loads. This restriction was removed by Bleich, who also introduced the notion of "Selbstspannung", nowadays commonly used in its English half-adopted equivalent "eigenstress". One may assume that Bleich's work was inspired by Grüning not only through his publication but also by his presence in Vienna, but this is not confirmed. - With Bleich's work, we come already close to today's notion of shakedown theory.

Fig. 2 Hans Heinrich Bleich (24.3.1909 – 8.2.1985) was, like Grüning, civil engineer and had graduated in 1931 in Structural Engineering from TH Wien. In 1939, Bleich left Austria for England and moved in 1945 to the USA, where he continued his career first in industry and then as highly appreciated and prominent professor at Columbia University's school of Engineering.

Hans Bleich, together with his father Friedrich Bleich had been in close contact with Ernst Melan (Fig.3), publishing together in 1927 the book "Die gewöhnlichen und partiellen Differentialgleichungen der Baustatik" [10]. It might be this contact, but also the contact to Grüning, at that time professor in Vienna, that had inspired Melan to his path-breaking works in 1936 and 1938 in which he, rightly referring to Grüning and Bleich, formulated the general lower bound shakedown theorem in his key contribution "Der Spannungszustand eines Mises-Hencky'schen Kontinuums bei veränderlicher Belastung" [11]:

„Wir verzeichnen somit folgendes Ergebnis: Unter den gemachten Voraussetzungen besitzt das Integral (15) stets einen nicht negativen Wert. Dieser Wert bleibt unverändert, wenn das Material an keiner Stelle des Körpers fließt oder wenn bleibende Dehnungen auftreten, die keine Zwangsspannungen hervorrufen. Fließt aber das Material, so nimmt J wegen $\dot{J} < 0$ *immer nur ab" (page 86, lines 6-12)*

("We note therefore the following result: under the given assumptions, the integral (15) has always a non-negative value. This value remains unchanged, if the material does not flow in any point of the body or if permanent strains occur, that do not cause enforced stresses. If however the material flows, then the integral J decreases monotonically because of $\dot{J} < 0$*")*

Fig. 3 Ernst Melan (16.11.1890 – 10.12.1963) was Civil Engineer graduated from the German Technical University of Prague from where he also obtained his Dr-degree in 1917. In the same place he became extraordinary professor after holding 1916 and 1923 different positions in Austrian administration and industry. In the same place he became extraordinary professor after holding between 1916 and 1923 different positions in Austrian administration and industry.

It may surprise that, in particular, Melan's powerful general theorem from 1938 remained without resonance in the scientific community for many years. However, one must not forget the atrocious situation in Germany and, after WWII had started, in the entire world, at that time. Many brilliant German-Jewish scientists in mechanics had to struggle for their lives and to find new places to live for themselves and their families, if they had the chance to survive. In addition, the relevant papers had been published in the German language, which was evidently not very popular in these and the coming years. It was Prager (Fig.4), then at Brown University, who came back to the problem in his contribution "Problem Types in the Theory of Perfectly Plastic Materials" [12], presented at the Symposium on Plasticity held at Brown University in 1948. There, he refers to the less far reaching formulation of the lower bound theorem for bar-systems by Melan [13]. It was also Prager, at that time funded by the US Office of Naval Research, who introduced the denomination "Shakedown", which is known in shipyards to describe the process of accommodation of parts in new ships due to dynamic loading by engine vibration after putting them into operation for the first time (O. Mahrenholtz, private communication).

Fig. 4 William Prager (23.5.1903 – 16.3.1980) graduated as engineer from the Technical University Darmstadt, where he also obtained his Dr-degree. He then held positions in Göttingen and Karlsruhe (as youngest professor in Germany at that time) before being forced to leave Germany after the Nazi-regime came into power. After moving to Turkey he was appointed professor at Brown University in the USA, where, except for a short period at the University of California San Diego, he continued his brilliant career until retirement.

Brown University with its scientifically particularly fertile atmosphere at that time can be considered as a kind of cradle for the further development of shakedown theory. Besides Prager, also should be mentioned Paul Southworth Symonds *(20.8.1916 – 28.3.2005)* (Fig.5) and B.G. Neal as particularly active in the field of

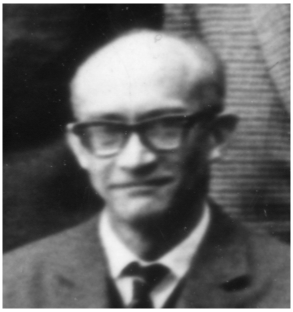

Fig. 5 P.S. Symonds

determination of limit loads for frame structures under proportional and variable loads. [14-19]. Special importance has their paper “Recent Progress in the Plastic Methods of Structural Analysis” [17] as it anticipates the general upper bound theorem of shakedown. In his contribution [20], Paul Symonds gives an enlightening personal insight into the developments of shakedown theory at that time at Brown University and Cambridge University.

Generally speaking, in the U.K. to our knowledge, it was basically the group by John Baker working in structural plasticity in the same period and their interest was mainly limit analysis (see also the multi-volume work “The Steel Skeleton”, by Baker, Horne and Heyman [21]).

Front row, left to right: E. H. Lee, W. Cohen, J. F. Baker, W. Prager
Rear, left to right: R. T. Shield, L. S. Beedle, J. M. Crowley, J. Heyman, assembled for the Design Session.

Fig. 6 This photograph was taken at Brown University in April 1960 on the occasion of the Second Plasticity Symposium organized by the US Office of Naval Research.

W.T. Koiter (Fig.7), who had been visitor to Brown University at that time, has the merit of not only formulating the upper bound theorem of shakedown in a general form in 1956 in his paper “A new general theorem on shakedown of elastic-plastic structures” [23], but also to revalue shakedown theory as a whole through his fundamental work “General Theorems for Elastic-Plastic Solids” [24] that summarises pretty well the state of the art at that time in a very accessible manner. We stress however that the contribution by Paul Symonds concerning the upper bound theorem should not be undervalued [25] as it has been in the past. He formulated in fact the first time an upper bound approach in shakedown theory, not in general form, as Koiter did, but for frames. Symonds also greatly simplified the proof of Melan’s lower bound theorem [15] in the general case in a form adopted by Koiter [24].

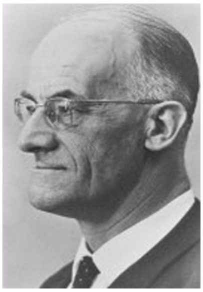

Fig. 7 Warner Tjardus Koiter (16.6.1915 – 2.9.1997), graduated as mechanical engineer from Delft University of Technology in 1936. He worked at the Dutch National Aeronautical Research Institute (NLL) in Amsterdam, the Government Patent Office and the Government Civil Aviation Office. In 1949, he was appointed Professor of Applied Mechanics in Delft where he stayed until his retirement in 1979. His contributions to shell theory and plasticity influenced significantly the development of modern mechanical sciences.

One can say, that with Koiter's formulation of the lower and the upper bound theorems of shakedown theory in the context of continuum mechanics the first chapter of development was closed, leaving evidently many questions open.

2 Theoretical Extensions and Development of Numerical Methods

Open questions left by pioneers are also opportunities for interesting research subjects for the younger generation. In case of shakedown theory, the interest was double: On one side, the theory had an evidently high practical potential and on the other side, basic assumptions were rather restrictive and the application to industrial problems not easy. One can grossly summarize this to two major questions:

(i) How can these powerful theorems be applied in practical engineering?
(ii) How to get rid of the very coercive assumptions on which the classical proofs of shakedown theory were based?

