Some Fixed-Point Issues in PPTL*

Zhenhua Duan**, Qian Ma, Cong Tian, and Nan Zhang

Institute of Computing Theory and Technology and ISN Laboratory
P.O. Box 177, Xidian University, Xi'an 710071, P.R. China
{zhhduan,qma,ctian,nzhang}@mail.xidian.edu.cn

Abstract. In Propositional Projection Temporal Logic (PPTL), a *well-formed* formula is generally formed by applying rules of its syntax finitely many times. However, under some circumstances, although formulas such as ones expressed by *index set expressions*, are constructed via applying rules of the syntax infinitely many times, they are possibly still well-formed. With this motivation, this paper investigates the relationship between formulas specified by index set expressions and concise syntax expressions by means of fixed-point induction approach. Firstly, we present two kinds of formulas, namely $\bigvee_{i\in N_0} \bigcirc^i P$ and $\bigvee_{i\in N_0} P^i$, and prove they are indeed well-formed by demonstrating their equivalence to formulas $\Diamond P$ and P^+ respectively. Further, we generalize $\bigvee_{i\in N_0} \bigcirc^i Q$ to $\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q$ and explore solutions of an abstract equation $X \equiv Q \vee P \wedge \bigcirc X$. Moreover, we equivalently represent 'U' (strong until) and 'W' (weak until) constructs in Propositional Linear Temporal Logic within PPTL using the index set expression techniques.

1 Introduction

Temporal Logic (TL) [11] is a useful formalism for specifying properties of concurrent systems. Variants of TL have been proposed, such as Linear Temporal logic (LTL) [13], Computational Tree Logic (CTL) [1], Interval Temporal Logic (ITL) [12], and Projection Temporal Logic (PTL) [3,4] etc. Propositional PTL (PPTL) [3] is a propositional subset of PTL with a usual next construct $\bigcirc P$ and a new projection construct $(P_1, \ldots, P_m)$ prj Q as its basic constructs. At present, a decision procedure [6] and an axiomatic system [7] for PPTL are available, which enables PPTL to be utilized in both model checking [2] and theorem proving [9,8].

In general, a *well-formed* formula in PPTL is obtained through applying rules of its syntax finitely many times. However, under some circumstances, although formulas such as $\bigvee_{i\in N_0} \bigcirc^i P$ with N_0 the set of non-negative integers (called *index set expression*), are formed via applying rules of the syntax countably infinitely many times, they are actually well-formed since their equivalent well-formed PPTL formulas can be found. Thus, we are motivated to identify some such formulas and prove they are indeed well-formed.

* This research is supported by the NSFC Grant Nos. 61133001, 61272118, 61272117, 61202038, 91218301 and National Program on Key Basic Research Project (973 Program) Grant No. 2010CB328102.

** Corresponding author.

Z. Liu, J. Woodcock, and H. Zhu (Eds.): He Festschrift, LNCS 8051, pp. 151–165, 2013.

Our contributions are three-fold: (1) We present two kinds of formulas with index set expressions, namely $\bigvee_{i \in N_0} \bigcirc^i P$ and $\bigvee_{i \in N_0} P^i$, and prove they are indeed well-formed by means of demonstrating their equivalence to $\Diamond P$ and P^+ respectively with fixed-point induction method [15]. (2) We generalize $\bigvee_{i \in N_0} \bigcirc^i Q$ to $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$ and explore the least and great fixed-points of the abstract equation $X \equiv Q \vee P \wedge \bigcirc X$. (3) We equivalently represent the operator 'U' (strong until) and 'W' (weak until) of Propositional LTL (PLTL) within PPTL using the index set expression technique.

This paper is organized as follows. Section 2 briefly introduces PPTL. In Section 3, some fixed-point issues concerning $\bigvee_{i \in N_0} \bigcirc^i P$, $\bigvee_{i \in N_0} P^i$ and $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$ are given. Moreover, Section 4 is devoted to equivalently denoting 'U' and 'W' constructs of PLTL within PPTL. Finally, conclusions are drawn in Section 5.

2 Propositional Projection Temporal Logic

Propositional Projection Temporal Logic (PPTL) [3,4] is an extension of Propositional ITL (PITL) [12] with a new projection construct [5]. Let *Prop* be a countable set of atomic propositions and $B = \{true, false\}$ the boolean domain. Usually, we use small letters, possibly with subscripts, like *p,q,r* to denote atomic propositions and capital letters, possibly with subscripts, like *P, Q, R* to represent general PPTL formulas. Then the formulas of PPTL are defined by the following grammar:

$$P ::= p \mid \neg P \mid P_1 \wedge P_2 \mid \bigcirc P \mid (P_1, \ldots, P_m)\, \mathsf{prj}\ P \mid P^+$$

where $p \in Prop$, $\bigcirc$ (next), $+$ (chop-plus) and prj (projection) are temporal operators, and $\neg, \wedge$ are similar as that in classical propositional logic.

We define a *state* s over *Prop* to be a mapping from *Prop* to B, $s : Prop \to B$. We write $s[p]$ to denote the valuation of p at state s. An *interval* $\sigma = \langle s_0, s_1, ...\rangle$ is a non-empty sequence of states, which can be finite or infinite. The length of σ, $|\sigma|$, is the number of states in σ minus one if σ is finite; otherwise it is ω. To have a uniform notation for both finite and infinite intervals, we will use *extended integers* as indices, that is, $N_\omega = N_0 \cup \{\omega\}$ and extend the comparison operators, $=, <, \leq$, to N_ω by considering $\omega = \omega$ and for all $i \in N_0, i < \omega$. Moreover, we write $\preceq$ as $\leq -\{(\omega, \omega)\}$. Let $\sigma = \langle s_0, s_1, \ldots\rangle$ be an interval and $r_1, \ldots, r_h$ be integers ($h \geq 1$) such that $0 \leq r_1 \leq \ldots \leq r_h \preceq |\sigma|$. The projection of σ onto $r_1, \ldots, r_h$ is the *projected interval*, $\sigma \downarrow (r_1, \ldots, r_h) \stackrel{\text{def}}{=} \langle s_{t_1}, s_{t_2}, \ldots, s_{t_l}\rangle$, where $t_1, \ldots, t_l$ are attained from $r_1, \ldots, r_h$ by deleting all duplicates. In other words, $t_1, \ldots, t_l$ is the longest strictly increasing subsequence of $r_1, \ldots, r_h$. The concatenation($\cdot$) of an interval σ with another interval σ' is represented by $\sigma \cdot \sigma'$ (not sharing any states).

An *interpretation* is a tuple $\mathcal{I} = (\sigma, k, j)$, where $\sigma = \langle s_0, s_1, \ldots\rangle$ is an interval, k is a non-negative integer, and j is an integer or ω, such that $0 \leq k \preceq j \leq |\sigma|$. We write (σ, k, j) to mean that a formula is interpreted over a subinterval $\sigma_{k,\ldots,j}$ with the current state being s_k. We utilize I^k_{prop} to stand for the state interpretation at state s_k. The satisfaction relation $\models$ for formulas is given as follows:

$$\begin{array}{l} \mathcal{I} \models p \text{ iff } s_k[p] = I^k_{prop}[p] = \ true \\ \mathcal{I} \models \neg P \text{ iff } \mathcal{I} \not\models P \\ \mathcal{I} \models P_1 \wedge P_2 \text{ iff } \mathcal{I} \models P_1 \text{ and } \mathcal{I} \models P_2 \end{array}$$

$\mathcal{I} \models \bigcirc P$ iff $k < j$ and $(\sigma, k+1, j) \models P$
$\mathcal{I} \models (P_1, \ldots, P_m)$ prj P iff there exist integers $r_0, \ldots, r_m$, and $k = r_0 \leq \ldots \leq r_{m-1} \preceq r_m \leq j$ such that $(\sigma, r_{l-1}, r_l) \models P_l$ for all $1 \leq l \leq m$ and $(\sigma', 0, |\sigma'|) \models P$ for σ' given by :
(1) $r_m < j$ and $\sigma' = \sigma \downarrow (r_0, \ldots, r_m) \cdot \sigma_{(r_m+1, \ldots, j)}$
(2) $r_m = j$ and $\sigma' = \sigma \downarrow (r_0, \ldots, r_h)$ for some $0 \leq h \leq m$
$\mathcal{I} \models P^+$ iff there are finitely many integers $r_0, \ldots, r_n$ and $k = r_0 \leq r_1 \leq \ldots \leq r_{n-1} \preceq r_n = j$ $(n \geq 1)$ such that $(\sigma, r_{l-1}, r_l) \models P$ for all $1 \leq l \leq n$; or $j = \omega$ and there are infinitely many integers $k = r_0 \leq r_1 \leq r_2 \leq \ldots$ such that $\lim_{i \to \infty} r_i = \omega$ and $(\sigma, r_{l-1}, r_l) \models P$ and for all $l \geq 1$.

