Chapter 3
Projective Dimension in Filtrated K-Theory

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Abstract Under mild assumptions, we characterise modules with projective resolutions of length $n \in \mathbb{N}$ in the target category of filtrated K-theory over a finite topological space in terms of two conditions involving certain Tor-groups. We show that the filtrated K-theory of any separable C^*-algebra over any topological space with at most four points has projective dimension 2 or less. We observe that this implies a universal coefficient theorem for rational equivariant KK-theory over these spaces. As a contrasting example, we find a separable C^*-algebra in the bootstrap class over a certain five-point space, the filtrated K-theory of which has projective dimension 3. Finally, as an application of our investigations, we exhibit Cuntz-Krieger algebras which have projective dimension 2 in filtrated K-theory over their respective primitive spectrum.

Keywords K-theory • Filtered K-theory • Ideal-related KK-theory • Universal coefficient theorem

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3.1 Introduction

A far-reaching classification theorem in [7] motivates the computation of Eberhard Kirchberg's ideal-related Kasparov groups $\mathrm{KK}(X; A, B)$ for separable C^*-algebras A and B over a non-Hausdorff topological space X by means of

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K-theoretic invariants. We are interested in the specific case of finite spaces here. In [10, 11], Ralf Meyer and Ryszard Nest laid out a theoretic framework that allows for a generalisation of Jonathan Rosenberg's and Claude Schochet's universal coefficient theorem [16] to the equivariant setting. Starting from a set of generators of the equivariant bootstrap class, they define a homology theory with a certain universality property, which computes $\mathrm{KK}(X)$-theory via a spectral sequence. In order for this *universal coefficient* spectral sequence to degenerate to a short exact sequence, it remains to be checked *by hand* that objects in the range of the homology theory admit projective resolutions of length 1 in the Abelian target category.

Generalising earlier results from [3, 11, 15] the verification of the condition mentioned above for *filtrated* K*-theory* was achieved in [2] for the case that the underlying space is a disjoint union of so-called accordion spaces. A finite connected T_0-space X is an accordion space if and only if the directed graph corresponding to its specialisation pre-order is a Dynkin quiver of type A. Moreover, it was shown in [2, 11] that, if X is a finite T_0-space which is not a disjoint union of accordion spaces, then the projective dimension of filtrated K-theory over X is *not* bounded by 1 and objects in the equivariant bootstrap class are *not* classified by filtrated K-theory. The assumption of the separation axiom T_0 is not a loss of generality in this context (see [9, §2.5]).

There are two natural approaches to tackle the problem arising for non-accordion spaces: one can either try to refine the invariant—this has been done with some success in [11] and [1]; or one can hold onto the invariant and try to establish projective resolutions of length 1 on suitable subcategories or localisations of the category $\mathfrak{KK}(X)$, in which X-equivariant KK-theory is organised. The latter is the course we pursue in this note. We state our results in the next section.

3.2 Statement of Results

The definition of filtrated K-theory and related notation are recalled in Sect. 3.3.

Proposition 1. *Let X be a finite topological space. Assume that the ideal $\mathcal{NT}_{\mathrm{nil}} \subset \mathcal{NT}^*(X)$ is nilpotent and that the decomposition $\mathcal{NT}^*(X) = \mathcal{NT}_{\mathrm{nil}} \rtimes \mathcal{NT}_{\mathrm{ss}}$ holds. Fix $n \in \mathbb{N}$. For an $\mathcal{NT}^*(X)$-module M, the following assertions are equivalent:*

1. *M has a projective resolution of length n.*
2. *The Abelian group $\mathrm{Tor}_n^{\mathcal{NT}^*(X)}(\mathcal{NT}_{\mathrm{ss}}, M)$ is free and the Abelian group $\mathrm{Tor}_{n+1}^{\mathcal{NT}^*(X)}(\mathcal{NT}_{\mathrm{ss}}, M)$ vanishes.*

The basic idea of this paper is to compute the Tor-groups above by writing down projective resolutions for the fixed right-module $\mathcal{NT}_{\mathrm{ss}}$.

Let Z_m be the $(m+1)$-point space on the set $\{1, 2, \ldots, m+1\}$ such that $Y \subseteq Z_m$ is open if and only if $Y \ni m+1$ or $Y = \emptyset$. A C^*-algebra over Z_m is a C^*-algebra A

with a distinguished ideal such that the corresponding quotient decomposes as a direct sum of m orthogonal ideals. Let S be the set $\{1,2,3,4\}$ equipped with the topology $\{\emptyset, 4, 24, 34, 234, 1234\}$, where we write $24 := \{2,4\}$ etc. A C^*-algebra over S is a C^*-algebra together with two distinguished ideals which need not satisfy any further conditions; see [9, Lemma 2.35].

Proposition 2. *Let X be a topological space with at most 4 points. Let $M = \mathrm{FK}(A)$ for some C^*-algebra A over X. Then M has a projective resolution of length 2 and* $\mathrm{Tor}_2^{\mathcal{N}\mathcal{T}^*}(\mathcal{N}\mathcal{T}_{\mathrm{ss}}, M) = 0$.

Moreover, we can find explicit formulas for $\mathrm{Tor}_1^{\mathcal{N}\mathcal{T}^*}(\mathcal{N}\mathcal{T}_{\mathrm{ss}}, M)$*; for instance,* $\mathrm{Tor}_1^{\mathcal{N}\mathcal{T}^*(Z_3)}(\mathcal{N}\mathcal{T}_{\mathrm{ss}}, M)$ *is isomorphic to the homology of the complex*

$$\bigoplus_{j=1}^{3} M(j4) \xrightarrow{\begin{pmatrix} i & -i & 0 \\ -i & 0 & i \\ 0 & i & -i \end{pmatrix}} \bigoplus_{k=1}^{3} M(1234 \setminus k) \xrightarrow{(i\ i\ i)} M(1234)\,. \tag{3.1}$$

A similar formula holds for the space S; see (3.6).

The situation simplifies if we consider *rational* $\mathrm{KK}(X)$-theory, whose morphism groups are given by $\mathrm{KK}(X; A, B) \otimes \mathbb{Q}$; see [6]. This is a $\mathbb{Q}$-linear triangulated category which can be constructed as a localisation of $\mathfrak{KK}(X)$; the corresponding localisation of filtrated K-theory is given by $A \mapsto \mathrm{FK}(A) \otimes \mathbb{Q}$ and takes values in the category of modules over the $\mathbb{Q}$-linear category $\mathcal{N}\mathcal{T}^*(X) \otimes \mathbb{Q}$.

Proposition 3. *Let X be a topological space with at most 4 points. Let A and B be C^*-algebras over X. If A belongs to the equivariant bootstrap class $\mathcal{B}(X)$, then there is a natural short exact universal coefficient sequence*

$$\mathrm{Ext}^1_{\mathcal{N}\mathcal{T}^*(X)\otimes\mathbb{Q}}\big(\mathrm{FK}_{*+1}(A)\otimes\mathbb{Q}, \mathrm{FK}_*(B)\otimes\mathbb{Q}\big) \rightarrowtail \mathrm{KK}_*(X; A, B)\otimes\mathbb{Q}$$
$$\twoheadrightarrow \mathrm{Hom}_{\mathcal{N}\mathcal{T}^*(X)\otimes\mathbb{Q}}\big(\mathrm{FK}_*(A)\otimes\mathbb{Q}, \mathrm{FK}_*(B)\otimes\mathbb{Q}\big)\,.$$

In [6], a long exact sequence is constructed which in our setting, by the above proposition, reduces the computation of $\mathrm{KK}_*(X; A, B)$, up to extension problems, to the computation of a certain torsion theory $\mathrm{KK}_*(X; A, B; \mathbb{Q}/\mathbb{Z})$.

The next proposition says that the upper bound of 2 for the projective dimension in Proposition 2 does not hold for all finite spaces.

Proposition 4. *There is an $\mathcal{N}\mathcal{T}^*(Z_4)$-module M of projective dimension 2 with free entries and* $\mathrm{Tor}_2^{\mathcal{N}\mathcal{T}^*}(\mathcal{N}\mathcal{T}_{\mathrm{ss}}, M) \neq 0$. *The module $M \otimes_{\mathbb{Z}} \mathbb{Z}/k$ has projective dimension 3 for every $k \in \mathbb{N}_{\geq 2}$. Both M and $M \otimes_{\mathbb{Z}} \mathbb{Z}/k$ can be realised as the filtrated* K*-theory of an object in the equivariant bootstrap class $\mathcal{B}(X)$.*

As an application of Proposition 2 we investigate in Sect. 3.10 the obstruction term $\mathrm{Tor}_1^{\mathcal{N}\mathcal{T}^*}\big(\mathcal{N}\mathcal{T}_{\mathrm{ss}}, \mathrm{FK}(A)\big)$ for certain Cuntz-Krieger algebras with four-point primitive ideal spaces. We find:

Proposition 5. *There is a Cuntz-Krieger algebra with primitive ideal space homeomorphic to Z_3 which fulfills Cuntz's condition (II) and has projective dimension 2 in filtrated* K*-theory over Z_3. The analogous statement for the space S holds as well.*

The relevance of this observation lies in the following: *if* Cuntz-Krieger algebras *had* projective dimension at most 1 in filtrated K-theory over their primitive ideal space, this would lead to a strengthened version of Gunnar Restorff's classification result [14] with a proof avoiding reference to results from symbolic dynamics.

