Estimation of Transport Systems Capacity

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Abstract A transport system capacity is introduced as maximal car flow density compatible with a desired quality of system performance. As an objective function one can choose mean car velocity or mean travel time dealing with highway capacity. Mean number of cars waiting before crossroads is useful to analyze the traffic lights capacity. Probability of a line of stationary or very slow moving traffic with length exceeding a given threshold can also serve for estimation of a transport system capacity. We consider three examples of transport systems (a highway without traffic lights with two car types, a single crossroads as well as hierarchical networks) and estimate their capacities.

1 Introduction

It is well known that in order to investigate a real-life process or system one has to construct an appropriate mathematical model. Interest in transport models is old enough, the first one was introduced by Greenshields, see [1], in 1935. Now it is impossible even to mention all the researchers who contributed to traffic modeling, see, e.g., [2–5] and references therein. Various methods such as cellular automata, statistical mechanics, mathematical physics or queueing theory are widely used.

One of the main problems is how to deal with traffic congestion, the condition on road networks that occurs as their use increases. It is characterized by slower speeds, longer trip times and increased vehicular queues, leading to loss of time, air pollution and many other bad consequences.

We define a transport system capacity as the maximal traffic intensity providing fulfillment of certain conditions concerning parameters characterizing its performance.

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The paper is organized as follows. In Sect. 2 we estimate the capacity of highway without traffic lights under assumption of two car types (quick and slow). The Sect. 3 deals with a single crossroads having traffic lights. A hierarchical transport network is treated in Sect. 4, whereas conclusions and further research directions are given in Sect. 5. Due to lack of space almost all proofs, as well as numerical results, are omitted.

2 Highway Without Traffic Lights (Two Types of Cars)

2.1 Model Description

Consider a one-lane highway without traffic lights. There are cars of two types (quick and slow) moving in the same direction. Let $\lambda_i \, dx \, dt$ be the probability that a car of type i appears in the interval $(x, x + dx)$ during the time interval $(t, t + dt)$, $i = 1, 2$. Velocity of a i-type car is V_i and $V_1 < V_2$. The distance covered by a car of type i on the highway is exponential random variable (r.v.) with parameter μ_i.

We make two assumptions:

- A car of type 2 (quick) on catching up with a car of type 1 (slow) begins to move with velocity V_1 until either of them leaves the road.
- The car size is not taken into account.

Let $X(t)$ be a stochastic process with values in the space of configurations $\mathcal{X} = \{(x_s, n_s, e_s)_{s=-\infty}^{s=+\infty}\}$, here x_s is the position of the sth group of cars ($x_s \in R^1$), n_s is the size of the sth group (cars number) and $e_s = 1$ if the group contains a slow car, $e_s = 2$ otherwise.

According to the model assumptions $X(t) = \{(x_s(t), n_s(t), e_s(t))_{s=-\infty}^{s=+\infty}\}$ is a homogenous Markov process with values in the space $\mathcal{X}$.

Theorem 1. *If $\mu_i > 0$, $V_i > 0$, $i = 1, 2$, then $X(t)$ is ergodic.*

A *proof* can be found in [2].

We suppose further on that $X(t)$ is stationary.

2.2 Car Flow Densities

Let I be a finite interval in R^1 and $|I|$ its length, whereas $\mathbb{I}(A)$ is indicator of the event A.

Definition 1. The density of cars moving with velocity V_i, $i = 1, 2$, at time t is given by

$$\delta_i = \lim_{|I|\to\infty} |I|^{-1} \sum_{s=-\infty}^{+\infty} n_s(t)\mathbb{I}(e_s(t) = i, x_s(t) \in I), \quad \text{a.s.,}$$

the mean velocity on highway is

$$\tilde{V} = V_1 + (V_2 - V_1)\frac{\delta_2}{\delta_1 + \delta_2} \tag{1}$$

and the density of car flow is $\delta_0 = \delta_1 + \delta_2$.

The main result of this section is the following

Theorem 2. *The densities δ_i, $i = 1, 2$, have the form*

$$\delta_1 = \lambda_1\zeta_1 + \lambda_1\lambda_2 C(\mu_2 V_1(1 + \lambda_1 C))^{-1}, \quad \delta_2 = \lambda_2\zeta_2(1 + \lambda_1 C)^{-1} \tag{2}$$

where $\zeta_i = (V_i\mu_i)^{-1}$ and $C = \zeta_1\zeta_2(V_2 - V_1)\mu_2(\mu_1 + \mu_2)^{-1}$.

Proof. Let $p_j dx$ be the probability that interval $(x, x+dx)$ contains j cars of type 2 and one car of type 1, $j = 0, 1, 2, \ldots$, whereas $q_j dx$ the probability that interval $(x, x + dx)$ contains only j cars of type 2, $j = 1, 2, \ldots$. Put $\alpha = \sum_{j=0}^{\infty} p_j$ and $\beta = \sum_{j=1}^{\infty} q_j$.