Many of the young scientists who had been on visit at Brown University in the 50 and 60ties had been inspired by the strong plasticity group and carried on their research in this field after returning home. They tried to find answers to these questions. Among them, Giulio Maier *(*8.3.1931)* (Fig.8), back from Brown University where he had been as visiting scholar in 1964, continued his work in Milano, Italy, focusing together with his co-workers on the problems of non-associated flow rules [26], geometrical non-linearities [27] and dynamic effects [28]. Non-associated flow rules are particularly important for frictional materials such as soils, masonry, but also, more recently investigated, porous and heterogeneous materials [33-35]. Also, accounting for the progress in numerical methods and in

Fig. 8 G. Maier

view of industrial applications, Maier adapted shakedown theory to the so-called linear programming methods [26]. His very rich scientific oeuvre in the field of shakedown theory, partly referred to in this chapter (ref. [26-41]) was initially stimulated by problems in civil engineering but triggered many research activities in the field of limit analysis in general with greatest impact worldwide, not only in Italy. To mention, from Italian side, are in this context the contributions e.g. by L. Corradi, R. Contro, F. Genna, A. Corigliano, U. Perego, and in recent years, V. Carvelli and G. Cocchetti, in the same group as Giulio Maier.

Fig. 9 A. Ponter

Equally back from Brown University where he stayed 1964-1965, first to Cambridge and later to Leicester, Alan Ponter *(*13.2.1940)* (Fig.9) as applied mathematician interested in problems of mechanics engineering design extended the lower bound theorem to creep behaviour of materials. Due to the explicit time-dependence, absent in plasticity, this problem was (and still is) particularly challenging. He showed that if the cycle time is regarded as a variable, solutions for very short and very long cycle times possessed bounding properties for intermediate finite cycle times. For short cycle times, the residual stress remained constant and hence behaviour could be related to shakedown, resulting in the Creep Modified Shakedown Limit [43-46]. He also derived displacement bounds in shakedown conditions [47] and more general bounding theorems in plasticity [48,49]. These works were carried out in cooperation with Fred Leckie, John Martin and others and influenced both high temperature design and life assessment methods considerably. We note that Martin had also been at Brown University and spent a year at Leicester with Ponter while he wrote his book [42]. Bree's complete solution for a particular problem involving thermal and loading of a tube [50] was an important impulse for the design community to grasp the complete range of possible behaviour, including shakedown, reverse plasticity and ratchetting. This decisively paved the way for the acceptance of shakedown limit solutions as a basis for design in the nuclear design community.

Ponter was then attracted by the problem of mechanical parts in power plants operating at elevated temperatures, related to the European Fast Reactor-project, which was later cancelled. This involved extending shakedown theory to the evaluation of the ratchet limit, and this was achieved, in an approximate way, for the characteristic problems of Fast Reactor design [51-53]. Methods for obtaining the rate of ratchet strain growth in excess of shakedown were also obtained [54,55]. He wrote together with Sami Karadeniz, Keith Carter and Alan Cocks four reports for the EC and many papers, including both creep deformation and creep rupture effects. The final EC report [56], describes the background to a design code for thermal loading as part of the European Fast Reactor Project.

In more recent years, Ponter focused on rolling contact problems [57], composites [58,59] and the development of methods of how to solve most efficiently shakedown and related problems by numerical methods. He introduced the so-called "Linear Matching Method" [60,61] and implemented it successfully into commercial software for design purposes [62,63]. As well as shakedown limits, ratchet limits in excess of shakedown are also obtained. This work continues at the University of Strathclyde by a research group led by H. F. Chen.

Variable thermal loads was also the field David Aronovich Gokhfeld *(31.7.1919 - 14.3.2004)* (Fig.10) from the South Ural University, USSR, was deeply involved in. He studied in the beginning of the 60ties of the last century ratchetting of mechanical parts in furnaces and observed that under moving thermal loads even without any mechanical loading large deformations may occur that render the considered part unusable [64]. These studies, always joining theoretical and experimental work, were extended to other types of elements such as turbine blades, parts in nuclear reactors and pressure vessels in chemical processing. Aware of the work of Ponter, Williams and Leckie on creep, he and his co-workers developed an alternative way of taking creep into account by adjusting appropriately the yield limit of the material. - This idea resembles the concept of "sanctuary of elasticity" by Nayroles and Weichert, put forward many years after in another context [168]. - Gokhfeld, O.F.Cherniavsky and co-workers published more than 100 papers in the field, almost all of them in Russian language [64-66] and therefore largely unknown outside of the Russian speaking scientific community. Special mention deserves the book "Theory of shakedown and strain accumulation under thermal cycling" from 1980 in English which is still today an invaluable source of information. – The work from this group is today successfully continued in a modern computational environment by A. Cherniavsky and co-workers, with new fields of application like hydrogenated metals, crack forming and use of traveling heat sources for controlled metal forming.

Fig. 10 D. Gokhfeld

Another question, not addressed by classical theory, was the problem of dynamic (inertia) effects. Here also, in a different way than in case of viscoplasticity, time enters the formulation of the problem explicitly. This subject was first addressed first by Giulio Ceradini from Rome in his theorem on dynamic shakedown in 1969 [68], still of great importance in particular in earthquake engineering.

To come back to Italy, Castrenze Polizzotto from Palermo who came from a more mathematically than engineering-driven background, contributed to the problems of variational formulations, extended classes of material behaviour, dynamics and bounding theorems [69-78]. His work is continued by the groups of Guido Borino, Paolo Fuschi, Aurora Pisano and others.

Fig. 11 T.M. Huber

Particularly important for the development of shakedown theory was research conducted at the Polish Academy of Sciences: There was a long Polish tradition since the times of Tytus Maksymilian Huber *(4.1.1872 – 9.12.1950)* (Fig.11) in the field of Plasticity in Poland that was continued after WWII in particular related to limit states of plastic structures and phenomenological modeling of plastic behavior. Wacław Olszak (Fig.12) had been the initiator of developing the research team in plasticity theory and applications in engineering including both structural mechanics and technology of metal processing. In 1955 he initiated weekly scientific seminars at the Polish Academy of Sciences. The principles of limit state analysis, metal forming, yield conditions, anisotropy, inhomogeneity and flow rules were the main topics of discussion and research.

Fig. 12 W. Olszak

In 1957 a set of lectures was presented by Koiter, who discussed his kinematic approach to shakedown and the upper bound theorem illustrated by some examples. Olszak, who made his PhD in Vienna Polytechnic in 1935, knew well Melan and was familiar with his theorem on shakedown. It can be supposed that this was the main inspiration to develop more intensive study of cyclic loading and shakedown as fundamental for application in structural mechanics. In the sequel, Antoni Sawczuk *(16.1.1927 – 27.5.1984)* (Fig.13), primarily working on limit analysis of plates and shells wrote several papers on application of kinematic theorem to specify upper bounds on load amplitudes.

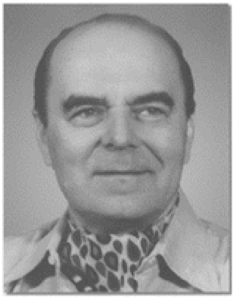

Fig. 13 A. Sawczuk

It was however Jan Andrej König *(16.5.1937 – 8.12.1990)* (Fig.14), his doctoral student, who contributed most significantly to the advancement of shakedown theory by a large number of theoretical papers on hardening material behavior [80], thermal problems [81], bounding methods [82], structures [83] and numerical methods [84]. His book "Shakedown of Elastic-Plastic Structures" [85] from 1987 is still of great actuality and a prime reference; it is concise and easy to read.

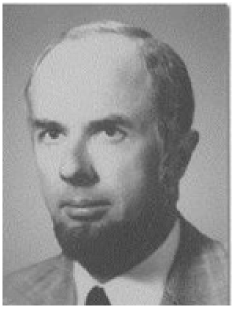

Fig. 14 A. König

Among the great number of Polish scientists working successfully in the field of shakedown between roughly 1960 and 2000, we only mention A. Borkowski, M. Kleiber, Z. Mróz, A. Sawczuk, M. Janas, St. Dorosz, A. Siemazsko, S. Pycko, J. Skrzypek, B. Skoczeń, J. Orkisz, J. Zwoliński, B. Bielawski, among others [86-110]. The book by Michal Życzkowski *(12.4.1930 – 24.5.2006)* (Fig.15) "Combined Loadings in the Theory of Plasticity" [79] gives an excellent account on research on plastic structures with more than 3000 entries of references.