3.3 Preliminaries

Let X be a finite topological space. A subset $Y \subseteq X$ is called *locally closed* if it is the difference $U \setminus V$ of two open subsets U and V of X; in this case, U and V can always be chosen such that $V \subseteq U$. The set of locally closed subsets of X is denoted by $\mathbb{LC}(X)$. By $\mathbb{LC}(X)^*$, we denote the set of *non-empty, connected* locally closed subsets of X.

Recall from [9] that a *C^*-algebra over X* is pair (A, ψ) consisting of a C^*-algebra A and a continuous map $\psi\colon \mathrm{Prim}(A) \to X$. A C^*-algebra (A, ψ) over X is called *tight* if the map ψ is a homeomorphism. A C^*-algebra (A, ψ) over X comes with *distinguished subquotients* $A(Y)$ for every $Y \in \mathbb{LC}(X)$.

There is an appropriate version $\mathrm{KK}(X)$ of bivariant K-theory for C^*-algebras over X (see [7, 9]). The corresponding category, denoted by $\mathfrak{KK}(X)$, is equipped with the structure of a triangulated category (see [12]); moreover, there is an equivariant analogue $\mathscr{B}(X) \subseteq \mathfrak{KK}(X)$ of the bootstrap class [9].

Recall that a triangulated category comes with a class of distinguished candidate triangles. An *anti-distinguished* triangle is a candidate triangle which can be obtained from a distinguished triangle by reversing the sign of one of its three morphisms. Both distinguished and anti-distinguished triangles induce long exact Hom-sequences.

As defined in [11], for $Y \in \mathbb{LC}(X)$, we let $\mathrm{FK}_Y(A) := \mathrm{K}_*\big(A(Y)\big)$ denote the $\mathbb{Z}/2$-graded K-group of the subquotient of A associated to Y. Let $\mathscr{N}\mathscr{T}(X)$ be the $\mathbb{Z}/2$-graded pre-additive category whose object set is $\mathbb{LC}(X)$ and whose space of morphisms from Y to Z is $\mathscr{N}\mathscr{T}_*(X)(Y, Z)$—the $\mathbb{Z}/2$-graded Abelian group of all natural transformations $\mathrm{FK}_Y \Rightarrow \mathrm{FK}_Z$. Let $\mathscr{N}\mathscr{T}^*(X)$ be the full subcategory with object set $\mathbb{LC}(X)^*$. We often abbreviate $\mathscr{N}\mathscr{T}^*(X)$ by $\mathscr{N}\mathscr{T}^*$.

Every open subset of a locally closed subset of X gives rise to an extension of distinguished subquotients. The corresponding natural maps in the associated six-term exact sequence yield morphisms in the category $\mathscr{N}\mathscr{T}$, which we briefly denote by i, r and δ.

A *(left-)module* over $\mathscr{N}\mathscr{T}(X)$ is a grading-preserving, additive functor from $\mathscr{N}\mathscr{T}(X)$ to the category $\mathfrak{Ab}^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$-graded Abelian groups. A morphism of $\mathscr{N}\mathscr{T}(X)$-modules is a natural transformation of functors. Similarly, we define left-modules over $\mathcal{NT}^*(X)$. By $\mathfrak{Mod}\big(\mathcal{NT}^*(X)\big)_{\mathrm{c}}$ we denote the category of countable $\mathcal{NT}^*(X)$-modules.

Filtrated K-*theory* is the functor $\mathfrak{KK}(X) \to \mathfrak{Mod}\big(\mathcal{NT}^*(X)\big)_{\mathrm{c}}$ which takes a C^*-algebra A over X to the collection $\big(\mathrm{K}_*(A(Y))\big)_{Y\in\mathbb{LC}(X)^*}$ equipped with the obvious $\mathcal{NT}^*(X)$-module structure.

Let $\mathcal{NT}_{\mathrm{nil}} \subset \mathcal{NT}^*$ be the ideal generated by all natural transformations between different objects, and let $\mathcal{NT}_{\mathrm{ss}} \subset \mathcal{NT}^*$ be the subgroup spanned by the identity transformations id_Y^Y for objects $Y \in \mathbb{LC}(X)^*$. The subgroup $\mathcal{NT}_{\mathrm{ss}}$ is in fact a subring of $\mathcal{NT}^*$ isomorphic to $\mathbb{Z}^{\mathbb{LC}(X)^*}$. We say that $\mathcal{NT}^*$ decomposes as semi-direct product $\mathcal{NT}^* = \mathcal{NT}_{\mathrm{nil}} \rtimes \mathcal{NT}_{\mathrm{ss}}$ if $\mathcal{NT}^*$ as an Abelian group is the inner direct sum of $\mathcal{NT}_{\mathrm{nil}}$ and $\mathcal{NT}_{\mathrm{ss}}$; see [2, 11]. We do not know if this fails for any finite space.

We define *right-modules* over $\mathcal{NT}^*(X)$ as *contravariant*, grading-preserving, additive functors $\mathcal{NT}^*(X) \to \mathfrak{Ab}^{\mathbb{Z}/2}$. If we do not specify between left and right, then we always mean left-modules. The subring $\mathcal{NT}_{\mathrm{ss}} \subset \mathcal{NT}^*$ is regarded as an $\mathcal{NT}^*$-right-module by the obvious action: The ideal $\mathcal{NT}_{\mathrm{nil}} \subset \mathcal{NT}^*$ acts trivially, while $\mathcal{NT}_{\mathrm{ss}}$ acts via right-multiplication in $\mathcal{NT}_{\mathrm{ss}} \cong \mathbb{Z}^{\mathbb{LC}(X)^*}$. For an $\mathcal{NT}^*$-module M, we set $M_{\mathrm{ss}} := M/\mathcal{NT}_{\mathrm{nil}} \cdot M$.

For $Y \in \mathbb{LC}(X)^*$ we define the *free $\mathcal{NT}^*$-left-module on Y* by $P_Y(Z) := \mathcal{NT}(Y, Z)$ for all $Z \in \mathbb{LC}(X)^*$ and similarly for morphisms $Z \to Z'$ in $\mathcal{NT}^*$. Analogously, we define the *free $\mathcal{NT}^*$-right-module on Y* by $Q_Y(Z) := \mathcal{NT}(Z, Y)$ for all $Z \in \mathbb{LC}(X)^*$. An $\mathcal{NT}^*$-left/right-module is called *free* if it is isomorphic to a direct sum of degree-shifted free left/right-modules on objects $Y \in \mathbb{LC}(X)^*$. It follows directly from Yoneda's Lemma that free $\mathcal{NT}^*$-left/right-modules are projective.

An $\mathcal{NT}$-module M is called *exact* if the $\mathbb{Z}/2$-graded chain complexes

$$\cdots \to M(U) \xrightarrow{i_U^Y} M(Y) \xrightarrow{r_Y^{Y\setminus U}} M(Y \setminus U) \xrightarrow{\delta_{Y\setminus U}^U} M(U)[1] \to \cdots$$

are exact for all $U, Y \in \mathbb{LC}(X)$ with U open in Y. An $\mathcal{NT}^*$-module M is called *exact* if the corresponding $\mathcal{NT}$-module is exact (see [2]).

We use the notation $C \in\in \mathscr{C}$ to denote that C is an object in a category $\mathscr{C}$.

In [11], the functors FK_Y are shown to be representable, that is, there are objects $\mathcal{R}_Y \in\in \mathfrak{KK}(X)$ and isomorphisms of functors $\mathrm{FK}_Y \cong \mathrm{KK}_*(X; \mathcal{R}_Y, \sqcup)$. We let $\widehat{\mathrm{FK}}$ denote the stable *co*homological functor on $\mathfrak{KK}(X)$ represented by the same set of objects $\{\mathcal{R}_Y \mid Y \in \mathbb{LC}(X)^*\}$; it takes values in $\mathcal{NT}^*$-*right*-modules. We warn that $\mathrm{KK}_*(X; A, \mathcal{R}_Y)$ does not identify with the K-homology of $A(Y)$. By Yoneda's lemma, we have $\mathrm{FK}(\mathcal{R}_Y) \cong P_Y$ and $\widehat{\mathrm{FK}}(\mathcal{R}_Y) \cong Q_Y$.