Equations satisfied by p_j and q_j were established in [2]. They have the form

$$(\mu_1 V_1 + \beta(V_2 - V_1))p_0 = \mu_2 V_1 p_1 + \lambda_1, \tag{3}$$

$$(j\mu_2 V_1 + \mu_1 V_1 + \beta(V_2 - V_1))p_j = \mu_2 V_1(j + 1)p_{j+1} + (V_2 - V_1)\sum_{i=0}^{j-1} p_i q_{j-i}, \quad j > 0,$$

$$(\mu_2 V_2 + \alpha(V_2 - V_1))q_1 = 2\mu_2 V_2 q_2 + \mu_1 V_1 p_1 + \lambda_2, \tag{4}$$

$$(j\mu_2 V_2 + \alpha(V_2 - V_1))q_j = (j + 1)\mu_2 V_2 q_{j+1} + \mu_1 V_1 p_j, \quad j > 1.$$

These equations can be solved numerically, the algorithm was proposed in [2]. Since $\delta_1 = \sum_{j=0}^{\infty}(j + 1)p_j$ and $\delta_2 = \sum_{j=1}^{\infty} jq_j$, we introduce here probability generating functions $P(z) = \sum_{j=0}^{\infty} p_j z^j$ and $Q(z) = \sum_{j=1}^{\infty} q_j z^j$. Then we have

$$\delta_1 = P(1) + P'(1), \quad \text{whereas} \quad \delta_2 = Q'(1). \tag{5}$$

Moreover, in Eqs. (3) and (4) we have $\alpha = P(1)$ and $\beta = Q(1)$.

Multiplying the jth equation of (3) (and (4), resp.) by z^j, $j = 0, 1, \ldots$, ($j = 1, 2, \ldots$, resp.) and summing them, we get the following equations for generating functions

$$\mu_2 V_1(z - 1)P'(z) = P(z)[(V_2 - V_1)(Q(z) - Q(1)) - \mu_1 V_1] + \lambda_1, \tag{6}$$

$$\mu_2 V_2(z - 1)Q'(z) = \mu_1 V_1[P(z) - p_0] - P(1)(V_2 - V_1)Q(z) - \mu_2 V_2 q_1 + \lambda_2 z. \tag{7}$$

The left-hand sides of (6) and (7) are equal to zero for $z = 1$, the same is true of the right-hand sides. Thus, we get immediately $P(1) = \lambda_1\zeta_1$. Furthermore, using the L'Hopital rule we obtain $P'(1)$ and $Q'(1)$ as follows

$$P'(1) = \lim_{z\to 1} \frac{P(z)[(V_2 - V_1)(Q(z) - Q(1)) - \mu_1 V_1] + \lambda_1}{\mu_2 V_1(z-1)}$$
$$= (\mu_2 V_1)^{-1}[(V_2 - V_1)Q'(1)P(1) - \mu_1 V_1 P'(1)],$$
$$Q'(1) = \lim_{z\to 1} \frac{\mu_1 V_1[P(z) - p_0] - P(1)(V_2 - V_1)Q(z) - \mu_2 V_2 q_1 + \lambda_2 z}{\mu_2 V_2(z-1)}$$
$$= (\mu_2 V_2)^{-1}[\lambda_2 + \mu_1 V_1 P'(1) - \lambda_1\zeta_1(V_2 - V_1)Q'(1)].$$

Solving this system of equations for $P'(1)$ and $Q'(1)$ we get from (5) the desired expressions for densities δ_i, $i = 0, 1, 2$. □

2.3 Highway Capacity

Definition 2. The highway capacity is $\Lambda = \max\{\lambda_1 + \lambda_2 : \tilde{V} > V_1 + \Delta\}$, where $\Delta \in (0, V_2 - V_1)$ is specified.

Let the relationship between intensities of slow and quick cars be known, i.e. $\lambda_1 = a\lambda_2$. Then the densities δ_i are functions of λ_2, namely, $\delta_i(\lambda_2)$, $i = 1, 2$. Set $g(\lambda_2) = \delta_2(\lambda_2)[\delta_1(\lambda_2) + \delta_2(\lambda_2)]^{-1}$, hence, $\tilde{V}(\lambda_2) = V_1 + (V_2 - V_1)g(\lambda_2)$. Thus, according to Definition 2, the highway capacity is $\Lambda = (1 + a)\widetilde{\lambda_2}$ with $\widetilde{\lambda_2} = \max\{\lambda_2 : \tilde{V}(\lambda_2) \geq V_1 + \Delta\}$, whence we get the following equation for getting $\widetilde{\lambda_2}$

$$(V_2 - V_1)g(\lambda_2) = \Delta. \tag{8}$$

From Theorem 2 one easily obtains

Corollary 1. *The function $g(\lambda_2)$ has the form*

$$g(\lambda_2) = \zeta_2[a\zeta_1 + \zeta_2 + \lambda_2 a C(a\zeta_1 + (\mu_2 V_1)^{-1})]^{-1}.$$

Since $g(\lambda_2)$ is continuous non-increasing, $g(\infty) = 0$ and $g(0) = \zeta_2(\zeta_2 + a\zeta_1)^{-1}$, Eq. (8) has a solution $\widetilde{\lambda_2}$ if $\Delta(V_2 - V_1)^{-1} < \zeta_2(\zeta_2 + a\zeta_1)^{-1}$. Therefore, for a fixed $\Delta \in (0, (V_2 - V_1))$, it follows that condition

$$a < \tilde{a} = \zeta_2(V_2 - V_1 - \Delta)(\zeta_1\Delta)^{-1}$$

is necessary for existence of $\widetilde{\lambda_2}$. Using (2) it is easy to get the explicit form of solution

$$\widetilde{\lambda_2} = \frac{\zeta_2(V_2 - V_1)\Delta^{-1} - a\zeta_1 - \zeta_2}{a\zeta_1^2\zeta_2(V_2 - V_1)} \cdot \frac{\mu_1 + \mu_2}{\mu_1 + a\mu_2} \tag{9}$$

and highway capacity Λ.