It was also König who brought shakedown theory back to Germany: In the beginning of the 80ties, Oskar Mahrenholtz *(*17.5.1931)* (Fig.16), then at Hannover University, was involved in studies on failure of zirconium tubes under variable thermal and mechanical loads, which is a typical problem from nuclear power engineering. König, at that time visiting scholar in Hannover, suggested solving this problem by applying the upper bound shakedown theorem and carrying out experiments simultaneously.

Fig. 15 M. Życzkowski

Fig. 16 O. Mahrenholtz

The experimental and numerical results they obtained were in quite good agreement [111-113]. Until today, in particular the experimental results from this study are used as benchmark. – Some years later, König stayed again at Hannover University, this time with Erwin Stein *(*5.7.1931)* (Fig.17). There he initiated a series of studies focusing on the development of numerical shakedown analysis involving material hardening, cracks and structural optimization [113-122]. These works were successfully continued by Stein, Zhang, Mahnken, Wiechmann and others in the following years. Of particular interest is the "reduced base"-technique, developed by Stein. This technique reduces significantly the numerical efforts to construct numerical solutions and is used in industrial applications.

Fig. 17 E. Stein

It should be noted that in the German community of civil engineers, to which Stein belongs, the application of shakedown theory was even in the 90ties far from being commonly accepted and was discussed quite controversially [123].

Another strand of development, the initial boundary value problem of plasticity, was the entry point for Dieter Weichert *(* 5.3.1948)* (Fig.18) to discover indirectly shakedown theory through the influence of Pawel Rafalski [124], another Polish scholar from Warsaw, visiting Bochum University in the beginning of the 80ties. Weichert studied in the sequel first the problem of geometrical nonlinearities in the context of shakedown theory in the framework of continuum mechanics [125], investigated then the problem of generalized material laws according to the standard material model by [126] with applications to thin-walled structures. Through his stay at the American University of Beirut, he started to work with Lutfi Raad and others on problems in pavement mechanics [127-131]. Later, at Lille University in France, Weichert initiated and carried out a number of studies on numerical methods, dynamic shakedown and the problem of shakedown including material damage and cracked bodies [132-137]. He continued this work back to Germany at Aachen University of Technology with applications to composites [138-140] and with the aim to apply shakedown theory to large scale industrial problems [141-146]. Weichert and his co-workers concentrated on the lower bound theorem and used in the beginning stress-based numerical approaches, which delivered very good results, but which are difficult to combine with displacement-based commercial finite element codes. Today, his group works primarily with displacement-based approaches.

Fig. 18 D. Weichert

Force and stress-based methods have been intensively developed at Liège University, starting in the 60ties. There was a strong group around Baudouin Fraeijs de Veubeke *(3.8.1917 – 16.9.1976)* (Fig.19), Ch. Massonnet, G. Sanders, C. Fleury, mainly involved in general mechanics, limit analysis of plates and shells, and optimization. M. Save and G. Guerlement continued the work at the Polytechnic School of Mons.

Fig. 19 B.F. de Veubeke

Later, in particular Patrick Morelle and Nguyen Dang Hung applied the tools that had been developed in the innovative scientific environment of Liège University to shakedown analysis [147-152]. Their and their co-workers efforts were aimed at the numerical exploitation of duality principles. Manfred Staat from Jülich Research Center continued successfully this work in recent years [153-157].

The path of force methods has been followed in an original manner by Kostas Spiliopoulos, with application to frame structures, based on graph theory and linear programming [158-160].

Géry de Saxcé, who had also started his career at Liège University [161] before moving first to the Polytechnic School of Mons and then to Lille University, introduced the so-called bi-potential theory [162-163] as generalization of Fenchel's inequality, opening new doors to take into account more complex, friction-type material laws in limit analysis and shakedown theory [164]. This novel approach has a high potential and is far from being fully exploited at the time being.

The research by de Saxcé is therefore linked to the scientific tradition of the Belgian group, but also to the French, mathematically inclined community of mechanics, strongly involved in the 70s and 80s of the last century in the development and application of Convex Analysis which is based on the Fenchel inequality and so carries further the idea of classical potential-based principles. Convex Analysis had been developed in the 70s by T. Rockafellar in the context of operational research and by J.J. Moreau [165] in the context of mechanics. B. Nayroles together with O. Debordes, both at that time in Marseille, applied this to the shakedown problem [166,167] and contributed essentially to the strengthening of the mathematical basis of the theory. In this context, the important work by Quoc-Son Nguyen has to be mentioned, who contributed strongly to the understanding of the effect of hardening from mathematical point of view [169-171]. Very fruitful because well suited to extend the classical theorems to larger classes of material behavior was the introduction of the so-called Standard Material Model by B. Halphen and Nguyen Quoc Son [172] as had already been shown by Mandel in 1976 [173]. Radenkovic's work on non-associated flow rules from 1961 [174], although basically related to limit analysis played an important role in the sequel also in shakedown theory and should be mentioned in this place.

Independently and application oriented, Joseph Zarka and his group developed the so-called Simplified Method [175,176], particularly useful for applications involving alternating plasticity and fatigue problems in mechanical engineering as has been shown by Geneviève Inglebert and her co-workers [177]).

Coming back to typical problems of civil engineering (which, as mentioned in the beginning, in some sense has triggered limit- and shakedown analysis), there were some important but for long time spared-off areas, which are mechanics of soils, foundations and pavements. Here, just as in case of concrete and reinforced concrete structures, the complexity of the material behavior and the difficulty to develop realistic material models that fit the framework of shakedown theory are important obstacles. In particular for pavements, other effects like rutting, crack development, moisture, freezing and thawing cycles are very important aside of plasticity as to their long-time behavior; in case of foundations, mostly the fluctuation of loads is by far less important than gravitational "dead loads".

Apart from the groups mentioned before, there is a tradition in this field of research in Australia and New Zealand. Pioneering work on pavements goes back to John Robert Booker *(24.7.1942 – 13.1.1998)* (Fig.20) and R.W. Sharp from Sydney University in 1984 [178], basing their approach on the particular stress pattern that develops in the rolling contact on roads.

Fig. 20 J.R. Booker

This work was followed by others, like Scott Sloan *(* 2.7.1954)* (Fig.21) from Newcastle University and his group [179-181], Ian Collins from Auckland University New Zealand [182,183,131]. In his later work, there was a link to Ponter, Weichert and Raad through the fact that Mostapha Boulbibane, a former PhD-student of Weichert, had been active in all three groups. But also shakedown of structures such as bridges has been widely investigated in Australia: It was Paul Grundy from Monash University who started as early as 1969 to study in many papers the shakedown behavior of mechanical elements, in particular linked to bridge constructions [184,185]. His work is continued by Francis Tin Loi and his coworkers at the University of New South Wales. - It is from Australia, Newcastle University that from Sloans group Hai-Sui Yu brought back to the U.K. at Nottingham University shakedown. Yu and his co-workers concentrate on the numerical application of Melan's theorem to compute lower bounds to the shakedown limit for rolling contact problems for Mohr-Coulomb type yield conditions for road pavement design [186-190].

Fig. 21 S. Sloan

We can however have another way to look at the historical evolution of shakedown analysis, detached from the individual researcher and research groups and their connections and relations: The onset was the observation that residual stresses due to plasticity in redundant elements of hyperstatic bar-structures are beneficial for their survival under variable loads, what differentiates shakedown theory essentially from limit load theory. Application did not appear for long time due to the lack of means how to translate the theory to calculation methods. First applications appear in special types of structures like beams, plates and shells, where by appropriate assumptions and semi-analytical methods the complexity of the problem can be reduced drastically. As the theory of plasticity is genuinely linked to metals, the fields of application were on the side of mechanical engineering pressure vessels and pipes, on the side of civil engineering steel frame structures. This first "bifurcation" was not methodological, but naturally imposed by engineering practice. In the sequel, on both sides, application-driven theoretical extensions were carried out: More complex material models for metals, for concrete and for soil-like materials, material damage and cracks, temperature influence, geometrical non-linearity are the major strands, accompanied by the development of appropriate and more rigorous mathematical foundations, such as the proper formulation of the theorems as optimization problems. The breakthrough to modern engineering however is due to the tremendous development of numerical methods and computer technology: Discretization of structures of almost arbitrary shape connected with fast linear solvers and highly performing optimization algorithms render the theorems of shakedown today easily applicable in practical engineering.

