Chapter 11
C^*-Algebras Associated with a-adic Numbers

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Abstract By a crossed product construction, we produce a family of (stabilized) Cuntz-Li algebras associated with the a-adic numbers. Moreover, we present an a-adic duality theorem.

Keywords C^*-dynamical system • Cuntz-Li algebras • a-adic numbers

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11.1 Introduction

In [1] Cuntz introduces the C^*-algebra $\mathcal{Q}_{\mathbb{N}}$ associated with the $ax + b$-semigroup over the natural numbers, that is $\mathbb{Z} \rtimes \mathbb{N}^{\times}$, where $\mathbb{N}^{\times}$ acts on $\mathbb{Z}$ by multiplication. It is defined as the universal C^*-algebra generated by isometries $\{s_n\}_{n \in \mathbb{N}^{\times}}$ and a unitary u satisfying the relations

$$s_m s_n = s_{mn}, \quad s_n u = u^n s_n, \quad \text{and} \quad \sum_{k=0}^{n-1} u^k s_n s_n^* u^{-k} = 1 \quad \text{for } m, n \in \mathbb{N}^{\times}.$$

Furthermore, $\mathcal{Q}_{\mathbb{N}}$ is shown to be simple and purely infinite and can also be obtained as a semigroup crossed product

$$C(\hat{\mathbb{Z}}) \rtimes (\mathbb{Z} \rtimes \mathbb{N}^{\times})$$

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for the natural $ax+b$-semigroup action of $\mathbb{Z} \rtimes \mathbb{N}^\times$ on the finite integral adeles $\hat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$ (i.e. $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$). Its stabilization $\overline{\mathcal{Q}}_\mathbb{N}$ is isomorphic to the ordinary crossed product

$$C_0(\mathscr{A}_f) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^\times)$$

where $\mathbb{Q}_+^\times$ denotes the multiplicative group of positive rationals and $\mathscr{A}_f$ denotes the finite adeles, i.e. the restricted product $\prod'_{p \text{ prime}} \mathbb{Q}_p = \prod_{p \text{ prime}}(\mathbb{Q}_p, \mathbb{Z}_p)$. The action of $\mathbb{Q} \rtimes \mathbb{Q}_+^\times$ on $\mathscr{A}_f$ is the natural $ax+b$-action. This crossed product is the minimal automorphic dilation of the semigroup crossed product above (see Laca [8]).

Replacing $\mathbb{N}^\times$ with $\mathbb{Z}^\times$ gives rise to the C^*-algebra $\mathcal{Q}_\mathbb{Z}$ of the ring $\mathbb{Z}$. This approach is generalized to certain integral domains by Cuntz and Li [3] and then to more general rings by Li [10].

In [9] Larsen and Li define the 2-adic ring algebra of the integers $\mathcal{Q}_2$, attached to the semigroup $\mathbb{Z} \rtimes |2\rangle$, where $|2\rangle = \{2^i : i \geq 0\} \subset \mathbb{N}^\times$ acts on $\mathbb{Z}$ by multiplication. It is the universal C^*-algebra generated by an isometry s_2 and a unitary u satisfying the relations

$$s_2 u^k = u^{2k} s_2 \quad \text{and} \quad s_2 s_2^* + u s_2 s_2^* u^* = 1.$$

The algebra $\mathcal{Q}_2$ shares many structural properties with $\mathcal{Q}_\mathbb{N}$. It is simple, purely infinite and has a semigroup crossed product description. Its stabilization $\overline{\mathcal{Q}}_2$ is isomorphic to its minimal automorphic dilation, which is the crossed product

$$C_0(\mathbb{Q}_2) \rtimes \big(\mathbb{Z}[\tfrac{1}{2}] \rtimes \langle 2 \rangle\big).$$

Here $\mathbb{Z}[\frac{1}{2}]$ denotes the ring extension of $\mathbb{Z}$ by $\frac{1}{2}$, $\langle 2 \rangle$ the subgroup of the positive rationals $\mathbb{Q}_+^\times$ generated by 2 and the action of $\mathbb{Z}[\frac{1}{2}] \rtimes \langle 2 \rangle$ on $\mathbb{Q}_2$ is the natural $ax+b$-action.