It is clear that $\widetilde{\lambda_2}(a)$ given by (9) is a monotone decreasing function of a, it tends to infinity, as $a \to 0$, and equals zero for $a = \tilde{a}$. Moreover, this expression shows how to organize the traffic in order to increase the highway capacity. For example, building special exits for slow cars, we increase μ_1 thus decreasing ζ_1, and that results in Λ growth.

3 Crossroads

3.1 Model Description

Consider a crossroads with cars arriving in two perpendicular directions and introduce the following notation. Let $A_i(t)$ be a Poisson flow of cars arriving in the ith direction, λ_i intensity of the flow and $\{t_n^{(i)}\}_{n=1}^{\infty}$ a sequence of green light switching on times for the ith direction, $i = 1, 2$. Let $0 = t_1^{(1)} < t_1^{(2)} < \ldots$ then $\tau_n^{(1)} = t_n^{(2)} - t_n^{(1)}$, $\tau_n^{(2)} = t_{n+1}^{(1)} - t_n^{(2)}$, $n = 1, 2, \ldots$, are interswitching times. Their distribution functions are $G_i(x) = \mathsf{P}(\tau_n^{(i)} \le x)$ with means $\gamma_i^{-1} = \int_0^{\infty} x\, dG_i(x)$, $i = 1, 2$, whereas $\theta = \gamma_1^{-1} + \gamma_2^{-1}$ is the period length. Obviously, the intervals $\tau_n^{(i)}$ when the green light is switched on for the ith direction can be interpreted as a working state of the server, while the red interval can be considered as a repair state.

The time of passing crossroads by a car is usually supposed exponentially distributed, see, e.g., [6] or [7]. In practice the distribution of the crossing time for cars arriving to traffic lights during a green interval differs from that for cars already waiting before traffic lights. To take into account this difference we assume that a car arriving during a green interval passes the crossroads immediately if there are no waiting cars (the service time of such cars is zero). The other cars have exponential distribution with parameter ν. All the random variables involved in model description are mutually independent.

3.2 Ergodic Theorem

Let $X_i(t)$ be the car number in the ith direction before the traffic lights. Below we consider a stochastic process $X(t) = (X_1(t), X_2(t))$, that is, a continuous-time two-dimensional random walk. If this process is Markovian its ergodicity conditions are well known, see, e.g., [8]. However in our setting $X(t)$ is not a Markov process. Thus, we need to use another approach.

Theorem 3. *The limits*

$$p_j^{(i)} = \lim_{t\to\infty} \mathsf{P}(X_i(t) = j), \quad i = 1,2, \quad j = 0,1,2,\dots, \tag{10}$$

with $p_j^{(i)} > 0$ *and* $\sum_{j=0}^{\infty} p_j^{(i)} = 1$ *exist iff*

$$\rho_i = \lambda_i \gamma_i \theta \nu^{-1} < 1. \tag{11}$$

The *proof* is based on the fact that in our case $X(t)$ is regenerative. Its regeneration points are the times $t_j^{(1)}$ with $X_i(t_j^{(1)} - 0) = 0$, $i = 1,2$, namely, there is no queue before the traffic lights in both directions and green light is switched on for the first direction. Existence of limits (10) follows immediately from the Smith theorem, see, e.g., [9]. More precisely, it is not difficult to establish that these limits form a probability distribution iff the process $X(t)$ is stochastically bounded. Conditions (11) are necessary and sufficient for stochastic boundedness. That can be proved by constructing two auxiliary processes, upper and lower bounds (in stochastic sense) of the process under consideration. The details are given in [5].

3.3 Estimation of Traffic Lights Capacity (Means as Criteria)

Definition 3. Traffic lights capacity in the ith direction $i = 1,2$, is the maximal intensity λ_i such that a chosen characteristic of performance quality does not exceed a certain threshold and the ergodicity condition (11) is fulfilled.

In particular, (11) provides the upper bound of capacity $\lambda_i < \nu(\gamma_i\theta)^{-1}$. It is possible to consider one of the following parameters as playing the main role in traffic lights performance (in a steady state):

1. Mean number $m_i(\lambda_i)$ of cars before the traffic lights in the ith direction,
2. Mean time $t_i(\lambda_i)$ of passing crossroads by a car.

Fixing the acceptable values of these parameters m^0 or t^0 the capacity can be obtained as solution of one of the equations

$$1.\ m_i(\lambda_i) = m^0 \quad \text{or} \quad 2.\ t_i(\lambda_i) = t^0. \tag{12}$$

Stationary Distribution Calculation

To implement the procedure of solving (12) we have to calculate the limits (10), since $m_i(\lambda_i) = \sum_{j=1}^{\infty} j p_j^{(i)}$.