And it is the modern formulation of the theory which makes that the differences between shakedown theory and theory of limit analysis in practice almost vanishes: Limit analysis became a particular case of shakedown analysis and today both are subsumed under the notion of "Direct Methods".

3 Final Remarks

To keep “the beauty of shakedown theory” evoked by Giulio Maier, can be understood as a warning against theoretical overstretching. The beauty of the theory is obvious: without information about the path of loading in an arbitrarily complex loading space, one predicts, if a dissipative mechanical system will fail or not. No need to walk step-by-step through the evolution of a system, which is not only cumbersome but in many cases simply impossible, because the loading history of the considered element is unknown.

What sounds a bit like a miracle, is, from mathematical point of view, due to convexity of the potentials involved and the existence of subgradients describing the evolution of the material. This reflects the physical features of small geometrical transformations and stable material behavior, as particular case of the validity of the second law of thermodynamics in form of the Clausius-Duhem inequality.

The price to pay for this miracle is loss of information about the evolution of local quantities during the process and, most important, about deformation, although certain bounding properties have been proven. This price comes along with rather harsh assumptions about material behavior, in many cases too rough for sophisticated investigations and, if one applies the theorems directly, a rather complex resolution methodology. And here is the risk of overstretching: If the evolution of a system inherently depends on the evolution of local quantities, it becomes very tricky, if not impossible, to find adequate theoretical extensions of shakedown theory. So, to find the border line of usefulness is an important issue.

But this is true for any kind of modeling and we have to keep in mind the issue of our efforts: confronted to a concrete problem, one has to decide about the adequate methodology to solve it. And in this sense, there is plenty of room for the further development and application of shakedown theory.

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References

[1] Benvenuto, E.: An Introduction to the History of Structural Mechanics, Part I, Statics and Resistance of Solids. Springer (1991)

[2] Poleni, G.: Memorie istoriche della gran cupola del Tempio Vaticano, Padova (1748)

[3] Heyman, J.: Poleni’s problem. Proc. Instn Civ. Engrs, Part 1 84, 737–759 (1988)

[4] Maier-Leibnitz, H.: Beitrag zur Frage der tatsächlichen Tragfähigkeit einfacher, durchlaufender Balkenträger aus Baustahl St. 37 und Holz, Bautechnik, 6. Jg. Heft 1, 11–14 und Heft 2, 27–31 (1928)

[5] Schaim, J.H.: Der durchlaufende Träger unter Berücksichtigung der Plastizität, Stahlbau (1930)

[6] Fritzsche, J.: Die Tragfähigkeit von Balken aus Baustahl bei beliebig oft wiederholter Belastung. Bauingenieur 12, 827 (1931)
[7] Grüning, M.: Die Statik des ebenen Tragwerks. Springer, Berlin (1925)
[8] Grüning, M.: Die Tragfähigkeit statisch unbestimmter Tragwerke aus Stahl bei beliebig häufig wiederholter Belastung. Springer, Berlin (1926)
[9] Bleich, H.: Über die Bemessung statisch unbestimmter Stahltragwerke unter Berücksichtigung des elastisch-plastischen Verhaltens des Baustoffes. Der Bauingenieur, Heft 19/20, 261–267 (1932)
[10] Bleich, F., Melan, E.: Die gewöhnlichen und partiellen Differentialgleichungen der Baustatik. Springer (1927)
[11] Melan, E.: Der Spannungszustand eines Mises-Hencky'schen Kontinuums bei veränderlicher Belastung, Sitzungsberichte, vol. 147, pp. 73–87. Akademie der Wissenschaften, Wien (1938)
[12] Prager, W.: Problem Types in the Theory of Perfectly Plastic Materials. Journal of the Aeronautical Sciences, 337–341 (June 1948)
[13] Melan, E.: Theorie unbestimmter Systeme aus idealplastischem Baustoff. Sitzungsberichte, Akademie der Wissenschaften 145, 195–218 (1936)
[14] Symonds, P.S., Prager, W.: Elastic-plastic analysis of structures subjected to loads varying arbitrarily between prescribed limits. J. Appl. Mech. 17, 315–324 (1950)
[15] Symonds, P.S.: Shakedown in Continuous Media. J. Appl. Mech. 17, 85–89 (1951)
[16] Neal, B.G.: Plastic-Collapse and Shake-Down Theorems for Structures of Strain Hardening Material. J. Aero. Sci. 17, 297–307 (1950)
[17] Symonds, P.S., Neal, B.G.: Recent progress in the plastic methods of structural analysis. J. Franklin Inst. 252(6), 469–492 (1951)
[18] Symonds, P.S., Neal, B.G.: The Calculation of Failure Loads on Plane Frames unbbder Arbitrary Loading Programs. J. Inst. Civil Engrs. 35, 41–61 (1951)
[19] Neal, B.G., Symonds, P.S.: A Method for Calculating the Failure Load for a Framed Structure Subjected to Fluctuating Loads. J. Inst. Civil Engrs. 35, 186 (1951)
[20] Symonds, P.S.: An early upper bound method for shakedown. In: Weichert, D., Maier, G. (eds.) Inelastic Analysis of Structures under Variable Loads – Theory and Engineering Applications. Series, Solid Mechanics and its Applications, vol. 83, pp. 1–9. Kluwer Academic Publishers, Dordrecht (2000)
[21] Baker, J., Horne, M.R., Heyman, J.: The steel Skeleton, vol. II, ch. 9. Cambridge University Press (1956)
[22] Horne, M.R.: The Effect of Variable Repeated Loads in the Plastic Theory of Structures. Research, Engineering Structures Supplement, Colston Papers II, 141 (1949)
[23] Koiter, W.T.: A new general theorem on shakedown of elastic-plastic structures. Proc. Kon. Ne. Ak. Wet. B59, 24–34 (1956)
[24] Koiter, W.T.: General theorems for elastic-plastic structures. In: Sneddon, I.N., Hill, R. (eds.) Progress in Solid Mechanics, ch. IV, pp. 165–221. North-Holland Publ. Co., Amsterdam (1960)
[25] Symonds, P.S.: Basic Theorems in the Plastic Theory of Structures. J. Aero. Sci. 17, 669–670 (1950)
[26] Maier, G.: Shakedown theory in perfect elastoplasticity with associated and non-associated flow-laws, a finite element, linear programming approach. Meccanica 4(3), 250–260 (1969)