A formula *P* is satisfied by an interval σ, signified by $\sigma \models P$ if $(\sigma, 0, |\sigma|) \models P$. A formula *P* is called *satisfiable* if $\sigma \models P$ for some σ. Furthermore, *P* is said to be *valid*, denoted by $\models P$, if $\sigma \models P$ for all intervals σ.

Some derived formulas of PPTL are shown below, which are explained in [4,3]. The abbreviations true, false, $\vee$, $\rightarrow$ and $\leftrightarrow$ are defined as usual.

$$
\begin{array}{llllll}
\varepsilon & \stackrel{\text{def}}{=} & \neg \bigcirc \mathsf{true} & P^* & \stackrel{\text{def}}{=} & P^+ \vee \varepsilon \\
\Diamond P & \stackrel{\text{def}}{=} & (\mathsf{true}, P) \ \mathsf{prj} \ \varepsilon & \mathsf{more} & \stackrel{\text{def}}{=} & \neg \varepsilon \\
\Box P & \stackrel{\text{def}}{=} & \neg \Diamond \neg P & \mathsf{fin}(P) & \stackrel{\text{def}}{=} & \Box(\varepsilon \rightarrow P) \\
\mathsf{halt}(P) & \stackrel{\text{def}}{=} & \Box(\varepsilon \leftrightarrow P) & \mathsf{keep}(P) & \stackrel{\text{def}}{=} & \Box(\neg\varepsilon \rightarrow P) \\
P\,;Q & \stackrel{\text{def}}{=} & (P, Q) \ \mathsf{prj} \ \varepsilon & P\,;_{\mathsf{w}}Q & \stackrel{\text{def}}{=} & (P\,;Q) \vee (P \wedge \Box\mathsf{more}) \\
\mathsf{fin} & \stackrel{\text{def}}{=} & \Diamond\varepsilon & \mathsf{len}(n) & \stackrel{\text{def}}{=} & \begin{cases} \varepsilon & \text{if } n = 0 \\ \bigcirc\mathsf{len}(n-1) & \text{if } n > 1 \end{cases} \\
\mathsf{inf} & \stackrel{\text{def}}{=} & \Box\mathsf{more} & P \parallel Q & \stackrel{\text{def}}{=} & (P \wedge (Q\,;\mathsf{true})) \vee (Q \wedge (P\,;\mathsf{true}))
\end{array}
$$

Commonly, $\models \Box(P \leftrightarrow Q)$ is represented by $P \equiv Q$ (*strong equivalence*), meaning that *P* and *Q* have the same truth values at all states in every model.

3 Fixed-Point Issues

A *well-formed* formula in PPTL is generally constructed by applying rules of the syntax finitely many times. However, although some formulas are formed via applying rules of the syntax countably infinitely many times, such as *index set expressions* (e.g. $\bigvee_{i \in N_0} \bigcirc^i P$), they are still well-formed due to the existence of their equivalent well-formed formulas. In this section, we identify two types of such formulas and prove they are indeed well-formed by means of the fixed-point induction approach [15]. Besides, we generalize one of them to a more generic form and investigate some related properties of an abstract equation $X \equiv Q \vee P \wedge \bigcirc X$.

3.1 Two Kinds of Index Set Expressions

(1) $\bigvee_{i \in N_0} \bigcirc^i P$

On one hand, $\bigvee_{i \in N_0} \bigcirc^i P \equiv P \vee \bigcirc P \vee \bigcirc^2 P \vee \bigcirc^3 P \vee \ldots$, is a disjunction of countably infinitely many $\bigcirc^i P$, where $\bigcirc^0 P \equiv P$. Intuitively, this formula means P

necessarily holds at some state from now on over an interval, which might be specified by the operator $\Diamond$. On the other hand, $\Diamond P$ indeed can be rewritten as:

$$\begin{aligned}\Diamond P &\equiv P \vee \bigcirc \Diamond P \\ &\equiv P \vee \bigcirc(P \vee \bigcirc \Diamond P) \\ &\equiv P \vee \bigcirc P \vee \bigcirc^2 \Diamond P \\ &\quad \dots \\ &\equiv P \vee \bigcirc P \vee \bigcirc^2 P \vee \bigcirc^3 P \vee \dots \qquad (\star)\end{aligned}$$

From the above, we observe that $\bigvee_{i \in N_0} \bigcirc^i P$ seems to be equivalent to $\Diamond P$, which will be affirmed in Theorem 1.

(2) $\bigvee_{i \in N_0} P^i$

For a chop formula $P_1 \,;\, \dots \,;\, P_m$, if all $P_i \equiv P$ $(1 \leq i \leq m)$, we can acquire:

$$\underbrace{P \,;\, \dots \,;\, P}_{m \ \ times}$$

which is briefly represented as P^m. For instance, $P^1 \equiv P$, $P^2 \equiv P \,;\, P$, and particularly $P^0 \equiv \mathsf{false}$. Thus, $\bigvee_{i \in N_0} P^i$ denotes $P \vee (P \,;\, P) \vee (P \,;\, P \,;\, P) \vee \dots$. Further, we have the equation about P^+:

$$\begin{aligned}P^+ &\equiv P \vee (P \,;\, P^+) \\ &\equiv P \vee (P \,;\, (P \vee (P \,;\, P^+))) \\ &\equiv P \vee P \,;\, P \vee P \,;\, (P \,;\, P^+)) \\ &\quad \dots \\ &\equiv P \vee (P \,;\, P) \vee (P \,;\, P \,;\, P) \vee \dots\end{aligned}$$

Hence, we can declare that $\bigvee_{i \in N_0} P^i \equiv P^+$ in Theorem 1.

Theorem 1. *The following logical laws hold:*

1. $\bigvee_{i \in N_0} \bigcirc^i P \equiv \Diamond P$
2. $\bigvee_{i \in N_0} P^i \equiv P^+$

Proof. The two laws can be proved in an analogous way and we only prove $\bigvee_{i \in N_0} \bigcirc^i P \equiv \Diamond P$. The proof proceeds by fixed-point induction approach.