Hence, begin by treating imbedded Markov chains $X_n^{(i)} = X_i(t_n^{(i)})$ where $t_n^{(i)}$ is the moment of the nth switching on of the green light in the ith direction, $n \geq 1$, $i = 1,2$. Condition (11) is supposed to be valid. Our aim is to propose an algorithm for calculating the stationary distribution of $\{X_n^{(i)}\}$, $i = 1, 2$, as well as steady-state expected numbers of waiting cars. It is possible to deal only with $\{X_n^{(1)}\}$ because the results for $\{X_n^{(2)}\}$ are obtained by putting everywhere λ_2 instead of λ_1, G_2 instead of G_1 and vice versa.

To obtain the transition probabilities for imbedded Markov chain we introduce an auxiliary birth-and-death process $Z(t)$ with absorbing state $\{0\}$, birth intensity λ_1 and death intensity ν. Put $\varphi_{kj}(t) = \mathsf{P}(Z(t) = j | Z(0) = k)$, $k, j > 0$. These functions satisfy the following system

$$\varphi'_{k1}(t) = -(\lambda_1 + \nu)\varphi_{k1}(t) + \nu\varphi_{k2}(t),$$
$$\varphi'_{kj}(t) = -(\lambda_1 + \nu)\varphi_{kj}(t) + \nu\varphi_{k,j+1}(t) + \lambda_1\varphi_{k,j-1}(t), \quad j > 1, \tag{13}$$

with initial conditions $\varphi_{kk}(0) = 1$ and $\varphi_{kj}(0) = 0$ for $j \neq k$. The solution of system (13) is obtained by applying the Laplace transform and its subsequent inversion. Thus we get the explicit form of $\varphi_{kj}(t)$ in terms of generalized Bessel functions of the first kind. For $j > 0$,

$$\varphi_{kj}(t) = e^{-(\lambda_1+\nu)t}(\lambda_1/\nu)^{(j-2)/2}(J_{|j-k|}(2\sqrt{\lambda_1\nu t}) - J_{j+k}(2\sqrt{\lambda_1\nu t}))$$

where

$$J_l(u) = \sum_{m=0}^{\infty} \frac{(u/2)^{l+2m}}{m!\Gamma(l+m+1)}.$$

Moreover, $\varphi_{k0}(t) = 1 - \sum_{j=1}^{\infty} \varphi_{kj}(t)$.

Next, denote by d_j the probability that j cars arrive during a red interval (in the first direction), therefore

$$d_j = \int_0^\infty e^{-\lambda_1 y}(\lambda_1 y)^j (j!)^{-1}\, dG_2(y), \quad j = 0, 1, \ldots.$$

Introduce also $b_{00} = 1$. Furthermore, let b_{kj} be the probability that at time of red light switching on there are j cars before traffic lights under condition that at time of the previous green light switching on there were k cars, namely, $b_{kj} = \mathsf{P}(X_1(t_n^{(2)}) = j | X_1(t_n^{(1)}) = k)$, $k, j \geq 0$. Then

$$b_{kj} = \int_0^\infty \varphi_{kj}(y)\, dG_1(y), \quad k, j > 0, \quad b_{k0} = 1 - \sum_{j=1}^{\infty} b_{kj}.$$

The transition probabilities for the imbedded Markov chain $\{X_n^{(1)}\}$ are given by

$$\mathsf{P}_{kj}^{(1)} = \mathsf{P}(X_{n+1}^{(1)} = j | X_n^{(1)} = k) = \sum_{m=0}^{j} b_{km} d_{j-m}, \quad k, j = 0, 1, \ldots.$$

Hence the stationary distribution $\{\pi_k\}$ of the imbedded Markov chain is obtained by solving the system

$$\pi_j = \sum_{k=0}^{\infty} \pi_k \mathsf{P}_{kj}^{(1)}, \quad \sum_{j=0}^{\infty} \pi_j = 1.$$

So it is easy to calculate the mean number of cars in a steady state (for imbedded Markov chain). Finally, the stationary distribution $\{p_j^{(1)}\}$ of the process $X_1(t)$ has the form

$$p_j^{(1)} = \theta^{-1} \sum_{k=0}^{\infty} \pi_k \int_0^{\infty} \varphi_{kj}(y)[1 - G_1(y)]\, dy$$

$$+\theta^{-1} \sum_{i=0}^{j} \sum_{k=0}^{\infty} \pi_k b_{ki} \int_0^{\infty} e^{-\lambda_1 y}(\lambda_1 y)^{j-i}[(j-i)!]^{-1}[1 - G_2(y)]\, dy. \tag{14}$$

Example 1. Assume interswitching intervals to be exponentially distributed. Consider the first direction. Let p_j (q_j, resp.) denote the probability of car queue before traffic lights in a steady state having length j and green (red, resp.) light being switched on for the first direction.

Generating functions for these probabilities are obtained by authors in [5]. The mean queue length $m_1(\lambda_1) = \sum_{j=1}^{\infty} j(p_j + q_j)$ has the form

$$m_1(\lambda_1) = \frac{\lambda_1\gamma_1(1 + (\nu - \lambda_1)(\gamma_1 + \gamma_2)^{-1})}{(\gamma_1 + \gamma_2)(\nu - \lambda_1)(\gamma_2(\gamma_1 + \gamma_2)^{-1} - \lambda_1\nu^{-1})}. \tag{15}$$

Function $m_1(\lambda_1)$, for $\lambda_1 \in (0, \nu(\theta\gamma_1)^{-1})$, is continuous and monotone increasing. Since $m_1(0) = 0$ and $\lim_{\lambda_1 \uparrow \nu(\theta\gamma_1)^{-1}} m_1(\lambda_1) = +\infty$, equation 1. in (12) has a unique solution $\widetilde{\lambda_1}$ which can be considered as traffic lights capacity for the first direction.