[27] Maier, G.: A shakedown matrix theory allowing for workhardening and second-order geometric effects. In: Sawczuk, A. (ed.) International Symposium, Warsaw, August 30-September 2. Foundations of Plasticity, vol. 1, pp. 417–433. Noordhoff, Leyden (1972, 1973)
[28] Corradi, L., Maier, G.: Dynamic non-shakedown theorem for elastic perfectly-plastic continua. Journal of the Mechanics and Physics of Solids 22, 401–413 (1974)
[29] König, A., Maier, G.: Shakedown analysis of elastoplastic structures, a review of recent developments. Nuclear Engineering and Design 66, 81–95 (1981)
[30] Maier, G.: A generalization to nonlinear hardening of the first shakedown theorem for discrete elastic-plastic structures. Rendic. Acc. Naz. dei Lincei, Serie Ottava, 161–174 (1988)
[31] Maier, G., Novati, G.: Dynamic shakedown and bounding theory for a class of nonlinear hardening discrete structural models. Int. J. of Plasticity 6(5), 551–572 (1990)
[32] Maier, G., Pan, L., Perego, U.: Geometric effects on shakedown and ratchetting of axisymmetric cylindrical shells subjected to variable thermal loading. Engineering Structures 15(6), 453–466 (1993)
[33] Corigliano, A., Maier, G., Pycko, S.: Dynamic shakedown analysis and bounds for elastoplastic structures with nonassociative, internal variable constitutive laws. Int. J. Solids and Structures 32(21), 3145–3166 (1995)
[34] Cocchetti, G., Maier, G.: Static shakedown theorems in piecewise linearized poro-plasticity. Archive of Applied Mechanics 68, 651–661 (1998)
[35] Carvelli, V., Maier, G., Taliercio, A.: Shakedown analysis of periodic heterogeneous materials by a kinematic approach. Journal of Mechanical Engineering 50(4), 229–240 (1999)
[36] Carvelli, V., Cen, Z.Z., Liu, Y., Maier, G.: Shakedown analysis of defective pressure vessels by a kinematic approach. Archive of Applied Mechanics 69, 751–764 (1999)
[37] Cocchetti, G., Maier, G.: A shakedown theorem in poroplastic dynamics. Rend. Mat., Accademia Nazionale dei Lincei, s. 9 13, 43–53 (2002)
[38] Maier, G., Cocchetti, G.: Fundamentals of direct methods in poroplasticity. In: Weichert, D., Maier, G. (eds.) Inelastic Behaviour of Structures Under Variable Repeated Loads, Direct Analysis Methods, CISM, pp. 91–113. Springer, Wien (2002)
[39] Maier, G., Carvelli, V.: A kinematic method for shakedown and limit analysis of periodic composites. In: Weichert, D., Maier, G. (eds.) Inelastic Behaviour of Structures Under Variable Repeated Loads, Direct Analysis Methods, CISM, pp. 115–132. Springer, Wien (2002)
[40] Maier, G., Pastor, J., Ponter, A.R.S.: Direct Methods in Limit and Shakedown Analysis. In: De Borst, R., Mang, H.A. (eds.) Numerical and Computational Methods, vol. 3. Elsevier-Pergamon, Amsterdam (2003); Milne, I., Ritchie, R.O., Karihaloo, B. (eds.) Comprehensive Structural Integrity
[41] Cocchetti, G., Maier, G.: Elastic-plastic and limit-state analyses of frames with softening plastic-hinge models by mathematical programming. Int. J. Solids and Structures 40, 7219–7244 (2003)
[42] Martin, J.: Plasticity, Fundamentals and General Results. MIT Press (1975)
[43] Ponter, A.R.S., Williams, J.J.: Work bounds and the associated deformation of cyclically loaded creeping structures. ASME, J. Appl. Mech. 40, 921–927 (1973)
[44] Ponter, A.R.S.: On the relationship between plastic shakedown and the repeated loading of creeping structures. Trans ASME, J. Appl. Mech., Series E 38, 437–440 (1971)

[45] Ponter, A.R.S.: Deformation, displacement and work bounds for structures in a state of creep and subject to variable loading. Trans. ASME, J. Appl. Mech, 39, Series E, 953–959 (1972)
[46] Ponter, A.R.S.: On the creep modified shakedown limit. In: Ponter, A.R.S., Hayhurst, D.R. (eds.) 3rd IUTAM Symposium on "Creep in Structures", Leicester University (September 1980), pp. 264–278. Springer (1981)
[47] Ponter, A.R.S.: An upper bound on the small displacement of elastic-plastic structures. Trans. ASME, J. Appl. Mech. 39, Series E, 959–964 (1972)
[48] Ponter, A.R.S.: General displacement and work bounds for dynamically loaded bodies. J. Mech. Phys. Solids 23, 151–163 (1975)
[49] Ponter, A.R.S.: A general shakedown theorem for inelastic materials. In: Proc. SMiRT-3, London, paper L5/2 (1975)
[50] Bree, J.: Elasto-plastic behavior of thin tubes subjected to internal pressure and intermittent high-heat fluxes with applications to Fast Reactor Fuel Elements. J. Strain Analysis, 2(3), 226–238 (1967)
[51] Ponter, A.R.S., Karadeniz, S.: An extended shakedown theory for structures that suffer cyclic thermal loading, Part I: Theory. Trans. ASME, J. Appl. Mech. 52, 877–882 (1985)
[52] Ponter, A.R.S., Karadeniz, S.: An extended shakedown theory for structures that suffer cyclic thermal loading, Part II: Applications. Trans. ASME, J. of Appl. Mech. 52, 883–889 (1985)
[53] Ponter, A.R.S., Carter, K.F.: The ratchetting of shells subjected to severe thermal loading. In: Tooth, A.S., Spence, J. (eds.) Applied Solid Mechanics - 2, pp. 303–320. Elsevier Applied Science (1987)
[54] Ponter, A.R.S., Cocks, A.C.F.: The incremental strain growth of an elastic-plastic body loaded in excess of the shakedown limit. Trans. ASME, J. Appl. Mech. 51(3), 465–469 (1984)
[55] Ponter, A.R.S., Cocks, A.C.F.: The incremental strain growth of elastic-plastic bodies subjected to high levels of cyclic thermal loading. Trans. ASME J. Appl. Mech. 51(3), 470–474 (1984)
[56] Ponter, A.R.S., Karadeniz, S., Carter, K.F.: The computation of shakedown limits for structural components subjected to variable thermal loading - Brussels Diagrams. Directorate General for Science, Research and Development, Office for Official Publications of the E.C., Report EUR12686EN, Brussels, 170 pages (1990)
[57] Ponter, A.R.S., Chen, H.F., Chiavarella, M., Specchia, G.: Shakedown analysis for rolling and sliding contact problems. Int. J. Sol. Struct. 43, 4201–4219 (2001)
[58] Ponter, A.R.S., Leckie, F.A.: Bounding Properties of Metal Matrix composites subjected to Cyclic loading. Jn. Mech. Phys. Solids 46, 697–717 (1998)
[59] Ponter, A.R.S., Leckie, F.A.: On the behaviour of metal matrix composites subjected to cyclic thermal loading. Jn. Mech. Phys. Solids 46, 2183–2199 (1998)
[60] Ponter, A.R.S., Engelhardt, M.: Shakedown limits for a general yield condition. European Journal of Mechanics, A/Solids 19, 423–445 (2000)
[61] Ponter, A.R.S., Chen, H.F.: A Minimum theorem for cyclic load in excess of shakedown, with applications to the evaluation of a ratchet limit. Euro. Jn. Mech., A/Solids 20, 539–553 (2001)
[62] Chen, H.F., Ponter, A.R.S., Ainsworth, R.A.: The Linear Matching Method ap-plied to the high temperature life assessment of structures, Part 1. Assessments involving constant residual stress fields. Int. J. Pressure Vessels and Piping 83, 123–135 (2006)