We firstly define $D = \{d_{-1}, d_0, \dots, d_n, \dots, d_\omega\}$, where $d_{-1} = \bigcirc^{-1} P = \mathsf{false}$, $d_i = \bigcirc^0 P \vee \dots \vee \bigcirc^i P$ $(i \in N_0)$, $d_\omega = \bigvee_{i \in N_0} \bigcirc^i P$. Let $N_\omega = N_0 \cup \{\omega\}$ with $\omega = \omega, \omega + c = \omega$ (c is an integer) and for all $i \in N_0, i < \omega$. Further, a binary relation $\preccurlyeq$ over D is formalized as

$$d_i \preccurlyeq d_j \text{ iff } i \leq j \ (i, j \in N_\omega \cup \{-1\})$$

Moreover, let $f : D \to D$ be a function given by

$$f(d_i) = P \vee \bigcirc d_i$$

Then $f(d_i) = P \vee \bigcirc(P \vee \ldots \vee \bigcirc^i P) = d_{i+1}$ for $i \in \{-1\} \cup N_0$, and $f(d_\omega) = P \vee \bigcirc(\bigvee_{i\in N_0} \bigcirc^i P) = P \vee \bigvee_{i\in N_0} \bigcirc^{i+1} P = \bigvee_{i\in N_0} \bigcirc^i P = d_\omega$. Obviously, we have $d_i \sqcup d_j = d_i \vee d_j = d_j$ if $d_i \preccurlyeq d_j$.

1. $\bigvee_{i\in N_0} \bigcirc^i P$ is the least fixed-point of f
(1) $(D, \preccurlyeq)$ is a complete partial order
$(D, \preccurlyeq)$ is a partial order, since it satisfies the properties below:

- reflexivity: for all $d_i \in D$, clearly we have $d_i \preccurlyeq d_i$ due to $i \leq i$.
- anti-symmetry: if $d_i \preccurlyeq d_j$ and $d_j \preccurlyeq d_i$, then we obtain $i \leq j$ and $j \leq i$, leading to $i = j$. Hence $d_i = d_j$.
- transitivity: if $d_i \preccurlyeq d_j$, $d_j \preccurlyeq d_k$, then $i \leq j \leq k$. Thus, $d_i \preccurlyeq d_k$.

Furthermore, for any non-empty subset $S = \{d_{i_1}, \ldots, d_{i_n}\}$ of D, where $-1 \leq i_1 \leq \ldots \leq i_{n-1} \preceq i_n \leq \omega$, if S is finite, as $d_i \sqcup d_j = d_j$, we obtain the least upper bound $d_{i_n} \in D$, where i_n is the biggest index in S; otherwise, S is infinite, there exists a least upper bound $\bigsqcup_{i_n\in N_0} d_{i_n} = \bigvee_{i_n\in N_0} d_{i_n} = (P \vee \bigcirc P \vee \ldots \vee \bigcirc^{i_1} P) \vee (P \vee \bigcirc P \vee \ldots \vee \bigcirc^{i_2} P) \vee \ldots = d_\omega$, which obviously belongs to D. Thus, $(D, \preccurlyeq)$ is a complete partial order.

(2) f is a continuous function
Suppose that $d_i \preccurlyeq d_j$, then $i \leq j$, so $i + 1 \leq j + 1$. As a result,

$$f(d_i) = d_{i+1} \preccurlyeq d_{j+1} = f(d_j)$$

Hence, f is monotonic. Moreover, for an arbitrary ω-chain in D, $d_{i_1} \preccurlyeq d_{i_2} \preccurlyeq \ldots \preccurlyeq d_{i_n} \preccurlyeq \ldots$, if there exists an element d_{i_n} such that $d_{i_n} \preccurlyeq d_{i_n} \preccurlyeq \ldots$, it is apparent that d_{i_n} is the least upper bound of this ω-chain. Thus, we have

$$\begin{aligned} f(\bigsqcup_{i_n\in N_0} d_{i_n}) = f(d_{i_n}) = d_{i_n+1} = d_{i_n} \sqcup d_{i_n+1} &= \bigsqcup_{i_n\in N_0} d_{i_n} \sqcup d_{i_n+1} \\ &= d_0 \sqcup \bigsqcup_{i_n\in N_0} d_{i_n+1} \\ &= \bigsqcup_{i_n\in N_0} d_{i_n+1} = \bigsqcup_{i_n\in N_0} f(d_{i_n}) \end{aligned}$$

Otherwise, we can obtain the following:

$$\begin{aligned} \bigsqcup_{i_n\in N_0} f(d_{i_n}) = \bigsqcup_{i_n\in N_0} d_{i_n+1} = d_0 \sqcup \bigsqcup_{i_n\in N_0} d_{i_n+1} &= \bigsqcup_{i_n\in N_0} d_{i_n} \\ &= d_\omega = f(d_\omega) = f(\bigsqcup_{i_n\in N_0} d_{i_n}) \end{aligned}$$

Therefore, f is a continuous function. Hence, by Kleene Fixed-point Theorem [15,10], there exists a least fixed-point:

$$\begin{aligned} fix_\mu(f) = \bigsqcup_{n\in N_0} f^n(d_{-1}) = \bigsqcup_{n\in N_0} d_{n-1} &= d_{-1} \sqcup \bigsqcup_{n\in N_0} d_n \\ &= \bigsqcup_{n\in N_0} d_n = d_\omega = \bigvee_{i\in N_0} \bigcirc^i P \end{aligned}$$

2. $\bigvee_{i\in N_0} \bigcirc^i P$ is equivalent to $\Diamond P$

By the equation ($\star$), each $P \vee \bigcirc P \vee \ldots \vee \bigcirc^i P$ ($i \in N_0$) is called a *prefix* of $\Diamond P$. Particularly, false is also a prefix of $\Diamond P$. Then we construct a subset B of D as follows:

$$B = \{d_i |\ d_i \in D \text{ and } d_i \text{ is a prefix of the formula } \Diamond P\}$$

For any ω-chain $d_{i_1} \preccurlyeq d_{i_2} \preccurlyeq \ldots \preccurlyeq d_{i_n} \preccurlyeq \ldots$ in D, suppose each $d_{i_n} = P \vee \bigcirc P \vee \ldots \vee \bigcirc^{i_n} P$ ($i_n \in N_0$) is a prefix of $\Diamond P$, i.e. $d_{i_n} \in B$. Then as $d_i \sqcup d_{i+1} = d_{i+1}$, $\bigsqcup_{i_n \in N_0} d_{i_n} = P \vee \bigcirc P \vee \ldots \vee \bigcirc^i P \vee \ldots$ is also a prefix of $\Diamond P$ and belongs to B. Thus, we can obtain that B is an inclusive subset of D.

Moreover, the bottom element false is obviously a prefix of $\Diamond P$, thus $\mathsf{false} \in B$. With the assumption of $d_i \in B$, when $i \in N_0 \cup \{-1\}$, since $f(d_i) = P \vee \bigcirc(d_i) = P \vee \bigcirc(P \vee \ldots \vee \bigcirc^i P) = P \vee \bigcirc P \vee \ldots \vee \bigcirc^{i+1} P$, $f(d_i)$ is also a prefix of $\Diamond P$ and $f(d_i) \in B$; when $i = \omega$, $f(d_\omega) = d_\omega \in B$. According to Scott's fixed-point induction [15], $fix_\mu(f) = \bigvee_{i \in N_0} \bigcirc^i P$ belongs to B and is a prefix of $\Diamond P$. Besides, as $fix_\mu(f)$ is the upper bound of all the elements in D and B, $fix_\mu(f)$ is the longest prefix of $\Diamond P$. Therefore, $\bigvee_{i \in N_0} \bigcirc^i P \equiv \Diamond P$. □

It is clear that $\Diamond P$ and P^+ are well-formed formulas in accordance to the syntax of PPTL. Further, by Theorem 1, index set expressions $\bigvee_{i \in N_0} \bigcirc^i P$ and $\bigvee_{i \in N_0} P^i$ are equivalent to $\Diamond P$ and P^+ respectively. Hence, we can assert that $\bigvee_{i \in N_0} \bigcirc^i P$ and $\bigvee_{i \in N_0} P^i$ are well-formed formulas.