We occasionally use terminology from [10, 11] concerning homological algebra in $\mathfrak{KK}(X)$ relative to the ideal $\mathfrak{I} := \ker(\mathrm{FK})$ of morphisms in $\mathfrak{KK}(X)$ inducing trivial module maps on FK. An object $A \in\in \mathfrak{KK}(X)$ is called $\mathfrak{I}$-*projective* if $\mathfrak{I}(A, B) = 0$ for every $B \in\in \mathfrak{KK}(X)$. We recall from [10] that FK restricts to an equivalence of categories between the subcategories of $\mathfrak{I}$-projective objects in

$\mathfrak{KK}(X)$ and of projective objects in $\mathfrak{Mod}\big(\mathcal{NT}^*(X)\big)_{\mathrm{c}}$. Similarly, the functor $\widehat{\mathrm{FK}}$ induces a contravariant equivalence between the $\mathfrak{I}$-projective objects in $\mathfrak{KK}(X)$ and projective $\mathcal{NT}^*$-*right*-modules.

3.4 Proof of Proposition 1

Recall the following result from [11].

Lemma 1 ([11, Theorem 3.12]). *Let X be a finite topological space. Assume that the ideal $\mathcal{NT}_{\mathrm{nil}} \subset \mathcal{NT}^*(X)$ is nilpotent and that the decomposition $\mathcal{NT}^*(X) = \mathcal{NT}_{\mathrm{nil}} \rtimes \mathcal{NT}_{\mathrm{ss}}$ holds. Let M be an $\mathcal{NT}^*(X)$-module. The following assertions are equivalent:*

1. *M is a free $\mathcal{NT}^*(X)$-module.*
2. *M is a projective $\mathcal{NT}^*(X)$-module.*
3. *M_{ss} is a free Abelian group and* $\mathrm{Tor}_1^{\mathcal{NT}^*(X)}(\mathcal{NT}_{\mathrm{ss}}, M) = 0$.

Now we prove Proposition 1. We consider the case $n = 1$ first. Choose an epimorphism $f\colon P \twoheadrightarrow M$ for some projective module P, and let K be its kernel. M has a projective resolution of length 1 if and only if K is projective. By Lemma 1, this is equivalent to K_{ss} being a free Abelian group and $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, K) = 0$. We have $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, K) = 0$ if and only if $\mathrm{Tor}_2^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) = 0$ because these groups are isomorphic. We will show that K_{ss} is free if and only if $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M)$ is free. The extension $K \rightarrowtail P \twoheadrightarrow M$ induces the following long exact sequence:

$$0 \to \mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) \to K_{\mathrm{ss}} \to P_{\mathrm{ss}} \to M_{\mathrm{ss}} \to 0\,.$$

Assume that K_{ss} is free. Then its subgroup $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M)$ is free as well. Conversely, if $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M)$ is free, then K_{ss} is an extension of free Abelian groups and thus free. Notice that P_{ss} is free because P is projective. The general case $n \in \mathbb{N}$ follows by induction using an argument based on syzygies as above. This completes the proof of Proposition 1.

3.5 Free Resolutions for $\mathcal{NT}_{\mathrm{ss}}$

The $\mathcal{NT}^*$-right-module $\mathcal{NT}_{\mathrm{ss}}$ decomposes as a direct sum $\bigoplus_{Y\in\mathbb{LC}(X)^*} S_Y$ of the simple submodules S_Y which are given by $S_Y(Y) \cong \mathbb{Z}$ and $S_Y(Z) = 0$ for $Z \neq Y$. We obtain

$$\mathrm{Tor}_n^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) = \bigoplus_{Y\in\mathbb{LC}(X)^*} \mathrm{Tor}_n^{\mathcal{NT}}(S_Y, M)\,.$$

Our task is then to write down projective resolutions for the $\mathcal{N}\mathcal{T}^*$-right-modules S_Y. The first step is easy: we map Q_Y onto S_Y by mapping the class of the identity in $Q_Y(Y)$ to the generator of $S_Y(Y)$. Extended by zero, this yields an epimorphism $Q_Y \twoheadrightarrow S_Y$.

In order to surject onto the kernel of this epimorphism, we use the indecomposable transformations in $\mathcal{N}\mathcal{T}^*$ whose range is Y. Denoting these by $\eta_i\colon W_i \to Y$, $1 \le i \le n$, we obtain the two step resolution

$$\bigoplus_{i=1}^{n} Q_{W_i} \xrightarrow{(\eta_1\ \eta_2\ \cdots\ \eta_n)} Q_Y \twoheadrightarrow S_Y\ .$$

In the notation of [11], the map $\bigoplus_{i=1}^{n} Q_{W_i} \to Q_Y$ corresponds to a morphism $\phi\colon \mathscr{R}_Y \to \bigoplus_{i=1}^{n} \mathscr{R}_{W_i}$ of $\mathfrak{I}$-projectives in $\mathfrak{KK}(X)$. If the mapping cone C_ϕ of ϕ is again $\mathfrak{I}$-projective, the distinguished triangle $\Sigma C_\phi \to \mathscr{R}_Y \xrightarrow{\phi} \bigoplus_{i=1}^{n} \mathscr{R}_{W_i} \to C_\phi$ yields the projective resolution

$$\cdots \to Q_Y \to Q_\phi[1] \to \bigoplus_{i=1}^{n} Q_{W_i}[1] \to Q_Y[1] \to Q_\phi \to \bigoplus_{i=1}^{n} Q_{W_i} \to Q_Y \twoheadrightarrow S_Y\ ,$$

where $Q_\phi = \mathrm{FK}(C_\phi)$. We denote periodic resolutions like this by

$$Q_\phi \longrightarrow \bigoplus_{i=1}^{n} Q_{W_i} \longrightarrow Q_Y \to S_Y\ .$$

If the mapping cone C_ϕ is not $\mathfrak{I}$-projective, the situation has to be investigated individually. We will see examples of this in Sects. 3.7 and 3.9. The resolutions we construct in these cases exhibit a certain six-term periodicity as well. However, they begin with a finite number of “non-periodic steps” (one in Sect. 3.7 and two in Sect. 3.9), which can be considered as a symptom of the deficiency of the invariant filtrated K-theory over non-accordion spaces from the homological viewpoint. We remark without proof that the mapping cone of the morphism $\phi\colon \mathscr{R}_Y \to \bigoplus_{i=1}^{n} \mathscr{R}_{W_i}$ is $\mathfrak{I}$-projective for every $Y \in \mathbb{LC}(X)^*$ if and only if X is a disjoint union of accordion spaces.

3.6 Tensor Products with Free Right-Modules

Lemma 2. *Let M be an $\mathcal{N}\mathcal{T}^*$-left-module. There is an isomorphism $Q_Y \otimes_{\mathcal{N}\mathcal{T}^*} M \cong M(Y)$ of $\mathbb{Z}/2$-graded Abelian groups which is natural in $Y \in\in \mathcal{N}\mathcal{T}^*$.*

Proof. This is a simple consequence of Yoneda’s lemma and the tensor-hom adjunction.

Lemma 3. *Let*

$$\Sigma\mathscr{R}_{(3)} \xrightarrow{\gamma} \mathscr{R}_{(1)} \xrightarrow{\alpha} \mathscr{R}_{(2)} \xrightarrow{\beta} \mathscr{R}_{(3)}$$

be a distinguished or anti-distinguished triangle in $\mathfrak{KK}(X)$*, where*

$$\mathscr{R}_{(i)} = \bigoplus_{j=1}^{m_i} \mathscr{R}_{Y^i_j} \oplus \bigoplus_{k=1}^{n_i} \Sigma\mathscr{R}_{Z^i_k}$$

for $1 \leq i \leq 3$*,* $m_i, n_i \in \mathbb{N}$ *and* $Y^i_j, Z^i_k \in \mathbb{LC}(X)^*$*. Set* $Q_{(i)} = \widehat{\mathrm{FK}}(\mathscr{R}_{(i)})$*. If* $M = \mathrm{FK}(A)$ *for some* $A \in\in \mathfrak{KK}(X)$*, then the induced sequence*

$$\begin{array}{ccccc}
Q_{(1)} \otimes_{\mathscr{N}\mathscr{T}^*} M & \xrightarrow{\alpha^*\otimes\mathrm{id}_M} & Q_{(2)} \otimes_{\mathscr{N}\mathscr{T}^*} M & \xrightarrow{\beta^*\otimes\mathrm{id}_M} & Q_{(3)} \otimes_{\mathscr{N}\mathscr{T}^*} M \\
{\scriptstyle \gamma^*\otimes\mathrm{id}_M[1]}\uparrow & & & & \downarrow{\scriptstyle \gamma^*\otimes\mathrm{id}_M} \\
Q_{(3)} \otimes_{\mathscr{N}\mathscr{T}^*} M[1] & \xleftarrow[\beta^*\otimes\mathrm{id}_M[1]]{} & Q_{(2)} \otimes_{\mathscr{N}\mathscr{T}^*} M[1] & \xleftarrow[\alpha^*\otimes\mathrm{id}_M[1]]{} & Q_{(1)} \otimes_{\mathscr{N}\mathscr{T}^*} M[1]
\end{array} \tag{3.2}$$

is exact.