Without loss of generality put $\gamma_1 + \gamma_2 = 1$ and $x = \lambda_1\nu^{-1}$. Then it easily follows from (15) that $a_1x^2 - b_1x + m^0(1 - \gamma_1) = 0$, here $a_1 = m^0 + \gamma_1$, $b_1 = m^0(2 - \gamma_1) + \gamma_1(\nu + 1)$. Hence, $\widetilde{\lambda_1} = \nu(b_1 - \sqrt{b_1^2 - 4a_1m^0(1 - \gamma_1)})(2a_1)^{-1}$.

According to Little's formula, we calculate the mean passing time of crossroads as follows $t_1(\lambda_1) = \lambda_1^{-1}m_1(\lambda_1)$. Inserting this expression in equation 2. of (12) we establish that there exists a unique solution satisfying (11). It represents the traffic lights capacity according to this criterion.

3.4 Capacity According to Criterion of Level Crossing

Sometimes it is desirable that, for given $\varepsilon > 0$ and N_0, probability of waiting car queue length exceeding N_0 were less than ε.

Let $r_j^{(i)}$ be the probability that the number of cars waiting before traffic lights in the ith direction in a steady state is equal to j. The aim is to choose the maximal λ_i such that

$$\sum_{j=N_0+1}^{\infty} r_j^{(i)} < \varepsilon, \quad \lambda_i < \nu(\gamma_i\theta)^{-1}. \tag{16}$$

Using the results obtained in [2] it is possible to propose an algorithm for numerical estimation of traffic lights capacity in framework of this criterion.

For large N_0 one can also use the heavy-traffic asymptotics given by the following result proved in [5]. Put $\lambda_1^\delta = (1-\delta)\nu(\gamma_1\theta)^{-1}$ and let $X_1(\lambda_1^\delta)$ be the number of waiting cars in the first direction if the flow intensity is λ_1^δ.

Theorem 4. *If* $\mathsf{E}\tau_i^{2+\kappa} < \infty$, $i = 1, 2$, *for some* $\kappa > 0$ *then*

$$\mathsf{P}(\delta X_1(\lambda_1^\delta) > y) \to e^{-y/\sigma^2}, \quad \delta \to 0, \tag{17}$$

with $\sigma^2 = 1 + (\nu\gamma_1)(2\theta^2)^{-1}(\gamma_1^{-2}\mathsf{Var}\tau_1 + \gamma_2^{-2}\mathsf{Var}\tau_2)$, *here* τ_1 *and* τ_2 *are the green and red intervals respectively.*

Putting

$$\lambda_1 = (1 + N_0^{-1}\sigma^2 \ln\varepsilon)\nu(\gamma_1\theta)^{-1}, \tag{18}$$

one obtains from (17) that (16) is fulfilled, so for large N_0 the capacity is estimated by (18).

4 Transport Networks Analysis

4.1 Model Description

We study transport networks assuming that their nodes (vertices) are crossroads (with traffic lights) and arcs (edges) are the roads connecting the nodes. It is supposed that each node has two admissible motion directions. Moreover, the transport system under consideration is a superposition of two hierarchical networks S^+ and S^-. Hence, if S^+ contains a route from node A to node B then S^- has a route from B to A.

Time intervals when vehicular traffic in a given node is permitted in network S^+ (green light) correspond to the intervals when traffic is forbidden in S^- (red light) and vice versa. We establish conditions of stationary distributions existence for the processes describing the system performance under assumption that input flows are regenerative. This enables us to introduce the so-called traffic coefficients of nodes, to point out the most loaded ones and investigate the dependence of asymptotic behavior of car queues at crossroads on the traffic-lights performance. The estimates of system capacity are also provided.

We assume that the nodes of network S^+ (S^- resp.) consist of r (l resp.) levels (or classes). Thus, the cars arrive in the network through nodes of the first level, then they proceed through nodes of the second level etc. and leave the network through the nodes of the last level. Note that additional nodes with zero passing times at each level let consider the case with cars arriving (or leaving) the system on some intermediate level. The interswitching intervals for all the traffic lights are independent random variables with Erlang distribution. The crossroads passing times for the cars are exponentially distributed r.v.'s.

Consider S^+ and enumerate the nodes of each level in such a way that (i, j) means the jth node, $j = \overline{1, n_i}$, of the ith level, $i = \overline{1, r}$. The route is a sequence $I = (j_1, \dots, j_r)$ where $j_i = \overline{1, n_i}$, $i = \overline{1, r}$. Let $d^{(i)}_{j_i, j_{i+1}}$ be the distance between the node j_i of level i and the node j_{i+1} of the level $i + 1$, $j_i = \overline{1, n_i}$, $j_{i+1} = \overline{1, n_{i+1}}$, and $v^{(i)}_{j_i, j_{i+1}}$ the admissible velocity for cars moving on this link (arc). Suppose a probability measure $\{\mathsf{P}(I), I \in J\}$ is defined on the set J of all the routes and an arriving car chooses a route I with probability $\mathsf{P}(I)$ independently of others.