Corollary 1. *$\bigvee_{i \in N_0} \bigcirc^i P$ and $\bigvee_{i \in N_0} P^i$ are well-formed formulas.*

Proof. This is the direct consequence of Theorem 1. □

According to Theorem 1, we can also infer that P^+ can be represented by the projection construct prj since P^+ is equivalent to $\bigvee_{i \in N_0} P^i$ and P^i is an abbreviation of

$$\underbrace{P\ ;\ \ldots\ ;P}_{i\ \ times} \equiv \underbrace{(P, \ldots, P)}_{i\ \ times}\ \mathsf{prj}\ \varepsilon$$

Thus, with techniques in this paper, $+$ can be regarded as a derived operator within PPTL.

In order to show the practical use of such index set expressions, we give an example.

Example 1. Let $P \stackrel{\text{def}}{=} \varepsilon$ in $\bigvee_{i \in N_0} \bigcirc^i P$. Then

$$\bigvee_{i \in N_0} \bigcirc^i \varepsilon \equiv \Diamond \varepsilon \equiv \mathsf{fin}$$

which claims the interval is finite and will terminate at some point. Further, we can obtain $\bigvee_{i \in N_0} \bigcirc^i \varepsilon \vee \mathsf{inf} \equiv \Diamond \varepsilon \vee \Box \mathsf{more} \equiv \mathsf{true}$.

It is interesting to consider Theorem 1 from another viewpoint. Since $\Diamond P$ can be rewritten as $P \vee \bigcirc \Diamond P$, namely $\Diamond P \equiv P \vee \bigcirc \Diamond P$, we can abstract it as a recursive equation $X \equiv P \vee \bigcirc X$ with the equality '$\equiv$' and one solution $\Diamond P$. Then $\bigvee_{i \in N_0} \bigcirc^i P$ can also be treated as a solution of the recursive equation due to its equivalence to $\Diamond P$. It is similar for the recursive equation $X \equiv P \vee P\ ; X$, whose solution is P^+, i.e. $\bigvee_{i \in N_0} P^i$.

3.2 Generalization of $\bigvee_{i \in N_0} \bigcirc^i Q$

In this subsection, we generalize $\bigvee_{i \in N_0} \bigcirc^i Q$ to $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$, where $P^{(0)} = \mathsf{true}$, $P^{(1)} = P, P^{(2)} = P \wedge \bigcirc P, \ldots$ and $P^{(n)} = P \wedge \bigcirc P \wedge \ldots \wedge \bigcirc^{n-1} P$ $(n \in N_0)$. In particular, when $P \equiv \mathsf{true}$, $\bigvee_{i \in N_0} \mathsf{true}^{(i)} \wedge \bigcirc^i Q$ can be exactly reduced to $\bigvee_{i \in N_0} \bigcirc^i Q$.

Further, for the recursive equation $X \equiv Q \vee P \wedge \bigcirc X$, we have:

$$\begin{aligned}
X &\equiv Q \vee P \wedge \bigcirc X \\
&\equiv Q \vee P \wedge \bigcirc(Q \vee P \wedge \bigcirc X) \\
&\equiv Q \vee P \wedge \bigcirc Q \vee P \wedge \bigcirc P \wedge \bigcirc^2 X \\
&\quad \ldots \\
&\equiv Q \vee P \wedge \bigcirc Q \vee P \wedge \bigcirc P \wedge \bigcirc^2 Q \vee P \wedge \bigcirc P \wedge \bigcirc^2 P \wedge \bigcirc^3 Q \vee \ldots
\end{aligned}$$

We can see that $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$ might have something to do with the above equation, which is declared in Theorem 2.

Theorem 2. *For a recursive equation $X \equiv Q \vee P \wedge \bigcirc X$, where X, P and Q are PPTL formulas, its least fixed-point is $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$ and its greatest fixed-point is $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q \vee \Box(P \wedge \mathsf{more})$.*

Proof. At first, we prove that $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$ and $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q \vee \Box(P \wedge \mathsf{more})$ are two fixed-points of the equation $X \equiv Q \vee P \wedge \bigcirc X$ by means of simple replacement:

$$\begin{aligned}
(a) \quad & Q \vee P \wedge \bigcirc(\textstyle\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q) \\
&= Q \vee \textstyle\bigvee_{i \in N_0} P^{(i+1)} \wedge \bigcirc^{i+1} Q \\
&= \textstyle\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q \\
(b) \quad & Q \vee P \wedge \bigcirc(\textstyle\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q \vee \Box(P \wedge \mathsf{more})) \\
&= Q \vee P \wedge \bigcirc(\textstyle\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q) \vee P \wedge \bigcirc\Box(P \wedge \mathsf{more}) \\
&= \textstyle\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q \vee \Box(P \wedge \mathsf{more}) \ (P \wedge \bigcirc\Box(P \wedge \mathsf{more}) \equiv \Box(P \wedge \mathsf{more}))
\end{aligned}$$

Then we respectively employ Kleene Fixed-point Theorem [15,10] and Knaster-Tarski Fixed-point Theorem [15,14] to prove $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$ and $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q \vee \Box(P \wedge \mathsf{more})$ are the least and greatest fixed-points.

Let $d_{-1} = \mathsf{false}, d_0 = Q, d_n = Q \vee P \wedge \bigcirc Q \vee P \wedge \bigcirc P \wedge \bigcirc^2 Q \vee \ldots \vee P^{(n)} \wedge \bigcirc^n Q$ $(n \in N_0), d_{\omega_1} = \bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$ and $d_{\omega_2} = \bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q \vee \Box(P \wedge \mathsf{more})$. Then we define a set $D = \{d_{-1}, d_0, \ldots, d_n, \ldots\} \cup \{d_{\omega_1}, d_{\omega_2}\}$. Further, a binary relation $\preccurlyeq$ over D is formalized as

$$d_i \preccurlyeq d_j \ \text{ if } \begin{cases} (1)\ i \leq j \text{ and } i, j \in N_0 \cup \{-1\} \\ (2)\ i \in N_0 \cup \{-1\} \text{ and } j = \omega_1 \text{ or } \omega_2 \\ (3)\ i = \omega_1 \text{ and } j = \omega_1 \text{ or } \omega_2 \\ (4)\ i = \omega_2 \text{ and } j = \omega_2 \end{cases}$$

Moreover, let $g : D \to D$ be a function given below

$$g(d_i) = Q \vee P \wedge \bigcirc d_i$$

Then, for $i \in N_0 \cup \{-1\}$, $g(d_i) = Q \vee P \wedge \bigcirc(Q \vee P \wedge \bigcirc Q \vee \ldots \vee P^{(i)} \wedge \bigcirc^i Q) = Q \vee P \wedge \bigcirc Q \vee P \wedge \bigcirc P \wedge \bigcirc^2 Q \vee \ldots \vee P^{(i+1)} \wedge \bigcirc^{i+1} Q = d_{i+1}$; further, for

$i = \omega_1$ or ω_2, by (a)(b), we know that d_{ω_1} and d_{ω_2} are fixed-points of g, that is, $g(d_{\omega_1}) = d_{\omega_1}, g(d_{\omega_2}) = d_{\omega_2}$. Obviously, we have $d_i \sqcup d_j = d_i \vee d_j = d_j$ if $d_i \preccurlyeq d_j$. As a result, we can obtain the following two facts:

(1) $(D, \preccurlyeq)$ is a complete partial order and a complete lattice

$(D, \preccurlyeq)$ is a partial order, since it satisfies the properties given below:

- reflexivity: for $i \in N_0 \cup \{-1\}$, we have $d_i \preccurlyeq d_i$ due to $i \leq i$. Further, by the definition of $\preccurlyeq$, we obtain $d_{\omega_1} \preccurlyeq d_{\omega_1}, d_{\omega_2} \preccurlyeq d_{\omega_2}$.
- anti-symmetry: for $i, j \in N_0 \cup \{-1\}$, if $d_i \preccurlyeq d_j$ and $d_j \preccurlyeq d_i$, then we obtain $i \leq j$ and $j \leq i$, leading to $i = j$. Hence $d_i = d_j$. For other cases, according to the definition of $\preccurlyeq$, $d_i \preccurlyeq d_j \not\Longrightarrow d_j \preccurlyeq d_i$.
- transitivity: if $d_i \preccurlyeq d_j, d_j \preccurlyeq d_k$, (a) $i, j, k \in N_0 \cup \{-1\}$: by assumption, $i \leq j \leq k$, so $d_i \preccurlyeq d_k$; (b) $i \in N_0 \cup \{-1\}, j \in N_0 \cup \{-1, \omega_1, \omega_2\}, k \in \{\omega_1, \omega_2\}$: according to case (2) in the definition of $\preccurlyeq$, we can acquire $d_i \preccurlyeq d_k$; (c) $i, j, k \in \{\omega_1, \omega_2\}$, it is clear that $d_i \preccurlyeq d_k$ by the cases (3) and (4) in the definition of $\preccurlyeq$.