Proof. Using the previous lemma and the representability theorem, we naturally identify $Q_{(i)} \otimes_{\mathscr{N}\mathscr{T}^*} M \cong \mathrm{KK}_*(X; \mathscr{R}_{(i)}, A)$. Since, in triangulated categories, distinguished or anti-distinguished triangles induce long exact Hom-sequences, the sequence (3.2) is thus exact.

3.7 Proof of Proposition 2

We may restrict to connected T_0-spaces. In [9], a list of isomorphism classes of connected T_0-spaces with three or four points is given. If X is a disjoint union of accordion spaces, then the assertion follows from [2]. The remaining spaces fall into two classes:

1. All connected non-accordion four-point T_0-spaces except for the pseudocircle;
2. The pseudocircle (see Sect. 3.7.2).

The spaces in the first class have the following in common: If we fix two of them, say X, Y, then there is an ungraded isomorphism $\Phi\colon \mathscr{N}\mathscr{T}^*(X) \to \mathscr{N}\mathscr{T}^*(Y)$ between the categories of natural transformations on the respective filtrated K-theories such that the induced equivalence of ungraded module categories

$$\Phi^*\colon \mathfrak{Mod}^{\mathrm{ungr}}\big(\mathcal{NT}^*(Y)\big)_{\mathrm{c}} \to \mathfrak{Mod}^{\mathrm{ungr}}\big(\mathcal{NT}^*(X)\big)_{\mathrm{c}}$$

restricts to a bijective correspondence between exact ungraded $\mathcal{NT}^*(Y)$-modules and exact ungraded $\mathcal{NT}^*(X)$-modules. Moreover, the isomorphism Φ restricts to an isomorphism from $\mathcal{NT}_{\mathrm{ss}}(X)$ onto $\mathcal{NT}_{\mathrm{ss}}(Y)$ and one from $\mathcal{NT}_{\mathrm{nil}}(X)$ onto $\mathcal{NT}_{\mathrm{nil}}(Y)$. In particular, the assertion holds for X if and only if it holds for Y.

The above is a consequence of the investigations in [1, 2, 11]; the same kind of relation was found in [2] for the categories of natural transformations associated to accordion spaces with the same number of points. As a consequence, it suffices to verify the assertion for one representative of the first class—we choose Z_3—and for the pseudocircle.

3.7.1 Resolutions for the Space Z_3

We refer to [11] for a description of the category $\mathcal{NT}^*(Z_3)$, which in particular implies, that the space Z_3 satisfies the conditions of Proposition 1. Using the extension triangles from [11, (2.5)], the procedure described in Sect. 3.5 yields the following projective resolutions induced by distinguished triangles as in Lemma 3:

$$Q_1[1] \longrightarrow Q_4 \longrightarrow Q_{14} \to S_{14}\,, \qquad \text{and similarly for } S_{24},\, S_{34};$$

$$Q_{1234}[1] \longrightarrow Q_1[1] \oplus Q_2[1] \oplus Q_3[1] \longrightarrow Q_4 \to S_4\,;$$

$$Q_{234} \longrightarrow Q_{1234} \longrightarrow Q_1 \to S_1\,, \qquad \text{and similarly for } S_2,\, S_3.$$

Next we will deal with the modules S_{jk4}, where $1 \leq j < k \leq 3$. We observe that there is a Mayer-Vietoris type exact sequence of the form

$$Q_4 \longrightarrow Q_{j4} \oplus Q_{k4} \longrightarrow Q_{jk4}\ . \tag{3.3}$$

Lemma 4. *The candidate triangle $\Sigma\mathcal{R}_4 \to \mathcal{R}_{jk4} \to \mathcal{R}_{j4} \oplus \mathcal{R}_{k4} \to \mathcal{R}_4$ corresponding to the periodic part of the sequence (3.3) is distinguished or anti-distinguished (depending on the choice of signs for the maps in (3.3)).*

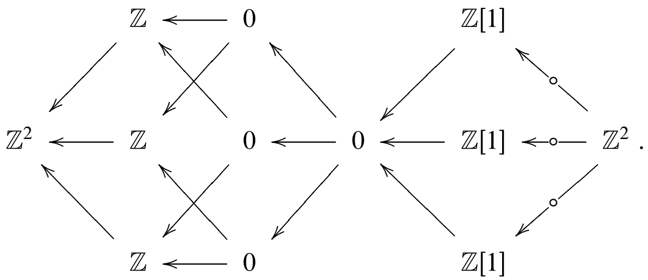
Proof. We give the proof for $j = 1$ and $k = 2$. The other cases follow from cyclicly permuting the indices 1, 2 and 3. We denote the morphism $\mathcal{R}_{124} \to \mathcal{R}_{14} \oplus \mathcal{R}_{24}$ by φ and the corresponding map $Q_{14} \oplus Q_{24} \to Q_{124}$ in (3.3) by φ^*. It suffices to check that $\widehat{\mathrm{FK}}(\mathrm{Cone}_\varphi)$ and Q_4 correspond, possibly up to a sign, to the same element in $\mathrm{Ext}^1_{\mathcal{NT}^*(Z_3)^{\mathrm{op}}}\big(\ker(\varphi^*), \mathrm{coker}(\varphi^*)[1]\big)$. We have $\mathrm{coker}(\varphi^*) \cong S_{124}$ and an extension $S_{124}[1] \rightarrowtail Q_4 \twoheadrightarrow \ker(\varphi^*)$. Since $\mathrm{Hom}(Q_4, S_{124}[1]) \cong S_{124}(4)[1] = 0$ and $\mathrm{Ext}^1(Q_4, S_{124}[1]) = 0$ because Q_4 is projective, the long exact Ext-sequence yields $\mathrm{Ext}^1\big(\ker(\varphi^*), \mathrm{coker}(\varphi^*)[1]\big) \cong \mathrm{Hom}(S_{124}[1], S_{124}[1]) \cong \mathbb{Z}$. Considering the sequence of transformations $3 \xrightarrow{\delta} 124 \xrightarrow{i} 1234 \xrightarrow{r} 3$, it is straight-forward to check that such an extension corresponds to one of the generators $\pm 1 \in \mathbb{Z}$ if and only if its underlying module is exact. This concludes the proof because both $\widehat{\mathrm{FK}}(\mathrm{Cone}_\varphi)$ and Q_4 are exact.

Hence we obtain the following projective resolutions induced by distinguished or anti-distinguished triangles as in Lemma 3:

$$Q_4 \longrightarrow Q_{j4} \oplus Q_{k4} \longrightarrow Q_{jk4} \to S_{jk4} \; .$$

To summarize, by Lemma 3, $\mathrm{Tor}_n^{\mathcal{N}\mathcal{T}^*}(S_Y, M) = 0$ for $Y \neq 1234$ and $n \geq 1$.