Both $\mathscr{A}_f$ and $\mathbb{Q}_2$ are examples of groups of so-called a-adic numbers, defined by a doubly infinite sequence $a = (\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots)$ with $a_i \geq 2$ for all $i \in \mathbb{Z}$. Our goal is to construct C^*-algebras associated with the a-adic numbers and show that these algebras provide a family of examples that under certain conditions share many structural properties with $\mathcal{Q}_2$, $\mathcal{Q}_\mathbb{N}$ and also the ring C^*-algebras of Cuntz and Li.

Our approach is inspired by [5], that is, we begin with a crossed product by a group and use the classical theory of C^*-dynamical systems to prove our results, instead of the generators and relations as in the papers of Cuntz, Li and Larsen. Therefore, our construction only gives analogs of the stabilized algebras $\overline{\mathcal{Q}}_\mathbb{N}$ and $\overline{\mathcal{Q}}_2$.

Even though the C^*-algebras associated with a-adic numbers are closely related to the ring C^*-algebras of Cuntz and Li, they are not a special case of these (except in the finite adeles case). Also, our approach does not fit in general into the framework of [5].

One of the main results in the paper is Theorem 3, which is a general a-adic duality theorem that encompasses the 2-adic duality theorem [9, Theorem 7.5] and the analogous result of Cuntz [1, Theorem 6.5]. In the proof, we only apply crossed product techniques, and not the groupoid equivalence as in [9].

11.2 The *a*-adic Numbers

Let $a = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$ be a doubly infinite sequence of natural numbers with $a_i \geq 2$ for all $i \in \mathbb{Z}$. Let the sequence a be arbitrary, but fixed.

We use Hewitt and Ross [4, Sects. 10 and 25] as our reference and define the a-adic numbers Ω as the group of sequences

$$\left\{ x = (x_i) \in \prod_{i=-\infty}^{\infty} \{0, 1, \ldots, a_i - 1\} : x_i = 0 \text{ for } i < j \text{ for some } j \in \mathbb{Z} \right\}$$

under addition with carry, that is, the sequences have a first nonzero entry and addition is defined inductively. Its topology is generated by the subgroups $\{\Lambda_j : j \in \mathbb{Z}\}$, where

$$\Lambda_j = \{x \in \Omega : x_i = 0 \text{ for } i < j\}.$$

This turns Ω into a totally disconnected, locally compact Hausdorff abelian group. The group Δ of a-adic integers is defined as $\Delta = \Lambda_0$. It is a compact, open subgroup, and a maximal compact ring in Ω with product given by multiplication with carry. On the other hand, Ω itself is not a ring in general (see (11.4) in Sect. 11.5).

Define the a-adic rationals N as the additive subgroup of $\mathbb{Q}$ given by

$$N = \left\{ \frac{j}{a_{-1} \cdots a_{-k}} : j \in \mathbb{Z}, k \geq 1 \right\}.$$

In fact, all noncyclic additive subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$ are of this form (see Lemma 2 below). There is an injective homomorphism

$$\iota : N \hookrightarrow \Omega$$

determined by

$$\left(\iota\Big(\frac{1}{a_{-1} \cdots a_{-k}} \Big) \right)_{-j} = \delta_{jk}.$$

Moreover, $\iota(N)$ is the dense subgroup of Ω comprising the sequences with only finitely many nonzero entries. This map restricts to an injective ring homomorphism denoted by the same symbol

$$\iota : \mathbb{Z} \hookrightarrow \Delta$$

with dense range. Henceforth, we will suppress the ι and identify N and $\mathbb{Z}$ with their image in Ω and Δ, respectively.

Now let $\mathscr{U}$ be the family of all subgroups of N of the form $\frac{m}{n}\mathbb{Z}$, where m and n are natural numbers such that m divides $a_0 \cdots a_j$ for some $j \geq 0$ and n divides $a_{-1} \cdots a_{-k}$ for some $k \geq 1$. Then $\mathscr{U}$