Furthermore, for any non-empty subset $S = \{d_{i_1}, \ldots, d_{i_n}\}$ of D, we consider the following cases:

(a) S includes neither d_{ω_1} nor d_{ω_2}:

In this case, if S is finite, as $d_i \sqcup d_j = d_j$ $(i \leq j)$, we obtain the least upper bound $d_{i_n} \in D$ with i_n the biggest index in S; otherwise, S is infinite, there exists a least upper bound $\bigsqcup_{i_n \in N_0} d_{i_n} = \bigvee_{i_n \in N_0} d_{i_n} = (Q \vee P \wedge \bigcirc Q \vee \ldots \vee P^{(i_1)} \wedge \bigcirc^{i_1} Q) \vee (Q \vee P \wedge \bigcirc Q \vee \ldots \vee P^{(i_2)} \wedge \bigcirc^{i_2} Q) \vee \ldots = Q \vee P \wedge \bigcirc Q \vee P \wedge \bigcirc P \wedge \bigcirc^2 Q \vee \ldots = \bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q = d_{\omega_1}$, which evidently belongs to D.

(b) S involves d_{ω_1} or d_{ω_2}:

If d_{ω_2} is contained in S, it is the least upper bound in S; otherwise, d_{ω_1} is the least upper bound in S.

Thus, $(D, \preccurlyeq)$ is a complete lattice, which is also a complete partial order.

(2) g is a continuous function

Suppose that $d_i \preccurlyeq d_j$, then (a) when $i, j \in N_0 \cup \{-1\}$: $i \leq j$, so $i + 1 \leq j + 1$. As a result, $g(d_i) = d_{i+1} \preccurlyeq d_{j+1} = g(d_j)$; (b) when $i \in N_0 \cup \{-1\}$ and $j = \omega_1$ or ω_2, $g(d_i) = d_{i+1} \preccurlyeq d_{\omega_t} = g(d_{\omega_t})$ $(t = 1, 2)$; (c) when $i, j \in \{\omega_1, \omega_2\}$, $g(d_i) = d_i \preccurlyeq d_j = g(d_j)$. Hence, g is monotonic.

Further, for an arbitrary ω-chain in D, $d_{i_1} \preccurlyeq d_{i_2} \preccurlyeq \ldots \preccurlyeq d_{i_n} \preccurlyeq \ldots$, if there exists an element d_{i_n} such that $d_{i_n} \preccurlyeq d_{i_n} \preccurlyeq \ldots$, it is obvious that d_{i_n} is the least upper bound of this ω-chain. Thus, we acquire

$$\begin{aligned} g(\bigsqcup_{i_n \in N_0} d_{i_n}) = g(d_{i_n}) = d_{i_n+1} = d_{i_n} \sqcup d_{i_n+1} &= (\bigsqcup_{i_n \in N_0} d_{i_n}) \sqcup d_{i_n+1} \\ &= \bigsqcup_{i_n \in N_0} d_{i_n+1} = \bigsqcup_{i_n \in N_0} g(d_{i_n}) \end{aligned}$$

Otherwise,

$$\begin{aligned} \bigsqcup_{i_n \in N_0} g(d_{i_n}) = \bigsqcup_{i_n \in N_0} d_{i_n+1} = d_0 \sqcup \bigsqcup_{i_n \in N_0} d_{i_n+1} &= \bigsqcup_{i_n \in N_0} d_{i_n} \\ &= d_{\omega_1} = g(d_{\omega_1}) = g(\bigsqcup_{i_n \in N_0} d_{i_n}) \end{aligned}$$

Therefore, g is a continuous function. Based on these, we can prove:

1. $\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q$ is the least fixed-point

Since $(D, \preccurlyeq)$ is a complete partial order, whose bottom element is false, and g is a continuous function, by Kleene fixed-point theorem, there exists a least fixed-point:

$$\begin{aligned} fix_\mu(g) = \bigsqcup_{n\in N_0} g^n(d_{-1}) = \bigsqcup_{n\in N_0} d_{n-1} &= d_{-1} \sqcup \bigsqcup_{n\in N_0} d_n \\ &= \bigsqcup_{n\in N_0} d_n \\ &= d_{\omega_1} = \bigvee\nolimits_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \end{aligned}$$

2. $\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \vee \Box(P \wedge \mathsf{more})$ is the greatest fixed-point

Let $\{x \in D | x \preccurlyeq g(x)\}$ be the set of post fixed-points of g. As $d_i \preccurlyeq d_{i+1} = g(d_i)$ $(i \in N_0 \cup \{-1\})$ and $d_{\omega_j} \preccurlyeq d_{\omega_j} = g(d_{\omega_j})$ $(j = 1, 2)$, we obtain $\{x \in D | x \preccurlyeq g(x)\} = D$. In other words, all the elements in D are post fixed-points of g. Further, $(D, \preccurlyeq)$ is a complete lattice, whose bottom element is false, and g is monotonic. According to Knaster-Tarski fixed-point theorem, the greatest fixed-point is:

$$\begin{aligned} fix_\nu(g) = \bigsqcup\{x \in D | x \preccurlyeq g(x)\} &= \bigsqcup D \\ &= d_{\omega_2} \\ &= \bigvee\nolimits_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \vee \Box(P \wedge \mathsf{more}) \end{aligned}$$

□

Corollary 2. *For a recursive equation $X \equiv Q \vee \bigcirc X$, where X and Q are PPTL formulas, its least fixed-point is $\Diamond Q$ and its greatest fixed-point is $\Diamond Q \vee \Box\mathsf{more}$.*

Proof. In Theorem 2, let $P \equiv \mathsf{true}$. Then for the equation $X \equiv Q \vee \bigcirc X$, its least fixed-point is $\bigvee_{i\in N_0} (\mathsf{true})^{(i)} \wedge \bigcirc^i Q \equiv \bigvee_{i\in N_0} \bigcirc^i Q$. Further, by Theorem 1, $\bigvee_{i\in N_0} \bigcirc^i Q \equiv \Diamond Q$, so $\Diamond Q$ is the least fixed-point of the equation $X \equiv Q \vee \bigcirc X$. Moreover, its greatest fixed-point is $\bigvee_{i\in N_0} (\mathsf{true})^{(i)} \wedge \bigcirc^i Q \vee \Box(\mathsf{true} \wedge \mathsf{more}) \equiv \Diamond Q \vee \Box\mathsf{more}$. □

In fact, Corollary 2 is a special case of Theorem 2 as well as the equation $X \equiv Q \vee \bigcirc X$ is an instance of $X \equiv Q \vee P \wedge \bigcirc X$ with $P \equiv \mathsf{true}$ but has well-formed formulas as its least and greatest fixed-points. In addition, Theorem 2 also tells us that there are at least two fixed-points for the equation $X \equiv Q \vee P \wedge \bigcirc X$. Actually, $X \equiv Q \vee P \wedge \bigcirc X$ has and only has two fixed-points, namely the least and greatest fixed-points, which is confirmed by Theorem 3.