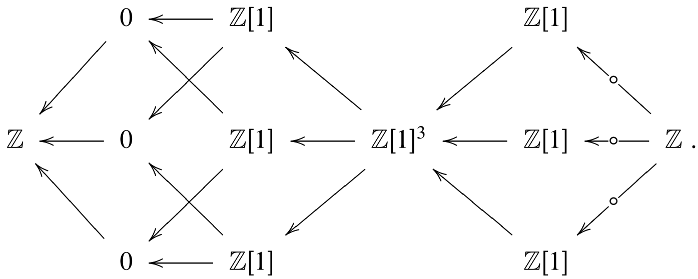
As we know from [11], the subset 1234 of Z_3 plays an exceptional role. In the notation of [11] (with the direction of the arrows reversed because we are dealing with *right*-modules), the kernel of the homomorphism $Q_{124} \oplus Q_{134} \oplus Q_{234} \xrightarrow{(i\ i\ i)} Q_{1234}$ is of the form



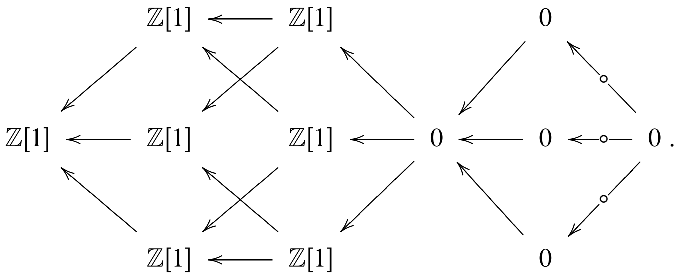
It is the image of the module homomorphism

$$Q_{14} \oplus Q_{24} \oplus Q_{34} \xrightarrow{\begin{pmatrix} i & -i & 0 \\ -i & 0 & i \\ 0 & i & -i \end{pmatrix}} Q_{124} \oplus Q_{134} \oplus Q_{234}, \tag{3.4}$$

the kernel of which, in turn, is of the form



A surjection from $Q_4 \oplus Q_{1234}[1]$ onto this module is given by $\left(\begin{smallmatrix} i & i & i \\ \delta_{1234}^{14} & 0 & 0 \end{smallmatrix}\right)$, where $\delta_{1234}^{14} := \delta_3^{14} \circ r_{1234}^{3}$. The kernel of this homomorphism has the form



This module is isomorphic to $\mathrm{Syz}_{1234}[1]$, where $\mathrm{Syz}_{1234} := \ker(Q_{1234} \twoheadrightarrow S_{1234})$. Therefore, we end up with the projective resolution

$$Q_4 \oplus Q_{1234}[1] \longrightarrow Q_{14} \oplus Q_{24} \oplus Q_{34} \longrightarrow Q_{124} \oplus Q_{134} \oplus Q_{234} \longrightarrow Q_{1234} \to S_{1234}\,. \tag{3.5}$$

The homomorphism from $Q_{124} \oplus Q_{134} \oplus Q_{234}$ to $Q_4 \oplus Q_{1234}[1]$ is given by

$$\begin{pmatrix} 0 & 0 & -\delta_{234}^{4} \\ i & i & i \end{pmatrix},$$

where $\delta_{234}^{4} := \delta_2^4 \circ r_{234}^{2}$.

Lemma 5. *The candidate triangle in $\mathfrak{KK}(X)$ corresponding to the periodic part of the sequence* (3.5) *is distinguished or anti-distinguished (depending on the choice of signs for the maps in* (3.5)*).*

Proof. The argument is analogous to the one in the proof of Lemma 4. Again, we consider the group $\mathrm{Ext}^1_{\mathcal{NT}^*(Z_3)^{\mathrm{op}}}\big(\ker(\varphi^*), \mathrm{coker}(\varphi^*)[1]\big)$ where φ^* now denotes the map (3.4). We have $\mathrm{coker}(\varphi^*) \cong \mathrm{Syz}_{1234}$ and an extension $Q_4 \rightarrowtail \ker(\varphi^*) \twoheadrightarrow S_{1234}[1]$. Using long exact sequences, we obtain

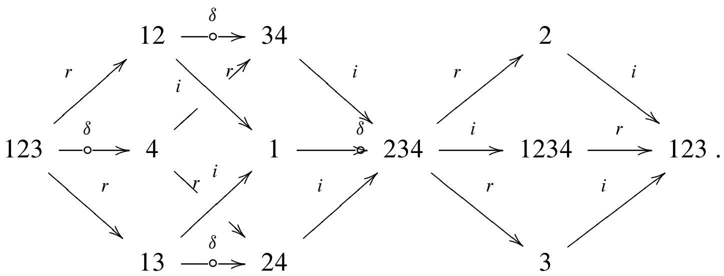
$$\begin{aligned} \mathrm{Ext}^1\big(\ker(\varphi^*), \mathrm{coker}(\varphi^*)[1]\big) &\cong \mathrm{Ext}^1(S_{1234}[1], \mathrm{Syz}_{1234}[1]) \\ &\cong \mathrm{Hom}(S_{1234}[1], S_{1234}[1]) \cong \mathbb{Z}. \end{aligned}$$

Again, an extension corresponds to a generator if and only if its underlying module is exact.

By the previous lemma and Sect. 3.6, computing the tensor product of this complex with M and taking homology shows that $\mathrm{Tor}_n^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) = 0$ for $n \geq 2$ and that $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M)$ is equal to $\mathrm{Tor}_1^{\mathcal{NT}^*}(S_{1234}, M)$ and isomorphic to the homology of the complex (3.1).

Example 1. For the filtrated K-module with projective dimension 2 constructed in [11, §5] we get $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) \cong \mathbb{Z}/k$.

Remark 1. As explicated in the beginning of this section, the category $\mathcal{NT}^*(S)$ corresponding to the four-point space S defined in the introduction is isomorphic in an appropriate sense to the category $\mathcal{NT}^*(Z_3)$. As has been established in [1], the indecomposable morphisms in $\mathcal{NT}^*(S)$ are organised in the diagram



In analogy to (3.1), we have that $\mathrm{Tor}_1^{\mathcal{NT}^*(S)}(\mathcal{NT}_{\mathrm{ss}}, M)$ is isomorphic to the homology of the complex

$$M(12)[1] \oplus M(4) \oplus M(13)[1] \xrightarrow{\begin{pmatrix} \delta & -r & 0 \\ -i & 0 & i \\ 0 & r & -\delta \end{pmatrix}} M(34) \oplus M(1)[1] \oplus M(24) \xrightarrow{(i\ \delta\ i)} M(234)\,, \tag{3.6}$$

where $M = \mathrm{FK}(A)$ for some separable C^*- algebra A over X.

3.7.2 Resolutions for the Pseudocircle

Let $C_2 = \{1, 2, 3, 4\}$ with the partial order defined by $1 < 3$, $1 < 4$, $2 < 3$, $2 < 4$. The topology on C_2 is thus given by $\{\emptyset, 3, 4, 34, 134, 234, 1234\}$. Hence the non-empty, connected, locally closed subsets are

$$\mathbb{LC}(C_2)^* = \{3, 4, 134, 234, 1234, 13, 14, 23, 24, 124, 123, 1, 2\}\,.$$

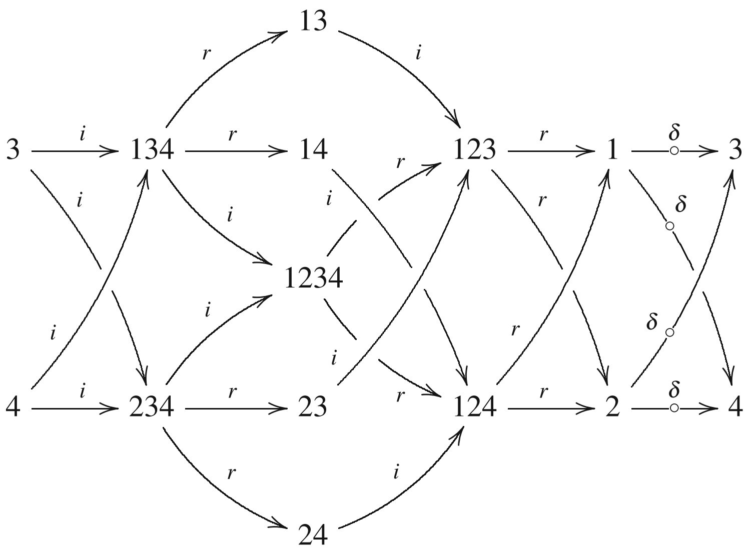
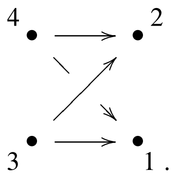


Fig. 3.1 Indecomposable natural transformations in $\mathscr{N}\mathscr{T}^*(C_2)$

The partial order on C_2 corresponds to the directed graph



The space C_2 is the only T_0-space with at most four points with the property that its order complex (see [11, Definition 2.6]) is not contractible; in fact, it is homeomorphic to the circle $\mathbb{S}^1$. Therefore, by the representability theorem [11, §2.1] we find

$$\mathscr{N}\mathscr{T}_*(C_2, C_2) \cong \mathrm{KK}_*(X; \mathscr{R}_{C_2}, \mathscr{R}_{C_2}) \cong \mathrm{K}_*\big(\mathscr{R}_{C_2}(C_2)\big) \cong \mathrm{K}^*\left(\mathbb{S}^1\right) \cong \mathbb{Z} \oplus \mathbb{Z}[1] ,$$

that is, there are non-trivial odd natural transformations $\mathrm{FK}_{C_2} \Rightarrow \mathrm{FK}_{C_2}$. These are generated, for instance, by the composition $C_2 \xrightarrow{r} 1 \xrightarrow{\delta} 3 \xrightarrow{i} C_2$. This follows from the description of the category $\mathscr{N}\mathscr{T}^*(C_2)$ below. Note that $\delta_{C_2}^{C_2} \circ \delta_{C_2}^{C_2}$ vanishes because it factors through $r_{13}^1 \circ i_3^{13} = 0$.