[63] Chen, H.F., Ponter, A.R.S., Ainsworth, R.A.: The Linear Matching Method applied to the high temperature life assessment of structures, Part 2. Assessments beyond shakedown involving changing residual stress fields. Int. J. Pressure Vessels and Piping 83, 136–147 (2006)
[64] Gokhfeld, D.A.: On the possibility of increase of plastic deformation due to thermal cycling effects. Calculations of Strength (7) (1961) (in Russian)
[65] Gokhfeld, D.A.: Bearing capacity of structures under thermal cycles. Mashinostroenie publ., Moscow, p. 259 (1970) (in Russian)
[66] Gokhfeld, D.A., Cherniavsky, O.F.: Theory of shakedown and strain accumulation under thermal cyclinges. In: Proceedings of the All-USSR. Symp. on Low-cycle Fatigue at Elevated Temperatures, Chelyabinsk, vol. (3), pp. C3–C31 (1974)
[67] Gokhfeld, D.A., FCherniavsky, O.: Limit analysis of structures at thermal cycling. Sijthoff and Noordhoff. Int. Publ. Alphen aan den Rijn, The Netherlands – Rockville, USA (1980)
[68] G. Ceradini, Sull' adattamento dei corpi elasto-plastic isoggetti ad azioni dinamiche. Giornaledel Genio Civile 415, 239–258 (1969)
[69] Polizzotto, C.: Workhardening adaptation of rigid-plastic structures. Meccanica 10, 280–288 (1975)
[70] Polizzotto, C.: A unified approach to quasi-static shakedown problems for elastoplastic solids with piecewise linear yield surfaces. Meccanica 13, 109–120 (1978)
[71] Polizzotto, C.: On workhardening adaptation of discrete structures under dynamic loadings. Arch. Mech. Stos. 32, 81–99 (1980)
[72] Polizzotto, C.: A unified treatment of shakedown theory and related bounding techniques. S.M. Arch. 7, 19–75 (1982)
[73] Polizzotto, C.: A convergent bounding principle for a class of elastoplastic strainhardening solids. Int. J. Plasticity 2, 357–370 (1986)
[74] Polizzotto, C.: On the condition to prevent plastic shakedown of structures ASME. J. Appl. Mech. I, II(60), 15–25, 318–330 (1993)
[75] Polizzotto, C., Borino, G.: Shakedown and steady state responses of elastic-plastic solids in large displacements. Int. J. Sol. Struct. 33, 3415–3437 (1996)
[76] Polizzotto, C., Borino, G., Caddemi, S., Fuschi, P.: Shakedown problems for material models with internal variables. Int. J. Mech. Sci. 35, 787–801 (1993)
[77] Polizzotto, C., Borino, G., Fuschi, P.: An extended shakedown theory for elastic-damaged models. Eur. J. Mech. - A/Solids 15, 825–858 (1996)
[78] Polizzotto, C., Borino, G., Fuschi, P.: Weak forms of shakedown for elastic-plastic structures exhibiting ductile damage. Meccanica 36, 49–66 (2001)
[79] Życzkowski, M.: Combined Loadings in the Theory of Plasticity. Polish-Scientific Publ. (1981)
[80] König, J.A.: Shakedown of strainhardening structures. 1st Canad. Cong. Appl. Mech., Quebec. (1967)
[81] König, J.A.: A shakedown theorem for temperature dependent elastic moduli. Bull. Acad. Polon. Sci. Sér. Sci. Tech. 17, 161–165 (1969)
[82] König, J.A.: Deflection bounding at shakedown under thermal and mechanical loadings. In: Second SMiRT Conf., Berlin, paper L7/3 (1973)
[83] König, J.A.: A method of shakedown analysis of frames and arches. Int. J. Sol. Struct., 327–344 (1971)
[84] König, J.A.: Shakedown deflections, a finite element approach. Teoret. I Prilož. Meh. 3, 65–69 (1972)
[85] König, J.A.: Shakedown of Elastic-Plastic Structures. Elsevier, Amsterdam (1987)

[86] Borkowski, A.: Analysis of Skeletal Structural Systems in the Elastic, and Plastic Range. PWN and Elsevier, Warsaw (1988)
[87] Borkowski, A., Kleiber, M.: On a numerical approach to shakedown analysis of structures. Comput. Methods Appl. Mech. Engng. 22, 101 (1980)
[88] Dorosz, S.: An improved upper bound to maximum deflections of elasticplastic structures at shakedown. J. Struct. Mech. 6, 267–287 (1978)
[89] Dorosz, S., König, J.A., Sawczuk, A., Biegus, A., Kowal, Z., Seidel, W.: Deflections of elastic-plastic hyperstatic beams under cyclic loading. Arch. Mech. 33, 611–624 (1981)
[90] Dorosz, S., König, J.A.: Iterative method of evaluation of elastic-plastic deflections of hyperelastic structures. Ing. Archiv. 55, 202–222 (1985)
[91] Janas, M., Pycko, S., Zwoliński, J.: A min-max procedure for the shakedown analysis of skeletal structures. Int. Journal of Mechanical Sciences 37, 629–649 (1995)
[92] Janas, M., König, J.A.: A cylindrical tank response as an example of shakedown of non-Clapeyronian systems. Arch. Mech. 43, 49–56 (1991)
[93] Kleiber, M., König, J.A.: Incremental shakedown analysis in the case of thermal effects. J. Num. Methods Engng. 20, 15–67 (1984)
[94] König, A., Kleiber, M.: On a new method of shakedown analysis. Bull. Acad. Pol.-Sci., Ser. Sci. Technol. 26, 165 (1978)
[95] König, J.A., Siemaszko, A.: Strainhardening effects in shakedown process. Ing.-Archiv 58, 58–66 (1988)
[96] König, J.A., Maier, G.: Shakedown of elastoplastic structures, a review of recent developments. Nuccl. Eng. Design 66, 81–95 (1981)
[97] König, J.A., Pycko, S.: Shakedown analysis in the case of imposed displacements. Mech., Teor. Stos. 28, 101–108 (1990)
[98] Mróz, Z.: On the theory of steady plastic cycles in structures. In: Proc. 1st SMIRT, L6, 489501 (1971)
[99] Orkisz, J., Orringer, O., Holowinski, M., Pazdanowski, M., Cecot, W.: Discrete analysis of actual residual stress resulting from cyclic loadings. Comput. Struct. 35, 397 (1990)
[100] Pycko, S., Mróz, Z.: Alternative approach to shakedown as a solution of min-max problem. Acta Mechanica 93, 205 (1992)
[101] Pycko, S., König, J.A.: Elastic-plastic structures subjected to variable repeated imposed displacements and mechanical loads. Int. J. Plasticity 8, 603–618 (1992)
[102] Sawczuk, A.: Evaluation of upper bounds to shakedown loads of shells. J. Mech. Phys. Solids 17, 291–301 (1969)
[103] Sawczuk, A.: Shakedown analysis of elastic-plastic structures. Nucl. Eng. Design 28, 121–136 (1974)
[104] Sawczuk, A.: Mechanics and Plasticity of Structures. Ellis-Horwood/PWN, Chichester/Warsaw (1989)
[105] Siemaszko, A., Mróz, Z.: Sensitivity of plastic optimal structures to imperfections and non-linear geometrical effects. Structural Opimization 3, 99–105 (1991)
[106] Siemaszko, A., König, J.A.: Analysis of stability of incremental collapse of skeletal structures. J. Struct. Mech. 13, 301–321 (1985)
[107] Siemaszko, A., König, J.A.: Shakedown optimisation accounting for non-linear geometrical effects. ZAMM 71, 294–296 (1991)
[108] Zwoliński, J., Bielawski, G.: An optimal selection of residual stress for determination limit and shakedown multiplier. In: Proc. of Conf. on Comp'. Meth. struct. Mecł., Jadwisin, p. a59 (1987) (in Polish)