Theorem 3. *The recursive equation $X \equiv Q \vee P \wedge \bigcirc X$ has and only has two solutions, i.e. $\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q$ and $\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \vee \Box(P \wedge \mathsf{more})$.*

Proof. It is clear that $\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q$ and $\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \vee \Box(P \wedge \mathsf{more})$ are two fixed-points of $X \equiv Q \vee P \wedge \bigcirc X$ by Theorem 2. Further, we prove the equation only has two fixed-points. We assume that there exists a third solution R such that $R \equiv Q \vee P \wedge \bigcirc R$. Since $\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q$ is the least fixed-point, $\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \preccurlyeq R$. Further, according to the proof of Theorem 2, we can acquire $\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \sqcup R =$

$\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \vee R = R$. Thus, R must be in the form of $\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \vee R'$. Therefore, we have,

$$\begin{aligned}
&\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \vee R' \\
&\equiv Q \vee P \wedge \bigcirc(\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \vee R') \\
&\equiv Q \vee P \wedge \bigcirc(\bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q) \vee P \wedge \bigcirc R' \\
&\equiv \bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \vee P \wedge \bigcirc R'
\end{aligned}$$

Accordingly, we can infer $R' \equiv P \wedge \bigcirc R'$, which can only be satisfied when $R' \equiv \mathsf{false}$ or $R' \equiv \Box(P \wedge \mathsf{more})$ within PPTL. As a result, if $R' \equiv \mathsf{false}$, $R \equiv \bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q$ while if $R' \equiv \Box(P \wedge \mathsf{more})$, $R \equiv \bigvee_{i\in N_0} P^{(i)} \wedge \bigcirc^i Q \vee \Box(P \wedge \mathsf{more})$. Hence, the equation only has two fixed-points. □

3.3 Examples

To intuitively understand the above theorems, we present some examples below.

Example 2. Let $P \stackrel{\text{def}}{=} R$ and $Q \stackrel{\text{def}}{=} R \wedge \varepsilon$. Then $X \equiv Q \vee P \wedge \bigcirc X$ can be instantiated as

$$X \equiv R \wedge \varepsilon \vee R \wedge \bigcirc X \tag{3.3.1}$$

According to Theorem 2, we can respectively obtain the least and greatest fixed-points of the equation (3.3.1) as

$$\bigvee_{i\in N_0} R^{(i)} \wedge \bigcirc^i (R \wedge \varepsilon) \quad \text{and} \quad \bigvee_{i\in N_0} R^{(i)} \wedge \bigcirc^i (R \wedge \varepsilon) \vee \Box(R \wedge \mathsf{more})$$

where the greatest fixed-point claims that R always holds either over an interval with the length i or over an infinite interval. On the other hand, since the logical law

$$\Box R \equiv R \wedge \varepsilon \vee R \wedge \bigcirc \Box R$$

can be satisfied, we can infer that $\Box R$ is one solution of the equation (3.3.1), and must be equivalent to either the least or the greatest fixed-points by Theorem 3. In accordance with the meaning of $\Box R$, we can see that the greatest fixed-point exactly characterizes $\Box R$ and acquire the following:

$$\bigvee_{i\in N_0} R^{(i)} \wedge \bigcirc^i (R \wedge \varepsilon) \vee \Box(R \wedge \mathsf{more}) \equiv \Box R$$

which convinces us $\bigvee_{i\in N_0} R^{(i)} \wedge \bigcirc^i (R \wedge \varepsilon)$ is well-formed.

Example 3. Let $P \stackrel{\text{def}}{=} \mathsf{true}$ and $Q \stackrel{\text{def}}{=} R \wedge \varepsilon$. Then $X \equiv Q \vee P \wedge \bigcirc X$ can be instantiated as

$$X \equiv R \wedge \varepsilon \vee \bigcirc X \tag{3.3.2}$$

whose least and greatest fixed-points respectively are:

$$\bigvee_{i\in N_0} \bigcirc^i (R \wedge \varepsilon) \equiv \Diamond(R \wedge \varepsilon) \;\; \text{and} \;\; \bigvee_{i\in N_0} \bigcirc^i (R \wedge \varepsilon) \vee \Box\mathsf{more} \equiv \Diamond(R \wedge \varepsilon) \vee \Box\mathsf{more}$$

Particularly, the greatest fixed-point states that R will hold and terminate at some point over a finite interval or the interval is infinite. Further, we have the logical law

$$\mathsf{fin}(R) \equiv R \wedge \varepsilon \vee \bigcirc \mathsf{fin}(R)$$

so $\mathsf{fin}(R)$ is one solution of the equation (3.3.2) and equivalent to the greatest fixed-point by its meaning. Thus, we can obtain:

$$\mathsf{fin}(R) \equiv \bigvee_{i \in N_0} \bigcirc^i (R \wedge \varepsilon) \vee \Box \mathsf{more} \equiv \Diamond(R \wedge \varepsilon) \vee \Box \mathsf{more}$$

Example 4. Let $P \stackrel{\text{def}}{=} R$ and $Q \stackrel{\text{def}}{=} \varepsilon$. Then $X \equiv Q \vee P \wedge \bigcirc X$ can be instantiated as

$$X \equiv \varepsilon \vee R \wedge \bigcirc X \tag{3.3.3}$$

whose least and greatest fixed-points can be attained as:

$$\bigvee_{i \in N_0} R^{(i)} \wedge \bigcirc^i \varepsilon \quad \text{and} \quad \bigvee_{i \in N_0} R^{(i)} \wedge \bigcirc^i \varepsilon \vee \Box(R \wedge \mathsf{more})$$

In particular, the greatest fixed-point tells us that R is true at every state over an infinite interval or over a finite interval with ignoring the final state. Further, the logical law

$$\mathsf{keep}(R) \equiv \varepsilon \vee R \wedge \bigcirc \mathsf{keep}(R)$$

holds and implies $\mathsf{keep}(R)$ is one solution of the equation (3.3.3). Since $\mathsf{keep}(R)$ precisely specifies the meaning of the greatest fixed-point, we have:

$$\mathsf{keep}(R) \equiv \bigvee_{i \in N_0} R^{(i)} \wedge \bigcirc^i \varepsilon \vee \Box(R \wedge \mathsf{more})$$

which makes $\bigvee_{i \in N_0} R^{(i)} \wedge \bigcirc^i \varepsilon$ well-formed.

Example 5. Let $P \stackrel{\text{def}}{=} \neg R$ and $Q \stackrel{\text{def}}{=} R \wedge \varepsilon$. Then $X \equiv Q \vee P \wedge \bigcirc X$ can be instantiated as

$$X \equiv R \wedge \varepsilon \vee \neg R \wedge \bigcirc X \tag{3.3.4}$$

By Theorem 2, its least and greatest fixed-points respectively are:

$$\bigvee_{i \in N_0} (\neg R)^{(i)} \wedge \bigcirc^i (R \wedge \varepsilon) \quad \text{and} \quad \bigvee_{i \in N_0} (\neg R)^{(i)} \wedge \bigcirc^i (R \wedge \varepsilon) \vee \Box(\neg R \wedge \mathsf{more})$$

where the greatest fixed-point asserts that R is only true at the final state over a finite interval or $\neg R$ always holds over an infinite interval. Moreover, we have known that

$$\mathsf{halt}(R) \equiv R \wedge \varepsilon \vee \neg R \wedge \bigcirc \mathsf{halt}(R)$$

which indicates $\mathsf{halt}(R)$ is one solution of the equation (3.3.4). As $\mathsf{halt}(R)$ exactly expresses the greatest fixed-point, we can get:

$$\mathsf{halt}(R) \equiv \bigvee_{i \in N_0} (\neg R)^{(i)} \wedge \bigcirc^i (R \wedge \varepsilon) \vee \Box(\neg R \wedge \mathsf{more})$$

which further convinces us $\bigvee_{i \in N_0} (\neg R)^{(i)} \wedge \bigcirc^i (R \wedge \varepsilon)$ is well-formed.