Figure 3.1 displays a set of indecomposable transformations generating the category $\mathscr{N}\mathscr{T}^*(C_2)$ determined in [1, §6.3.2], where also a list of relations generating the relations in the category $\mathscr{N}\mathscr{T}^*(C_2)$ can be found. From this, it is straight-forward to verify that the space C_2 satisfies the conditions of Proposition 1.

Proceeding as described in Sect. 3.5, we find projective resolutions of the following form (we omit explicit descriptions of the boundary maps):

$$Q_{123}[1] \longrightarrow Q_1[1] \oplus Q_2[1] \longrightarrow Q_3 \to S_3 \,, \qquad \text{and similarly for } S_4;$$

$$Q_1[1] \longrightarrow Q_3 \oplus Q_4 \longrightarrow Q_{134} \to S_{134} \,, \qquad \text{and similarly for } S_{234};$$

$$Q_4 \longrightarrow Q_{134} \longrightarrow Q_{13} \to S_{13} \,, \qquad \text{and similarly for } S_{14}, S_{23}, S_{24};$$

$$Q_3 \oplus Q_4 \longrightarrow Q_{134} \oplus Q_{234} \longrightarrow Q_{1234} \to S_{1234} \,;$$

$$Q_4 \oplus Q_{123}[1] \longrightarrow Q_{134} \oplus Q_{234} \longrightarrow Q_{1234} \oplus Q_{13} \oplus Q_{23} \to Q_{123} \to S_{123} \,,$$

and similarly for S_{124};

$$Q_{234} \oplus Q_1[1] \longrightarrow Q_{1234} \oplus Q_{23} \oplus Q_{24} \longrightarrow Q_{123} \oplus Q_{124} \to Q_1 \to S_1 \,,$$

and similarly for S_2. Again, the periodic part of each of these resolutions is induced by an extension triangle, a Mayer-Vietoris triangle as in Lemma 4 or a more exotic (anti-)distinguished triangle as in Lemma 5 (we omit the analogous computation here).

We get $\mathrm{Tor}_1^{\mathcal{N}\mathcal{T}^*}(S_Y, M) = 0$ for every $Y \in \mathbb{LC}(C_2)^* \setminus \{123, 124, 1, 2\}$, and further $\mathrm{Tor}_n^{\mathcal{N}\mathcal{T}^*}(S_Y, M) = 0$ for all $Y \in \mathbb{LC}(C_2)^*$ and $n \geq 2$. Therefore,

$$\mathrm{Tor}_1^{\mathcal{N}\mathcal{T}^*}(\mathcal{N}\mathcal{T}_{\mathrm{ss}}, M) \cong \bigoplus_{Y \in \{123,124,1,2\}} \mathrm{Tor}_1^{\mathcal{N}\mathcal{T}^*}(S_Y, M) \,.$$

The four groups $\mathrm{Tor}_1^{\mathcal{N}\mathcal{T}^*}(S_Y, M)$ with $Y \in \{123, 124, 1, 2\}$ can be described explicitly as in Sect. 3.7.1 using the above resolutions. This finishes the proof of Proposition 2.

3.8 Proof of Proposition 3

We apply the Meyer-Nest machinery to the homological functor $\mathrm{FK} \otimes \mathbb{Q}$ on the triangulated category $\mathfrak{KK}(X) \otimes \mathbb{Q}$. We need to show that every $\mathcal{N}\mathcal{T}^* \otimes \mathbb{Q}$ module of the form $M = \mathrm{FK}(A) \otimes \mathbb{Q}$ has a projective resolution of length 1. It is easy to see that analogues of Propositions 1 and 2 hold. In particular, the term $\mathrm{Tor}_2^{\mathcal{N}\mathcal{T}^* \otimes \mathbb{Q}}(\mathcal{N}\mathcal{T}_{\mathrm{ss}} \otimes \mathbb{Q}, M)$ always vanishes. Here we use that $\mathbb{Q}$ is a flat

$\mathbb{Z}$-module, so that tensoring with $\mathbb{Q}$ turns projective $\mathcal{NT}^*$-module resolutions into projective $\mathcal{NT}^* \otimes \mathbb{Q}$-module resolutions. Moreover, the freeness condition for the $\mathbb{Q}$-module $\operatorname{Tor}_1^{\mathcal{NT}^* \otimes \mathbb{Q}}(\mathcal{NT}_{\mathrm{ss}} \otimes \mathbb{Q}, M)$ is empty since $\mathbb{Q}$ is a field.

3.9 Proof of Proposition 4

The computations to determine the category $\mathcal{NT}^*(Z_4)$ are very similar to those for the category $\mathcal{NT}^*(Z_3)$ which were carried out in [11]. We summarise its structure in Fig. 3.2. The relations in $\mathcal{NT}^*(Z_4)$ are generated by the following:

- The hypercube with vertices $5, 15, 25, \ldots, 12345$ is a commuting diagram;
- The following compositions vanish:

$$1235 \xrightarrow{i} 12345 \xrightarrow{r} 4\,, \quad 1245 \xrightarrow{i} 12345 \xrightarrow{r} 3\,,$$

$$1345 \xrightarrow{i} 12345 \xrightarrow{r} 2\,, \quad 2345 \xrightarrow{i} 12345 \xrightarrow{r} 1\,,$$

$$1 \xrightarrow{\delta} 5 \xrightarrow{i} 15\,, \quad 2 \xrightarrow{\delta} 5 \xrightarrow{i} 25\,, \quad 3 \xrightarrow{\delta} 5 \xrightarrow{i} 35\,, \quad 4 \xrightarrow{\delta} 5 \xrightarrow{i} 45\,;$$

- The sum of the four maps $12345 \to 5$ via 1, 2, 3, and 4 vanishes.

This implies that the space Z_4 satisfies the conditions of Proposition 1.

In the following, we will define an exact $\mathcal{NT}^*$-left-module M and compute $\operatorname{Tor}_2^{\mathcal{NT}^*}(S_{12345}, M)$. By explicit computation, one finds a projective resolution of the simple $\mathcal{NT}^*$-right-module S_{12345} of the following form (again omitting explicit formulas for the boundary maps):

$$\xrightarrow{\circ} Q_5 \oplus \bigoplus_{1\le i\le 4} Q_{12345\setminus i}[1] \longrightarrow \bigoplus_{1\le l\le 4} Q_{l5} \oplus Q_{12345}[1] \longrightarrow \bigoplus_{1\le j<k\le 4} Q_{jk5} \longrightarrow$$
$$\longrightarrow \bigoplus_{1\le i\le 4} Q_{12345\setminus i} \longrightarrow Q_{12345} \longrightarrow S_{12345}.$$

Notice that this sequence is periodic as a cyclic six-term sequence except for the first *two* steps.

Consider the exact $\mathcal{NT}^*$-left-module M defined by the exact sequence

$$0 \to P_{12345} \xrightarrow{\begin{pmatrix} i\\ i\\ i\\ i \end{pmatrix}} \bigoplus_{1\le i\le 4} P_{12345\setminus i} \xrightarrow{\begin{pmatrix} i & -i & 0 & 0\\ -i & 0 & i & 0\\ 0 & i & -i & 0\\ i & 0 & 0 & -i\\ 0 & -i & 0 & i\\ 0 & 0 & i & -i \end{pmatrix}} \bigoplus_{1\le j<k\le 4} P_{jk5} \twoheadrightarrow M\,. \tag{3.7}$$

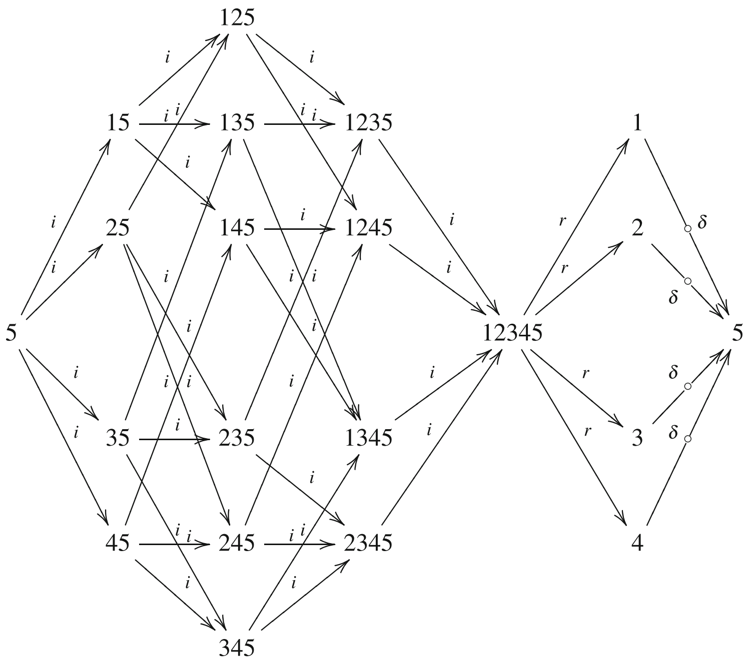


Fig. 3.2 Indecomposable natural transformations in $\mathscr{N}\mathscr{T}^*(Z_4)$