4.2 Ergodicity Conditions

Thus, we consider an hierarchical transport system with input flow $A(t)$. It is a stochastic process $\{A(t), t \geq 0\}$, $A(0) = 0$, taking values $0, 1, 2, \dots$ defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. The process has non-decreasing left-continuous trajectories with unit jumps.

Definition 4. The input flow $A(t)$ is regenerative if there exists an increasing sequence of r.v.'s $\{\theta_j, j \geq 0\}$, $\theta_0 = 0$, such that the sequence

$$\{\chi_j\}_{j=1}^{\infty} = \{\theta_j - \theta_{j-1}, A(\theta_{j-1} + t) - A(\theta_{j-1}), t \in [0, \theta_j - \theta_{j-1})\}_{j=1}^{\infty}$$

consists of i.i.d. random elements.

Note that for the most part flows used in queueing theory are regenerative. Doubly stochastic Poisson process, Markov modulated, semi-Markov flows and many others belong to this class, see, e.g., [3]. A very useful property of regenerative flows is their remaining regenerative (under some general assumptions) after passing a queueing system of any level of hierarchical networks. It plays important role in our reasoning.

We use the following notation. Let θ_i be the ith regeneration point, $\tau_i = \theta_i - \theta_{i-1}$ the ith regeneration interval, $\xi_i = A(\theta_i) - A(\theta_{i-1})$ the car number entering the network during this regeneration interval, $a = \mathsf{E}\xi_i$, $\mu = \mathsf{E}\tau_i$.

Intensity of input flow is given by $\lambda = \lim_{t\to\infty} A(t)/t = a/\mu$ a.s. The time of passing the traffic lights (i, k) has the mean ν_{ik}^{-1}, here i is the level of network and k is the number of the node on this level. We have assumed that this time has exponential distribution, however it is possible to consider an arbitrary distribution. The mean length of the green (red, resp.) interval for the traffic lights (i,k) is α_{ik}^{-1} (β_{ik}^{-1}, resp.), $k=\overline{1,n_i}$, $i=\overline{1,r}$.

Obviously, intensity of input in the node (i,k) is

$$\lambda_{ik}=\lambda\delta_{ik},\quad k=\overline{1,n_i},\ i=\overline{1,r},\quad \text{with}\quad \delta_{ik}=\sum_{I:j_i=k}\mathsf{P}(I). \tag{19}$$

The traffic coefficient of the node (i,k) is defined as

$$\rho_{ik}=\lambda_{ik}(\alpha_{ik}+\beta_{ik})(\nu_{ik}\beta_{ik})^{-1}. \tag{20}$$

It is well known that Erlang distribution can be considered as a convolution of exponential distributions. Namely, if the interswitching times $\eta_{ik}^{(j)}$, $j=1,2$, for the node (i,k) have Erlang distribution with parameters $(\gamma_{ik}^{(j)},l_{ik}^{(j)})$, then $\eta_{ik}^{(j)}=\zeta_{ik}^{(j)}(1)+\ldots+\zeta_{ik}^{(j)}(l_{ik}^{(j)})$ where $\{\zeta_{ik}^{(j)}(m)\}_{m=1}^{l_{ik}^{(j)}}$ are i.i.d. r.v.'s having exponential distribution with parameter $\gamma_{ik}^{(j)}$. It follows immediately that

$$\alpha_{ik}^{-1}=\mathsf{E}\eta_{ik}^{(1)}=l_{ik}^{(1)}[\gamma_{ik}^{(1)}]^{-1},\quad \beta_{ik}^{-1}=\mathsf{E}\eta_{ik}^{(2)}=l_{ik}^{(2)}[\gamma_{ik}^{(2)}]^{-1}.$$

We say that at time t the node (i,k) is in the mth phase, $m=\overline{1,l_{ik}^{(j)}}$, if $t\in(\zeta_{ik}^{(j)}(0)+\ldots+\zeta_{ik}^{(j)}(m-1),\zeta_{ik}^{(j)}(0)+\ldots+\zeta_{ik}^{(j)}(m))$ where $\zeta_{ik}^{(j)}(0)=0$. Moreover, for $j=1$ the green light is switched on, whereas for $j=2$ the red light is switched on.

Introduce a stochastic process

$$X(t)=(q_{ik}(t),w_{ik}(t),e_{ik}(t),u_{ik}(t),i=\overline{1,r},k=\overline{1,n_i}),$$

here $q_{ik}(t)$ is the number of cars waiting at the traffic lights (i,k), $w_{ik}(t)$ the number of cars on all the arcs going out of node (i,k), furthermore $e_{ik}(t)=1$ (or 2) in case of green (red, resp.) light switched on and $u_{ik}(t)$ is the number of phase in the node (i,k) after the previous switching of lights.