Example 6. Let $P \stackrel{\text{def}}{=} \mathsf{true}$ and $Q \stackrel{\text{def}}{=} \varepsilon$. Then $X \equiv Q \vee P \wedge \bigcirc X$ can be instantiated as

$$X \equiv \varepsilon \vee \bigcirc X \tag{3.3.5}$$

Further, the least and greatest fixed-points can be acquired as

$$\bigvee_{i \in N_0} \mathsf{true}^{(i)} \wedge \bigcirc^i \varepsilon \equiv \bigvee_{i \in N_0} \bigcirc^i \varepsilon \text{ and } \bigvee_{i \in N_0} \mathsf{true}^{(i)} \wedge \bigcirc^i \varepsilon \vee \Box\mathsf{more} \equiv \bigvee_{i \in N_0} \bigcirc^i \varepsilon \vee \Box\mathsf{more}$$

which respectively says that the interval is finite and the interval is finite or infinite. On the other hand, we have:

$$\mathsf{fin} \equiv \Diamond\varepsilon \equiv \varepsilon \vee \bigcirc\Diamond\varepsilon \text{ and } \mathsf{true} \equiv \varepsilon \vee \bigcirc\mathsf{true}$$

which suggests that $\Diamond\varepsilon$ and true are the solutions of the equation (3.3.5). Hence, according to their meanings, we can attain:

$$\bigvee_{i \in N_0} \bigcirc^i \varepsilon \equiv \Diamond\varepsilon \equiv \mathsf{fin} \text{ and } \bigvee_{i \in N_0} \bigcirc^i \varepsilon \vee \Box\mathsf{more} \equiv \mathsf{true}$$

which is consistent with Example 1.

4 Representation of ‘U’ and ‘W’ of PLTL within PPTL

Linear Temporal Logic (LTL) [13] is a well-known temporal logic, which is based on a linear-time perspective and often defined over an infinite path (i.e. an infinite interval). Propositional LTL (PLTL) is a propositional subset of LTL and has been widely used in practice. In PLTL, the most prominent operators are ‘U’ (strong until) and ‘W’ (weak until), where ‘W’ is a weak version of ‘U’. Their intuitive semantics are shown in Figure 1(a) and (b) respectively and more details can be found in [2]. Except U and W operators, other operators of PLTL can be directly formalized over an infinite interval in PPTL. In this section, we employ techniques proposed in Section 3 to equivalently express ‘U’ and ‘W’ constructs within PPTL.

In PLTL, the following laws have been proved:

$$\begin{array}{ll} P \,\mathsf{U}\, Q \equiv (P \wedge \neg Q) \,\mathsf{U}\, Q & P \,\mathsf{U}\, Q \equiv Q \vee P \wedge \bigcirc(P \,\mathsf{U}\, Q) \\ P \,\mathsf{W}\, Q \equiv (P \wedge \neg Q) \,\mathsf{W}\, Q & P \,\mathsf{W}\, Q \equiv Q \vee P \wedge \bigcirc(P \,\mathsf{W}\, Q) \\ P \,\mathsf{W}\, Q \equiv (P \,\mathsf{U}\, Q) \vee \Box P & \neg \bigcirc P \equiv \bigcirc \neg P \end{array}$$

Hence, $P \,\mathsf{U}\, Q$ can be reduced as follows:

$$\begin{aligned} P \,\mathsf{U}\, Q &\equiv (P \wedge \neg Q) \,\mathsf{U}\, Q \\ &\equiv Q \vee (P \wedge \neg Q) \wedge \bigcirc((P \wedge \neg Q) \,\mathsf{U}\, Q) \\ &\equiv Q \vee (P \wedge \neg Q) \wedge \bigcirc(Q \vee (P \wedge \neg Q) \wedge \bigcirc((P \wedge \neg Q) \,\mathsf{U}\, Q)) \\ &\equiv Q \vee (P \wedge \neg Q) \wedge \bigcirc Q \vee (P \wedge \neg Q) \wedge \bigcirc(P \wedge \neg Q) \wedge \bigcirc^2((P \wedge \neg Q) \,\mathsf{U}\, Q) \\ &\equiv Q \vee (P \wedge \neg Q) \wedge \bigcirc Q \vee (P \wedge \neg Q) \wedge \bigcirc(P \wedge \neg Q) \wedge \bigcirc^2 Q \vee \ldots \end{aligned}$$

From the above, we find that the recursive equation of $P \,\mathsf{U}\, Q$ can be treated as the form of $X \equiv Q \vee P \wedge \neg Q \wedge \bigcirc X$ $(\star\star)$ with one solution $P \,\mathsf{U}\, Q$. Further, by Theorem 2 and the semantics of $P \,\mathsf{U}\, Q$, we can obtain the least fixed-point $\bigvee_{i\in N_0}(P\wedge\neg Q)^{(i)}\wedge\bigcirc^i Q$ of the equation ($\star\star$), which corresponds to $P \,\mathsf{U}\, Q$. In other words, $P \,\mathsf{U}\, Q$ is equivalent to $\bigvee_{i\in N_0}(P\wedge\neg Q)^{(i)}\wedge\bigcirc^i Q$. However, formulas in PPTL can be interpreted over both infinite and finite intervals whereas formulas in PLTL can only be satisfied by infinite paths. Therefore, in order to force a PPTL formula to hold just over an infinite interval, an additionally PPTL formula $\Box\mathsf{more}$ is needed. Thus, we can equivalently represent $P \,\mathsf{U}\, Q$ within PPTL as follows:

$$P \,\mathsf{U}\, Q \stackrel{\text{def}}{=} (\bigvee_{i\in N_0}(P\wedge\neg Q)^{(i)}\wedge\bigcirc^i Q)\wedge\Box\mathsf{more} \quad (\star\star\star)$$

Similar to $P \,\mathsf{U}\, Q$, the recursive equation of $P \,\mathsf{W}\, Q$ is $P \,\mathsf{W}\, Q \equiv (P\wedge\neg Q) \,\mathsf{W}\, Q \equiv Q\vee(P\wedge\neg Q)\wedge\bigcirc((P\wedge\neg Q)\,\mathsf{W}\, Q)$ and also in the form of the equation ($\star\star$). Further, according to the semantics of $P \,\mathsf{W}\, Q$ and by Theorem 2, $P \,\mathsf{W}\, Q$ is equivalent to the greatest fixed-point $\bigvee_{i\in N_0}(P\wedge\neg Q)^{(i)}\wedge\bigcirc^i Q\vee\Box(P\wedge\neg Q\wedge\mathsf{more})$ of the equation ($\star\star$). Therefore, with the requirement of an infinite interval, we have the following:

$$\begin{aligned} P \,\mathsf{W}\, Q &\stackrel{\text{def}}{=} (\bigvee_{i\in N_0}(P\wedge\neg Q)^{(i)}\wedge\bigcirc^i Q\vee\Box(P\wedge\neg Q\wedge\mathsf{more}))\wedge\Box\mathsf{more} \\ &\equiv (\bigvee_{i\in N_0}(P\wedge\neg Q)^{(i)}\wedge\bigcirc^i Q)\wedge\Box\mathsf{more}\vee\Box(P\wedge\neg Q\wedge\mathsf{more}) \\ &\equiv P \,\mathsf{U}\, Q\vee\Box(P\wedge\neg Q\wedge\mathsf{more}) \end{aligned}$$

With techniques presented in this paper, we can see that $P \,\mathsf{U}\, Q$ is the least fixed-point while $P \,\mathsf{W}\, Q$ is the greatest fixed-point of the equation ($\star\star$), which is coherent with that in [2].