We have $\bigoplus_{1\leq l\leq 4} M(l5) \oplus M(12345)[1] \cong 0 \oplus \mathbb{Z}^3$, $\bigoplus_{1\leq j<k\leq 4} M(jk5) \cong \mathbb{Z}^6$, and $M(5) \oplus \bigoplus_{1\leq i\leq 4} M(12345\setminus i)[1] \cong \mathbb{Z}[1] \oplus \mathbb{Z}[1]^8$. Since

$$\begin{array}{ccc} \bigoplus\limits_{1\leq l\leq 4} M(l5) \oplus M(12345)[1] & \longrightarrow & \bigoplus\limits_{1\leq j<k\leq 4} M(jk5) \\ & \nwarrow & \big\downarrow \circ \\ & & M(5) \oplus \bigoplus\limits_{1\leq i\leq 4} M(12345\setminus i)[1] \end{array}$$

is exact, a rank argument shows that the map

$$\bigoplus_{1\leq l\leq 4} M(l5) \oplus M(12345)[1] \to \bigoplus_{1\leq j<k\leq 4} M(jk5)$$

is zero. On the other hand, the kernel of the map

$$\bigoplus_{1\leq j<k\leq 4} M(jk5) \xrightarrow{\begin{pmatrix} i & -i & 0 & i & 0 & 0 \\ -i & 0 & i & 0 & -i & 0 \\ 0 & i & -i & 0 & 0 & i \\ 0 & 0 & 0 & -i & i & -i \end{pmatrix}} \bigoplus_{1\leq i\leq 4} M(12345\setminus i)$$

is non-trivial; it consists precisely of the elements in

$$\bigoplus_{1\leq j<k\leq 4} M(jk5) \cong \bigoplus_{1\leq j<k\leq 4} \mathbb{Z}[\mathrm{id}_{jk5}^{jk5}]$$

which are multiples of $([\mathrm{id}_{jk5}^{jk5}])_{1\leq j<k\leq 4}$. This shows $\mathrm{Tor}_2^{\mathcal{NT}^*}(S_{12345}, M) \cong \mathbb{Z}$. Hence, by Proposition 1, the module M has projective dimension at least 2. On the other hand, (3.7) is a resolution of length 2. Therefore, the projective dimension of M is exactly 2.

Let $k \in \mathbb{N}_{\geq 2}$ and define $M_k = M \otimes_{\mathbb{Z}} \mathbb{Z}/k$. Since $\mathrm{Tor}_2^{\mathcal{NT}^*}(S_{12345}, M_k) \cong \mathbb{Z}/k$ is non-free, Proposition 1 shows that M_k has at least projective dimension 3. On the other hand, if we abbreviate the resolution (3.7) for M by

$$0 \to P^{(5)} \overset{\alpha}{\to} P^{(4)} \overset{\beta}{\to} P^{(3)} \twoheadrightarrow M \ , \tag{3.8}$$

a projective resolution of length 3 for M_k is given by

$$0 \to P^{(5)} \xrightarrow{\binom{k}{\alpha}} P^{(5)} \oplus P^{(4)} \xrightarrow{\begin{pmatrix} \alpha & -k \\ 0 & \beta \end{pmatrix}} P^{(4)} \oplus P^{(3)} \xrightarrow{(\beta \ k)} P^{(3)} \twoheadrightarrow M_k \ ,$$

where k denotes multiplication by k.

It remains to show that the modules M and M_k can be realised as the filtrated K-theory of objects in $\mathcal{B}(X)$. It suffices to prove this for the module M since tensoring with the Cuntz algebra $\mathcal{O}_{k+1}$ then yields a separable C^*- algebra with filtrated K-theory M_k by the Künneth Theorem.

The projective resolution (3.8) can be written as

$$0 \to \mathrm{FK}(P^2) \xrightarrow{\mathrm{FK}(f_2)} \mathrm{FK}(P^1) \xrightarrow{\mathrm{FK}(f_1)} \mathrm{FK}(P^0) \twoheadrightarrow M,$$

because of the equivalence of the category of projective $\mathcal{NT}^*$-modules and the category of $\mathfrak{J}$-projective objects in $\mathfrak{KK}(X)$. Let N be the cokernel of the module map $\mathrm{FK}(f_2)$. Using [11, Theorem 4.11], we obtain an object $A \in\in \mathcal{B}(X)$ with $\mathrm{FK}(A) \cong N$. We thus have a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{FK}(P^2) & \xrightarrow{\mathrm{FK}(f_2)} & \mathrm{FK}(P^1) & \xrightarrow{\qquad\mathrm{FK}(f_1)\qquad} & \mathrm{FK}(P^0) & \twoheadrightarrow & M \ . \\ & & & & & \searrow\!\!\!\!\searrow \qquad \nearrow \gamma & & & \\ & & & & & \mathrm{FK}(A) & & & \end{array}$$

Since A belongs to the bootstrap class $\mathcal{B}(X)$ and $\mathrm{FK}(A)$ has a projective resolution of length 1, we can apply the universal coefficient theorem to lift the homomorphism

γ to an element $f \in \mathrm{KK}(X; A, P^0)$. Now we can argue as in the proof of [11, Theorem 4.11]: since f is $\mathfrak{I}$-monic, the filtrated K-theory of its mapping cone is isomorphic to $\operatorname{coker}(\gamma) \cong M$. This completes the proof of Proposition 4.

3.10 Cuntz-Krieger Algebras with Projective Dimension 2

In this section we exhibit a Cuntz-Krieger algebra A which is a tight C^*-algebra over the space Z_3 and for which the odd part of $\mathrm{Tor}_1^{\mathcal{NT}^*(Z_3)}\big(\mathcal{NT}_{\mathrm{ss}}, \mathrm{FK}(A)\big)$—denoted $\mathrm{Tor}_1^{\mathrm{odd}}$ in the following—is not free. By Proposition 2 this C^*-algebra has projective dimension 2 in filtrated K-theory.

In the following we will adhere to the conventions for graph algebras and adjacency matrices from [4]. Let E be the finite graph with vertex set $E^0 = \{v_1, v_2, \ldots, v_8\}$ and edges corresponding to the adjacency matrix

$$\begin{pmatrix} B_4 & 0 & 0 & 0 \\ X_1 & B_1 & 0 & 0 \\ X_2 & 0 & B_2 & 0 \\ X_3 & 0 & 0 & B_3 \end{pmatrix} := \begin{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} & 0 & 0 & 0 \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} & 0 & 0 \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} & 0 \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & 0 & \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \end{pmatrix}. \tag{3.9}$$

Since this is a finite graph with no sinks and no sources, the associated graph C^*- algebra $C^*(E)$ is in fact a Cuntz-Krieger algebra (we can replace E with its *edge graph*; see [13, Remark 2.8]). Moreover, the graph E is easily seen to fulfill condition (K) because every vertex is the base of two or more simple cycles. As a consequence, the adjacency matrix of the edge graph of E fulfills condition (II) from [5]. In fact, condition (K) is designed as a generalisation of condition (II): see, for instance, [8].

Applying [13, Theorem 4.9]—and carefully translating between different graph algebra conventions—we find that the ideals of $C^*(E)$ correspond bijectively and in an inclusion-preserving manner to the open subsets of the space Z_3. By [9, Lemma 2.35], we may turn A into a tight C^*- algebra over Z_3 by declaring $A(\{4\}) = I_{\{v_1,v_2\}}$, $A(\{1,4\}) = I_{\{v_1,v_2,v_3,v_4\}}$, $A(\{2,4\}) = I_{\{v_1,v_2,v_5,v_6\}}$ as well as $A(\{3,4\}) = I_{\{v_1,v_2,v_7,v_8\}}$, where I_S denotes the ideal corresponding to the saturated hereditary subset S.