[109] Skoczeń, B., Skrzypek, J., Bielski, J.: Shakedown and inadaptation mechanisms of bellows subject to constant pressure and cyclic axial forces. Mech. Struct. Mach. 20, 119 (1992)
[110] Skoczeń, B., Skrzypek, J.: Inadaptation mechanisms in bellows subject to sustained pressure and cyclic axial loadings in terms of finite deformations. In: Mróz, Z., Weichert, D., Dorosz, S. (eds.) Inelastic Behaviour of Structures under Variable Loads, pp. 341–361. Kluwer Academic Publishers (1995)
[111] Mahrenholtz, O., Leers, K., König, J.A.: Shakedown of tubes: a theoretical analysis and experimental investigations. In: Reid, S.R. (ed.) Metal Forming and Impact Mechanics, W. Johnson Commemorative Volume, pp. 155–172. Pergamon Press (1984)
[112] Leers, K.: Experimentelle und theoretische Shakedownuntersuchung an Rohren. VDI-Verlag, Düsseldorf (1985)
[113] Leers, K., Klie, W., König, J.A., Mahrenholtz, O.: Experimental investigations of shakedown of tubes. In: Sawczuk, A., Bianchi, G. (eds.) Plasticity Today, pp. 259–275. Elevier Appl. Sci. Publ., London (1985)
[114] Stein, E., Zhang, G., Mahnken, R., König, J.A.: Micromechanical Modeling and Computation of Shakedown with Nonlinear Kinematic Hardening including examples for 2-D problems. In: Proc. CSME Mechanical Engineering Forum, Toronto, pp. 425–430 (1990)
[115] Stein, E., Zhang, G., König, J.A.: Shakedown with Nonlinear Hardening including Structural Computation using Finite Element Method. Int. J. Plasticity 8, 1–31 (1992)
[116] Stein, E., Zhang, G., Mahnken, R.: Shakedown Analysis for Perfectly Plastic and Kinematic Hardening Materials. In: Stein, E. (ed.) Progress in Computational Analysis of Inelastic Structures, pp. 175–244. Springer (1993)
[117] Stein, E., Zhang, G., Huang, Y.: Modeling and computation of shakedown problems for nonlinear hardening materials. Comput. Methods Appl. Mech. Engrg. 103, 247–272 (1993)
[118] Huang, Y., Stein, E.: Shakedown of a cracked body consisting of kinematic hardening material. Engineering Fracture Mechanics 54, 107–112 (1996)
[119] Huang, Y., Stein, E.: Prediction of the fatigue threshold for a cracked body by using shakedown theory. Fatigue & Fracture of Engineering Materials & Structures 18(3), 363–370 (1995)
[120] Huang, Y., Stein, E.: Shakedown of a CT-specimen with St52-steel: Experimental, analytical and numerical investigations. Journal of Strain Analysis 30(4), 283–289 (1995)
[121] Wiechmann, K., Barthold, F.-J., Stein, E.: Optimization of elasto-plastic structures using the finite element method. In: 2nd World Congress of Structural and Multidisciplinary Optimisation, pp. 1013–1018 (1997)
[122] Wiechmann, K., Barthold, F.-J., Stein, E.: Shape Optmization under Shakedown Constraints. In: Weichert, D., Maier, G. (eds.) Inelastic Analysis of Structures under Variable Loads, pp. 49–68. Kluwer Academic Publishers (2000)
[123] Stein, E.: Private communication
[124] Rafalski, P.: Minimun Principles in Plasticity, Mitteilungen aus dem Institut für Mechanik Bochum. Band 13 (1978)
[125] Weichert, D.: On the influence of geometrical nonlinearities on the shakedown of elastic-plastic structures. Int. J. Plast. 2(2), 135–148 (1986)

[126] Weichert, D., Groß-Weege, J.: The numerical assessment of elastic-plastic sheets under variable mechanical and thermal loads using a simplified two-surface yield-condition. Int. J. Mech. Sci. 30(10), 757–767 (1989)
[127] Raad, L., Weichert, D., Haidar, A.: Analysis of full-depth asphalt concrete pavements using shakedown-theory. Transactions of the Transportation Research Board (NRC), Transportation Research Record 1227, 53–65 (1989)
[128] Raad, L., Weichert, D., Najim, W.: Stability of multilayer systems under repeated loads. Transactions of Transportation Research Board (NRC), Transportation Research Record 1207, Pavement Design, 181–186 (1988)
[129] Weichert, D., Raad, L.: Extension of the static shakedown-theorem to a certain class of materials with variable elastic coefficients. Mech. Res. Comm. 19(6), 511–517 (1992)
[130] Boulbibane, M., Weichert, D., Raad, L.: Numerical application of shakedown theory to pavements with anisotropic layer properties, Paper No. 99-0342. Journal of the Transportation Research Board (NRC), Transportation Research Record 1687, 75–81 (1999)
[131] Boulbibane, M., Collins, I.F., Weichert, D., Raad, L.: Shakedown analysis of anisotropic asphalt concrete pavements with clay subgrade. Geotech. J. 37, 882–889 (2000)
[132] Hachemi, A., Weichert, D.: An extension of the static shakedown-theorem to a certain class of damaging inelastic material. Arch. of Mech. 44(5-6), 491–498 (1992)
[133] Hachemi, A., Weichert, D.: Application of shakedown theorems to damaging inelastic material under mechanical and thermal loads. Int. J. Mech. Sci. 39(9), 1067–1076 (1997)
[134] Belouchrani, M., Weichert, D.: An extension of the static shakedown theorem to inelastic cracked structures. Int. J. Mech. Sci. 41, 163–177 (1999)
[135] Belouchrani, M.A., Weichert, D., Hachemi, A.: Fatigue threshold computation by shakedown theory. Mech. Res. Comm. 27(3), 287–293 (2000)
[136] Hachemi, A., Weichert, D.: Numerical shakedown analysis of damaged structures. Comp. Meth. Appl. Mech. Engng. 160, 57–70 (1998)
[137] Hamadouche, M.A., Weichert, D.: Application of shakedown theory to soil dynamics. Mech. Res. Comm. 26(5), 565–574 (1999)
[138] Weichert, D., Schwabe, F., Hachemi, A.: Composite design by shakedown analysis for low cycle fatigue service conditions. In: Khan, A.S., Zhang, H., Yuan, Y. (eds.) Proceedings of the 8th Int. Symp. on Plasticity and its Current Applications (PLASTICITY 2000), held at Whistler, Canada, July 16-20, pp. 505–507. Neat Press (2000)
[139] Weichert, D., Hachemi, A.: Shakedown- and Limit Analysis of periodic composites. Journal of Theoretical and Applied Mechanics 40(1), 273–289 (2002)
[140] Hachemi, A., Weichert, D.: On the problem of interfacial damage in fibre-reinforced composites under variable loads. Mech. Res. Comm. 32, 15–23 (2005)
[141] Hachemi, A., Mouhtamid, S.D.: Weichert, Progress in shakedown analysis with applications to composites. Archives of Applied Mechanics, 1–11 (2005)
[142] Weichert, D., Hachemi, A.: A shakedown approach to the problem of damage of fiber-reinforced composites. In: Sadowski, T., (ed.) Proc. IUTAM-Symposium on “Multiscale Modelling of Damage and Fracture Processes in Composite Materials”, KazimierzDolny, Poland, May 23-27, Solid Mechanics and its Applications, vol. 135, pp. 41–48. Springer (2006)

[143] Nguyen, A.D., Hachemi, A., Weichert, D.: Application of the interior-point method to shakedown analysis of pavements. Int. J. Numer. Meth. Engng. 75, 414–439 (2008)

[144] Weichert, D., Hachemi, A.: Recent advances in lower bound shakedown analysis. In: Proceedings of Pressure Vessel and Piping Conference 2009, Prague, Czech Republic, July 26-30. ASME (2009)

[145] Simon, J.-W., Chen, M., Weichert, D.: Shakedown analysis combined with the problem of heat conduction. In: Proceedings of the ASME 2010 Pressure Vessels & Piping Division/K-PVP Conference, PVP 2010, Bellevue, Washington, USA, July 18-22 (2010)

[146] Simon, J.-W., Weichert, D.: An improved Interior-Point Algorithm for Large-Scale Shakedown Analysis PAMM. Proc. Appl. Math. Mech. 10, 223–224 (2010)

[147] Morelle, P.: Numerical shakedown analysis of axisymmetric sandwich shells: An upper bound formulation. Int. J. Num. Meth. 23(11), 2071–2088 (1986)

[148] Hung, N.-D., König, J.A.: Finite element formulation for shakedown problems using a yield criterion of the mean. Computer in Applied Mechanics and Engineering 8(2), 179–192 (1976)

[149] Hung, N.-D., Palgen, L.: Shakedown analysis by displacement method and equilibrium finite element. Transactions of the CSME 6(1), 32–39 (1980)

[150] Hung, N.-D.: Shakedown analysis by finite element method and linear programming techniques. Journal de Mécanique Appliquée 2(4), 587–599 (1983)

[151] Hung, N.-D., Yan, A.-M.: Direct finite element kinematical approaches in limit and shakedown analysis of shells and elbows. In: Inelastic Analysis of Structures under Variable Loads, Theory and Engineering Applications, pp. 233–254. Kluwer Academic Publishers (2000)