Example 7. We consider a LTL formula $\bigcirc^3 p \,\mathsf{U}\, \bigcirc q$, where p, q are atomic propositions. Three possible paths are shown in Figure 1(c)(1-3). Further, according to the equation ($\star\star\star$), an equivalent PPTL formula can be acquired as $(\bigvee_{i\in N_0}(\bigcirc^3 p\wedge\neg\bigcirc q)^{(i)}\wedge\bigcirc^i\bigcirc q)\wedge\Box\mathsf{more}$. Next we reduce the PPTL formula to obtain the three paths for showing the correctness of the equation ($\star\star\star$).

Firstly, we know that

$$\begin{aligned} &(\bigvee_{i\in N_0}(\bigcirc^3 p\wedge\neg\bigcirc q)^{(i)}\wedge\bigcirc^i\bigcirc q)\wedge\Box\mathsf{more} \\ \equiv\ &(\bigvee_{i\in N_0}(\bigcirc^3 p\wedge\bigcirc\neg q)^{(i)}\wedge\bigcirc^i\bigcirc q)\wedge\Box\mathsf{more} \\ \equiv\ &\bigcirc q\wedge\Box\mathsf{more}\vee(\bigcirc^3 p\wedge\bigcirc\neg q)\wedge\bigcirc\bigcirc q\wedge\Box\mathsf{more}\vee \\ &(\bigcirc^3 p\wedge\bigcirc\neg q)\wedge\bigcirc(\bigcirc^3 p\wedge\bigcirc\neg q)\wedge\bigcirc^2\bigcirc q\wedge\Box\mathsf{more}\vee\ldots \end{aligned}$$

Then $\bigcirc q\wedge\Box\mathsf{more}$ characterizes path (1), $(\bigcirc^3 p\wedge\bigcirc\neg q)\wedge\bigcirc^2 q\wedge\Box\mathsf{more}$ describes path (2) and $(\bigcirc^3 p\wedge\bigcirc\neg q)\wedge\bigcirc(\bigcirc^3 p\wedge\bigcirc\neg q)\wedge\bigcirc^3 q\wedge\Box\mathsf{more}$ specifies path (3). We merely reduce $(\bigcirc^3 p\wedge\bigcirc\neg q)\wedge\bigcirc(\bigcirc^3 p\wedge\bigcirc\neg q)\wedge\bigcirc^3 q\wedge\Box\mathsf{more}$ to illustrate how to get the relevant path and others can be obtained in a similar manner.

$$\begin{aligned} &(\bigcirc^3 p\wedge\bigcirc\neg q)\wedge\bigcirc(\bigcirc^3 p\wedge\bigcirc\neg q)\wedge\bigcirc^3 q\wedge\Box\mathsf{more} \\ \equiv\ &\bigcirc(\bigcirc^2 p\wedge\neg q\wedge\bigcirc^3 p\wedge\bigcirc\neg q\wedge\bigcirc^2 q\wedge\Box\mathsf{more}) \end{aligned}$$

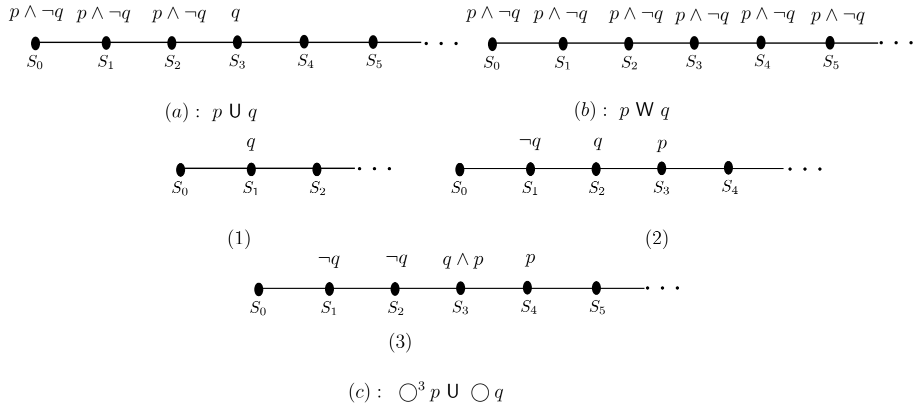

Fig. 1. Intuitive meaning of $p \mathsf{U} q$ and $p \mathsf{W} q$ and some models of $\bigcirc^3 p \mathsf{U} \bigcirc q$

Thus true holds at state s_0. Next, at state s_1, we continue to reduce

$$\bigcirc^2 p \wedge \neg q \wedge \bigcirc^3 p \wedge \bigcirc \neg q \wedge \bigcirc^2 q \wedge \Box \mathsf{more} \equiv \neg q \wedge \bigcirc(\bigcirc p \wedge \bigcirc^2 p \wedge \neg q \wedge \bigcirc q \wedge \Box \mathsf{more})$$

From this, we can see $\neg q$ holds at state s_1. Further, at state s_2,

$$\bigcirc p \wedge \bigcirc^2 p \wedge \neg q \wedge \bigcirc q \wedge \Box \mathsf{more} \equiv \neg q \wedge \bigcirc(p \wedge \bigcirc p \wedge q \wedge \Box \mathsf{more})$$

Therefore, $\neg q$ is satisfied by state s_2. At state s_3, we go on reducing and get below:

$$p \wedge \bigcirc p \wedge q \wedge \Box \mathsf{more} \equiv p \wedge q \wedge \bigcirc(p \wedge \Box \mathsf{more})$$

Then $p \wedge q$ holds at state s_3. Subsequently, at state s_4,

$$p \wedge \Box \mathsf{more} \equiv p \wedge \bigcirc(\Box \mathsf{more})$$

which makes p hold at state s_4 and all the successive states over an infinite path be satisfied by $true$ (i.e arbitrary propositions). Hence, we attain the path (3).

5 Conclusion

This paper investigated some fixed-point issues within PPTL. Particularly, we give two kinds of index set expressions $\bigvee_{i \in N_0} \bigcirc^i P$ and $\bigvee_{i \in N_0} P^i$, which are formed by applying rules of the PPTL syntax infinitely many times. Further, we proved that these formulas expressed by index set expressions are still well-formed PPTL formulas. Moreover, $\bigvee_{i \in N_0} \bigcirc^i Q$ is generalized to $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$ and the least and greatest fixed-points of the equation $X \equiv Q \vee P \wedge \bigcirc X$ are explored. In addition, the operators 'U' and 'W' in PLTL are equivalently represented within PPTL in terms of $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$.

In this paper, we only demonstrate some instances of the index set expression $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$ with specific formulas as P and Q are well-formed PPTL formulas but do not give its equivalent generic well-formed formulas. As a challenge, we will attempt to find out its concrete well-formed formula in the near future. Further, we will work out the conditions for the solutions of $X \equiv Q \vee P \wedge \bigcirc X$, so that we know when X takes the least fixed-point and when X takes the greatest fixed-point. Moreover, formulas under investigation possess a common feature that during their recursive rewriting, only one or two formulas appear repeatedly. For example, in $\bigvee_{i \in N_0} \bigcirc^i P$, P occurs iteratively for infinite number of times, and so do P and Q in $\bigvee_{i \in N_0} P^{(i)} \wedge \bigcirc^i Q$. However, for these formulas such as $\bigvee_{i \in N_0} \bigcirc^i P_i$, where P_i's might be different for distinct i, whether or not they are well-formed is an open question. We will study the problem in the future.

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