Theorem 5. *The process $X(t)$ is ergodic iff*

$$\rho_{ik}<1\quad \textit{for all}\quad k=\overline{1,n_i},\quad i=\overline{1,r}. \tag{21}$$

Sketch of proof. The process $X(t)$ is regenerative and its regeneration points are θ_j satisfying the following conditions $q_{ik}(\theta_j-0)=0$, $w_{ik}(\theta_j-0)=0$, $e_{ik}(\theta_j-0)=1$, $u_{ik}(\theta_j-0)=1$. We use the fact that input flow for each node is regenerative, a well-known Smith theorem and majorization procedure.

If for some node of network the ergodicity condition (21) is not fulfilled, that is, $\rho_{ik}\geq 1$, then $q_{ik}(t)$ will be stochastically unbounded, as $t\to\infty$.

4.3 Capacity of Transport Network

Conditions (21) enable us to estimate the maximal admissible intensity of car inflow in the network S^+ as follows

$$\lambda < \min_{k=\overline{1,n_i};\, i=\overline{1,r}} \nu_{ik}\beta_{ik}[(\alpha_{ik}+\beta_{ik})\delta_{ik}]^{-1}. \tag{22}$$

The same inequalities let us establish the network bottlenecks with ρ_{ik} defined by (20) very close to 1.

The objective function measuring the network performance we are going to consider is the mean time $T(\lambda)$ of passing the route by a car. Recall that λ is intensity of input flow to network.

Definition 5. The capacity of network is the maximal intensity of input flow λ satisfying the ergodicity condition (22) as well as inequality $T(\lambda) \le T_0$, where T_0 is the upper limit of desired network passing time by a car.

The question is how to determine T_0. At first we calculate the minimal mean time of passing the network by car. To this end consider a car arriving to empty network (that is, containing no other cars). Let $t(I)$ be the mean time of passing the route $I = (j_1, \ldots, j_r)$, then

$$t(I) = \sum_{k=1}^{r} \frac{d^{(k)}_{j_k,j_{k+1}}}{v^{(k)}_{j_k,j_{k+1}}} + \sum_{k=1}^{r} \frac{\alpha_{kj_k}}{\alpha_{kj_k}+\beta_{kj_k}}(\beta^{-1}_{kj_k} + \nu^{-1}_{kj_k}). \tag{23}$$

The first sum in (23) is the mean time of passing of all the arcs connecting the nodes of the chosen route, whereas the second sum is the mean time of nodes (traffic lights) passing by a car.

The mean velocity of passing the route I is given by $v(I) = ||I||/t(I)$, here $||I|| = \sum_{k=1}^{r} d^{(k)}_{j_k j_{k+1}}$ is the length of the route.

Thus, the minimal mean time of route passing and the maximal mean velocity are

$$T_{min} = \sum_I t(I)\mathsf{P}(I), \quad v_{max} = \sum_I \mathsf{P}(I)||I||/t(I), \tag{24}$$

respectively. Obviously, the network capacity can be determined by the condition $v(\lambda) \le v_0$ for mean velocity instead of $T(\lambda)$. Here $v(\lambda)$ is the mean car velocity in case of input flow intensity λ and v_0 the admissible bound.

Expressions (24) let us establish the possible values of T_{min} and v_{max}. Clearly they depend only on network structure and its parameters and are independent of the inflow intensity. It is possible to take $T_0 = \vartheta T_{min}$ and $v_0 = v_{max}\vartheta^{-1}$ for some $\vartheta > 1$.

4.4 Approaches to Estimation of Network Capacity

For capacity estimation we have to calculate (or estimate) the mean time or mean velocity of network passing. Recall that intensity of car inflow to node (i,k) is given by (19). Suppose for simplicity that input to network is Poisson with parameter λ. Then the inflows to the nodes of the first level of network are also Poisson with parameters $\lambda\delta_{1j}$, $j=\overline{1,n_1}$.

The inflows to the nodes of the second level are not Poisson even for exponential traffic lights passing times. They are doubly stochastic Poisson processes. The previous level forms the random environment for these processes. However for large networks these inflows may be considered approximately Poisson as the sums of large number of regenerative flows, see, e.g., [10]. Hence, assume that for a chosen route $I=(j_1,\ldots,j_r)$ we have Poisson inflows for all nodes with parameters $\lambda\delta_{1j_1},\ldots,\lambda\delta_{rj_r}$, respectively.

Consider some arc of the route, dropping all indices for brevity. Let d be the arc length and $\tilde\lambda$ the intensity of the Poisson input at the node beginning the arc. If $\tilde\lambda$ is small then a car does not disturb the others, that is, the mean car velocity does not depend on the intensity. Thus, to calculate the mean car velocity one can use the first term in (23). If $\tilde\lambda$ is large enough, a car on reaching a slow one either overtakes it or has to slow down. It is possible to use the model with two car types proposed in Sect. 1.

If the distance between the traffic lights (arc length) is not large we propose another model. Let $V_1<V_2<\ldots<V_l$ be the set of possible velocities and p_j the probability that a car intends to move with velocity V_j, $j=\overline{1,l}$. The main assumption is that if after passing the traffic lights a car finds some other car on the arc it begins to move with the same velocity. Otherwise its velocity equals V_j with probability p_j.