It is known how to compute the six-term sequence in K-theory for an extension of graph C^*- algebras: see [4]. Using this and Proposition 2, $\mathrm{Tor}_1^{\mathrm{odd}}$ is the homology of the complex

$$\ker(\phi_0) \xrightarrow{\left(\begin{smallmatrix} i & -i & 0 \\ -i & 0 & i \\ 0 & i & -i \end{smallmatrix}\right)} \ker(\phi_1) \xrightarrow{(\,i\ i\ i\,)} \ker(\phi_2)\ , \tag{3.10}$$

where $\phi_0 = \operatorname{diag}\left(\left(\begin{smallmatrix} B_4' & X_1^t \\ 0 & B_1' \end{smallmatrix}\right), \left(\begin{smallmatrix} B_4' & X_2^t \\ 0 & B_2' \end{smallmatrix}\right), \left(\begin{smallmatrix} B_4' & X_3^t \\ 0 & B_3' \end{smallmatrix}\right)\right)\ , \quad \phi_2 = \begin{pmatrix} B_4' & X_1^t & X_2^t & X_3^t \\ 0 & B_1' & 0 & 0 \\ 0 & 0 & B_2' & 0 \\ 0 & 0 & 0 & B_3' \end{pmatrix}\ ,$

$$\phi_1 = \operatorname{diag}\left(\begin{pmatrix} B_4' & X_1^t & X_2^t \\ 0 & B_1' & 0 \\ 0 & 0 & B_2' \end{pmatrix}, \begin{pmatrix} B_4' & X_1^t & X_3^t \\ 0 & B_1' & 0 \\ 0 & 0 & B_3' \end{pmatrix}, \begin{pmatrix} B_4' & X_2^t & X_3^t \\ 0 & B_2' & 0 \\ 0 & 0 & B_3' \end{pmatrix}\right)\ ,$$

and $B_4' = B_4^t - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)$ and $B_j' = B_j^t - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right)$ for $1 \leq j \leq 3$. We obtain a commutative diagram

$$\begin{array}{ccccc} \ker(\phi_0) & \rightarrowtail & (\mathbb{Z}^{\oplus 2})^{\oplus(2\cdot 3)} & \overset{\phi_0}{\twoheadrightarrow} & \operatorname{im}(\phi_0) \\ \downarrow{\scriptstyle f_K} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_I} \\ \ker(\phi_1) & \rightarrowtail & (\mathbb{Z}^{\oplus 2})^{\oplus(3\cdot 3)} & \overset{\phi_1}{\twoheadrightarrow} & \operatorname{im}(\phi_1) \\ \downarrow{\scriptstyle g_K} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_I} \\ \ker(\phi_2) & \rightarrowtail & (\mathbb{Z}^{\oplus 2})^{\oplus(4\cdot 1)} & \overset{\phi_2}{\twoheadrightarrow} & \operatorname{im}(\phi_2)\ , \end{array} \tag{3.11}$$

where f and g have the block forms

$$f = \begin{pmatrix} \mathrm{id} & 0 & -\mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{id} & 0 & 0 \\ -\mathrm{id} & 0 & 0 & 0 & \mathrm{id} & 0 \\ 0 & -\mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{id} \\ 0 & 0 & \mathrm{id} & 0 & -\mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{id} \end{pmatrix}\ , \qquad g = \begin{pmatrix} \mathrm{id} & 0 & 0 & \mathrm{id} & 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 & \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 & 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{id} & 0 & 0 & \mathrm{id} \end{pmatrix}\ ,$$

and $f_K := f|_{\ker(\phi_0)}$, $f_I := f|_{\operatorname{im}(\phi_0)}$, $g_K := g|_{\ker(\phi_1)}$, $g_I := g|_{\operatorname{im}(\phi_1)}$. Notice that f and g are defined in a way such that the restrictions $f|_{\ker(\phi_0)}$ and $g|_{\ker(\phi_1)}$ are exactly the maps from (3.10) in the identification made above.

We abbreviate the above short exact sequence of cochain complexes (3.11) as $K_\bullet \rightarrowtail Z_\bullet \twoheadrightarrow I_\bullet$. The part $\mathrm{H}^0(Z_\bullet) \to \mathrm{H}^0(I_\bullet) \to \mathrm{H}^1(K_\bullet) \to \mathrm{H}^1(Z_\bullet)$ in the corresponding long exact homology sequence can be identified with

$$\ker(f) \overset{\phi_0}{\longrightarrow} \ker(f_I) \to \frac{\ker(g_K)}{\operatorname{im}(f_K)} \to 0\ .$$

Hence

$$\mathrm{Tor}_1^{\mathrm{odd}} \cong \frac{\ker(g_K)}{\mathrm{im}(f_K)} \cong \frac{\ker(f_I)}{\phi_0\big(\ker(f)\big)} \cong \frac{\ker(f) \cap \mathrm{im}(\phi_0)}{\phi_0\big(\ker(f)\big)} \,.$$

We have $\ker(f) = \{(v,0,v,0,v,0) \mid v \in \mathbb{Z}^2\} \subset (\mathbb{Z}^{\oplus 2})^{\oplus(2\cdot 3)}$.

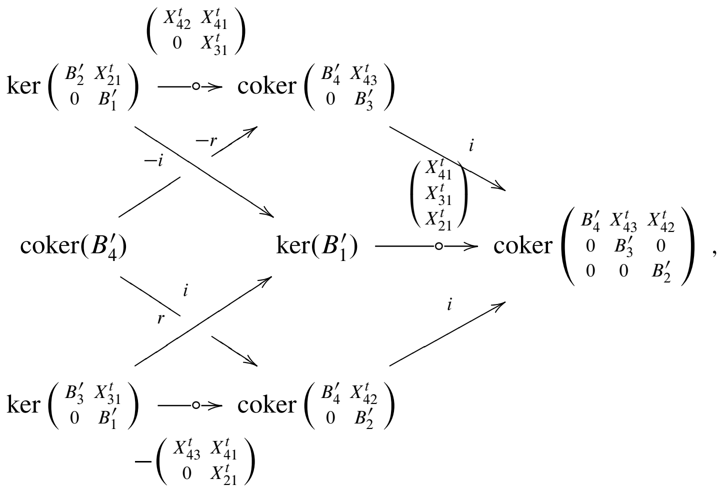
From the concrete form (3.9) of the adjacency matrix, we find that $\ker(f) \cap \mathrm{im}(\phi_0)$ is the free cyclic group generated by $(1,1,0,0,1,1,0,0,1,1,0,0)$, while $\phi_0\big(\ker(f)\big)$ is the subgroup generated by $(2,2,0,0,2,2,0,0,2,2,0,0)$. We see that $\mathrm{Tor}_1^{\mathrm{odd}} \cong \mathbb{Z}/2$ is not free.

Now we briefly indicate how to construct a similar counterexample for the space S. Consider the integer matrix

$$\begin{pmatrix} B_4 & 0 & 0 & 0 \\ X_{43} & B_3 & 0 & 0 \\ X_{42} & 0 & B_2 & 0 \\ X_{41} & X_{31} & X_{21} & B_1 \end{pmatrix} := \begin{pmatrix} (3) & 0 & 0 & 0 \\ (2) & (3) & 0 & 0 \\ (2) & 0 & (3) & 0 \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \end{pmatrix} .$$

The corresponding graph F fulfills condition (K) and has no sources or sinks. The associated graph C^*- algebra $C^*(F)$ is therefore a Cuntz-Krieger algebra satisfying condition (II). It is easily read from the block structure of the edge matrix that the primitive ideal space of $C^*(F)$ is homeomorphic to S. We are going to compute the even part of $\mathrm{Tor}_1^{\mathcal{NT}^*(S)}\big(\mathcal{NT}_{\mathrm{ss}}, \mathrm{FK}(C^*(F))\big)$. Since the nice computation methods from the previous example do not carry over, we carry out a more ad hoc calculation.

By Remark 1, the even part of our Tor-term is isomorphic to the homology of the complex



where column-wise direct sums are taken. Here $B'_1 = B^t_1 - \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ and $B'_j = B^t_j - \big(1\big) = \big(2\big)$ for $2 \le j \le 4$. This complex can be identified with

$$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z} \xrightarrow{\left(\begin{smallmatrix}0&1&0\\0&0&0\\-2&0&2\\0&1&0\\0&0&0\end{smallmatrix}\right)} (\mathbb{Z}/2)^2 \oplus \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \xrightarrow{\left(\begin{smallmatrix}1&0&0&1&0\\0&1&1&0&0\\0&0&1&0&1\end{smallmatrix}\right)} (\mathbb{Z}/2)^3 ,$$

the homology of which is isomorphic to $\mathbb{Z}/2$; a generator is given by the class of $(0,1,1,0,1) \in (\mathbb{Z}/2)^2 \oplus \mathbb{Z} \oplus (\mathbb{Z}/2)^2$. This concludes the proof of Proposition 5.

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