[152] Hung, N.-D., Vu, D.K.: Primal-dual algorithm for shakedown analysis of structures. Computer Methods in Applied Mechanics and Engineering 193(42-44), 4663–4674 (2004)

[153] Staat, M., Heitzer, M., Yan, A.M., Vu, D., Hung, N.-D., Voldoire, F., Lahousse, A.: Limit Analysis of Defects (In collaboration with, Berichte des Forschungszentrum Jülich 3746, Jül-3746) (2000)

[154] Vu, D., Staat, M., Tran, I.T.: Analysis of pressure equipment by application of the primal-dual theory of shakedown. Commun. Numer. Meth. Engng. 23(3), 213–225 (2007)

[155] Vu, D., Staat, M.: Shakedown analysis of structures made of materials with temperature-dependent yield stress. International Journal of Solids and Structures 44(13), 4524–4540 (2007)

[156] Heitzer, M., Staat, M.: Reliability analysis of elasto-plastic structures under variable loads. In: Weichert, D., Maier, G. (eds.) Inelastic Analysis of Structures under Variable Loads: Theory and Engineering Applications, pp. 269–288. Kluwer, Academic Press, Dordrecht (2000)

[157] Tran, T.N., Phạm, P.T., Vu, D.K., Staat, M.: Reliability analysis of ine-lastic shell structures under variable loads. In: Weichert, D., Ponter, A.R.S. (eds.) Limit States of Materials and Structures: Direct Methods, pp. 135–156. Springer, Netherlands (2009)

[158] Spiliopoulos, K.V.: On the automation of the force method in the optimal plastic design of frames. Comp. Meth. Appl. Mech. Eng. 141, 141–156 (1997)

[159] Spiliopoulos, K.V.: A fully automatic force method for the optimal shakedown design of frames. Comp. Mech. 23, 299–307 (1999)

[160] Spiliopoulos, K.V.: Force Method-based procedures in the limit equilibrium analysis of framed structures. In: Weichert, D., Ponter, A.R.S. (eds.) Limit States of Materials and Structures: Direct Methods, pp. 233–252. Springer, Netherlands (2009)
[161] de Saxcé, G., Massonet, C., Morelle, P.: Discussion of paper Plastic collapse, shakedown and hysteresis. Int. J. Struct. Eng. ASCE 112(9), 2177–2183 (1987); S.A. Grualnik, S. Singh, T. Erber (eds.)
[162] de Saxcé, G.: Une generalization de l'inégalité de Fenchel et ses applications aux lois constitutives. C.R. Acad. Sci. t.314. série II, 125–129 (1992)
[163] Bodoville, G., de Saxcé, G.: Plasticity with non linear kinematic hardening: modeling and shakedown analysis b the bipotential approach. Int. J. Plast. 17(1), 21–46 (2001)
[164] Bousshine, L., Chaaba, A., de Saxcé, G.: A new approach to shakedown analysis for non-standard elastoplastic material by the bipotential. Int. J. Plast. 19(5), 583–598 (2003)
[165] Moreau, J.J.: On unilateral constraints, friction and plasticity, in: new variational techniques in Mathematical Physics. CIME Course, pp. 173–322. Springer (1974)
[166] Debordes, O., Nayroles, B.: Sur la théorie et le calcul à l'adaptation des structures élasto-plastiques. J. Mécanique 20, 1–54 (1976)
[167] Nayroles, B.: Tendences récentes et perspectives à moyen terme en élastoplastici-té asymptotique des constructions. In: Congrès Français de Mécanique, Grenoble, France (1977)
[168] Nayroles, B., Weichert, La, D.: notion de sanctuaire d'élasticité et l'adaptation des structures. C.R. Acad. Sci. Paris 316, Série II, 1493–1498 (1993)
[169] Nguyen, Q.-S.: Extension des théorèmes d'adaptation et d'unicité en écrouissage non linéaire. C.R. Acad. Sc. 282, 755–758 (1976)
[170] Nguyen, Q.-S.: Min-Max Duality and shakedown theorems in plasticity. In: Alart, P., Maisonneuve, O., Rockafellar, R.T. (eds.) Nonsmooth Mechanics and Analysis, Theoretical and Numerical Advances, ch. 8. Springer (2006)
[171] Nguyen, Q.-S., Pham, D.: On shakedown theorems in hardening plasticity. C.R. Acad. Sci. 329, 307–314 (2001)
[172] Halphen, B.: Q-S Nguyen, Sur les matériaux standard généralisés. J. Mécanique 14, 1–37 (1975)
[173] Mandel, J.: Adaptation d'une structure plastique écrouissable. Mech. Res. Comm. 3, 251–256 (1976)
[174] Radenkovic, D.: Théorèmes limites pour un materiau de Coulomb à dilatation non-standardisée. Comptes Rendus de l'Académie des Sciences Paris 252, 4103–4104 (1961)
[175] Zarka, J., Frelat, J., Inglebert, G., Kasmaï-Navidi, P.: A new approach to inelastic analysis of structures, CADLM edition, France (1989)
[176] Inglebert, G., Zarka, J.: On a simplified inelastic analysis of structures. Nucl. Eng. 57, 333–368 (1980)
[177] Hassine, T., Inglebert, G., Pons, M.: shakedown and damage analysis applied to rocket machines. In: Weichert, D., Maier, G. (eds.) Inelastic Analysis of Structures under Variable Loads: Theory and Engineering Applications, pp. 255–2267. Kluwer, Academic Press, Dordrecht (2000)
[178] Sharp, R.W., Booker, J.R.: Shakedown of pavements under moving surface loads. Journal of Transportation Engng. ASCE 110(1), 1–14 (1984)
[179] Krabbenhøft, K., Lyamin, A.V., Sloan, S.W.: Bounds to shakedown loads for a class of deviatoric plasticity models. Comp. Mech. 39(6), 879–888 (2007)

[180] Krabbenhøft, K., Lyamin, A.V., Sloan, S.W.: Shakedown of a cohesive-frictional half-space subjected to rolling and sliding contact. Int. J. Solids and Structures 44(11-12), 3998–4008 (2007)
[181] Zhao, J.D., Sloan, S.W., Lyamin, A.V., Krabbenhøft, K.: Bounds for shakedown of cohesive-frictional materials under moving surface loads. Int. J. Solids and Structures 45(11-12), 3290–3312 (2008)
[182] Collins, I.F., Wang, A., Saunders, L.: Shakedown Theory and the design of unbound pavements. Road and Transport Research 2, 28–39 (1993)
[183] Collins, I.F., Wang, A., Saunders, L.: Shakedown in layered pavements under moving surface loads. Int. J. Numerical and Analytical Methods in Geomechanics 17, 165–174 (1993)
[184] Grundy, P.: Shakedown of Bars in Bending and Tension. J. Eng. Mech. Div. ASCE 95(EM3), 519–529 (1969)
[185] Alwis, W.A.M., Grundy, P.: Shakedown Analysis of Plates. Int. J. Mech. Sci. 27(1/2), 71–82 (1985)
[186] Shiau, S.H., Yu, H.S.: Finite element method for shakedown analysis of pavements. In: 16th Australasian Conference on the Mechanics of Structures and Materials, pp. 17–22 (1999)
[187] Shiau, S.H., Yu, H.S.: Shakedown analysis of flexible pavements. In: The John Booker Memorial Symposium, pp. 643–653 (2000)
[188] Yu, H.S.: Foreward: shakedown theory for pavement analysis. International Journal of Road Materials and Pavement Design 6(1), 7–9 (2005)
[189] Li, H.X., Yu, H.S.: A non-linear programming approach to kinematic shakedown analysis of composite materials. International Journal of Numerical Methods in Engineering 66(1), 117–146 (2006)
[190] Yu, H.S., Wang, J.: Three-dimensional shakedown solutions for cohesive-frictional materials under moving surface loads. Int. J. Sol. Struct. 49(26), 3797–3807 (2013)