To find the mean car velocity on the arc consider a queueing system $M|G|\infty$ with input intensity $\tilde\lambda$. Assume that the customer arriving to the empty system (that is, that beginning the busy period) has the service time d/V_j with probability p_j. All the customers arriving during this busy period have the same service time. We say that the busy period has type i if the service times are equal to d/V_i, $i=\overline{1,l}$. Denote by $T_b^{(i)}$ such a busy period. It is not difficult to establish that $\psi_i(s)=\mathsf{E}e^{-sT_b^{(i)}}$ has the form $(s+\tilde\lambda)(se^{(s+\tilde\lambda)d/V_i}+\tilde\lambda)-1$, $Re\, s\geq 0$. It follows immediately that $\mathsf{E}T_b^{(i)}=\tilde\lambda^{-1}(e^{\tilde\lambda d/V_i}-1)$. According to Little's formula, the mean number of customers served during this busy period is given by $\varkappa^{(i)}=e^{\tilde\lambda d/V_i}-1$. Hence, the mean number of customers served during a busy period is $\varkappa=\sum_{i=1}^{l}p_i\varkappa^{(i)}=\sum_{i=1}^{l}p_ie^{\tilde\lambda d/V_i}-1$. The probability of moving on arc with velocity V_i is equal to $p_i\varkappa^{(i)}/\varkappa$ and the mean velocity on the arc is

$$\bar V=\sum_{i=1}^{l}V_i(e^{\tilde\lambda d/V_i}-1)p_i\left(\sum_{i=1}^{l}p_ie^{\tilde\lambda d/V_i}-1\right)^{-1}. \tag{25}$$

For the arc (j_k, j_{k+1}) of the chosen route denote by $p_i^{(k)}$ the probability of velocity V_i, $i = \overline{1,l}$, $k = \overline{1,r}$. Finally, the mean total time of passing the route $I = (j_1, \ldots, j_r)$ is given by the first sum in (23) with $\tilde{v}^{(k)}_{j_k j_{k+1}}(\lambda)$ instead of $v^{(k)}_{j_k j_{k+1}}$ where

$$\tilde{v}^{(k)}_{j_k j_{k+1}}(\lambda) = \frac{\sum_{i=1}^{l} V_i p_i^{(k)}(e^{\lambda\delta_{kj_k} d_{j_k j_{k+1}}/V_i} - 1)}{\sum_{i=1}^{l} p_i^{(k)} e^{\lambda\delta_{kj_k} d^{(k)}_{j_k j_{k+1}}/V_i} - 1}. \tag{26}$$

Now it is necessary to find the mean waiting time before the traffic lights. For the simple case of exponential interswitching times (with parameters $\gamma_{jk}^{(i)}$, $i = 1, 2$, for the node (j, k) with $i = 1$ corresponding to the green light and $i = 2$ for the red one) one can rewrite expression (15) for the mean waiting time before traffic lights in the form

$$m_{jk}(x_{jk}) = \frac{x_{jk}(1 - c_{jk})(1 + d_{jk}(1 - x_{jk})(1 - c_{jk}))}{(1 - x_{jk})(c_{jk} - x_{jk})} \tag{27}$$

with $x_{jk} = \lambda_{jk}/\nu$, $c_{jk} = [\gamma_{jk}^{(1)}\theta_{jk}]^{-1}$, $\theta_{jk} = [\gamma_{jk}^{(1)}]^{-1} + [\gamma_{jk}^{(2)}]^{-1}$, $d_{jk} = \nu/\gamma_{jk}^{(1)}$. The mean time of traffic lights (j, k) passing by a car is found by means of Little's formula, that is, $t_{(jk)} = \lambda_{jk}^{-1} m_{jk}(x_{jk})$.

Using (23) we get the mean time of route $I = (j_1, \ldots, j_r)$ as

$$t(I) = \sum_{k=1}^{r} \frac{d^{(k)}_{j_k j_{k+1}}}{\tilde{v}^{(k)}_{j_k j_{k+1}}(\lambda)} + \sum_{k=1}^{r} \frac{m^{(k)}_{jk}(x_{kj_k})}{\lambda\delta_{kj_k}}$$

with $\tilde{v}^{(k)}_{j_k j_{k+1}}(\lambda)$ given by (26) and $m^{(k)}_{jk}$ by (27).

Then the mean time of passing is

$$T(\lambda) = \sum_I t(I)\mathsf{P}(I).$$

Thus the problem is to find λ satisfying (22) and being the solution of equation

$$T(\lambda) = \vartheta T_{min} \quad \text{for a chosen} \quad \vartheta.$$

5 Conclusion and Further Research Directions

A transport system capacity was introduced as maximal car flow intensity compatible with a desired quality of system performance. As an objective function we have chosen

- The mean car velocity or mean travel time dealing with highway capacity,

- The mean number of cars waiting before crossroads useful to analyze the traffic lights capacity,
- Probability of a line of stationary or very slow-moving traffic (caused by traffic lights, road works, an accident or heavy congestion) with length exceeding a given threshold.

We have considered several examples of transport models, namely, a highway (without traffic lights) with two types of cars, a single crossroads with traffic lights, as well as hierarchical networks and estimated their capacities. For this purpose we investigated the limit behavior of such systems (proving ergodic theorems) in light- and heavy-traffic conditions. Markov, Poisson and doubly stochastic Poisson processes, renewal and regenerative ones are the main tools in our study.

There arise interesting problems for further investigation concerning the optimal choice of routes and optimization of traffic lights performance under general assumptions about the random variables involved in models description. The results will be published elsewhere.

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