1. is downward directed, that is, for all $U, V \in \mathscr{U}$ there exists $W \in \mathscr{U}$ such that $W \subset U \cap V$,
2. is separating, that is,

$$\bigcap_{U \in \mathscr{U}} U = \{e\},$$

3. has finite quotients, that is, $|U/V| < \infty$ whenever $U, V \in \mathscr{U}$ and $V \subset U$,

and the same is also true for

$$\mathscr{V} = \{U \cap \mathbb{Z} : U \in \mathscr{U}\}.$$

In fact, both $\mathscr{U}$ and $\mathscr{V}$ are closed under intersections, since

$$\frac{m}{n}\mathbb{Z} \cap \frac{m'}{n'}\mathbb{Z} = \frac{\operatorname{lcm}(m, m')}{\gcd(n, n')}\mathbb{Z}. \tag{11.1}$$

It is a consequence of (1)–(3) above that the collection of subgroups $\mathscr{U}$ induces a locally compact Hausdorff topology on N. Denote the Hausdorff completion of N with respect to this topology by $\overline{N}$. Then

$$\overline{N} \cong \varprojlim_{U \in \mathscr{U}} N/U.$$

Next, let $U_0 = \mathbb{Z}$ and for $j \geq 1$ define $U_j = a_0 \cdots a_{j-1}\mathbb{Z}$ and set

$$\mathscr{W} = \{U_j : j \geq 0\} \subset \mathscr{V} \subset \mathscr{U}.$$

Note that $\mathscr{W}$ is also separating and closed under intersections. The closure of U_j in Ω is Λ_j, so

$$\Omega/\Lambda_j \cong N/U_j \quad \text{and} \quad \Delta/\Lambda_j \cong \mathbb{Z}/U_j \quad \text{for all } j \geq 0.$$

Next, let

$$\tau_j : \Omega \to N/U_j$$

denote the quotient map for $j \geq 0$, and identify $\tau_j(x)$ with the truncated sequence $x^{(j-1)}$, where $x^{(j)}$ is defined for all $j \in \mathbb{Z}$ by

$$(x^{(j)})_i = \begin{cases} x_i & \text{for } i \leq j, \\ 0 & \text{for } i > j. \end{cases}$$

We find it convenient to use the standard construction of the inverse limit of the system $\{N/U_j, \pmod{a_j}\}$:

$$\varprojlim_{j\geq 0} N/U_j = \left\{ x = (x_i) \in \prod_{i=0}^{\infty} N/U_i : x_i = x_{i+1} \pmod{a_i} \right\},$$

and then the product $\tau : \Omega \to \varprojlim_{j\geq 0} N/U_j$ of the truncation maps τ_j, given by

$$\tau(x) = (\tau_0(x), \tau_1(x), \tau_2(x), \dots) = (x^{(-1)}, x^{(0)}, x^{(1)}, \dots),$$

is an isomorphism.

Furthermore, we note that $\mathscr{W}$ is cofinal in $\mathscr{U}$. Indeed, for all $U = \frac{m}{n}\mathbb{Z} \in \mathscr{U}$, if we choose $j \geq 0$ such that m divides $a_0 \cdots a_j$ then we have $\mathscr{W} \ni U_{j+1} \subset U$.

Therefore,

$$\Omega \cong \varprojlim_{j\geq 0} N/U_j \cong \varprojlim_{U\in\mathscr{U}} N/U \cong \overline{N},$$

and similarly

$$\Delta \cong \varprojlim_{j\geq 0} \mathbb{Z}/U_j \cong \varprojlim_{V\in\mathscr{V}} \mathbb{Z}/V \cong \overline{\mathbb{Z}}.$$

In particular, Δ is a profinite group. In fact, every profinite group coming from a completion of $\mathbb{Z}$ occurs this way (see also Lemma 2).

The following is a consequence of (11.1) and should serve as motivation for our $\mathscr{U}$.

Lemma 1 ([6, Lemma 1.1]). *Every open subgroup of Ω is of the form*

$$\overline{\bigcup_{U\in\mathscr{C}} U}$$

for some increasing chain $\mathscr{C}$ in $\mathscr{U}$. In particular, every compact open subgroup of Ω is of the form $\overline{U}$ for some $U \in \mathscr{U}$.

Whenever any confusion is possible, we write Ω_a, Δ_a, N_a, etc. for the structures associated with the sequence a. If b is another sequence such that $\mathscr{U}_a = \mathscr{U}_b$, we write $a \sim b$. In this case also $N_a = N_b$. It is not hard to verify that $a \sim b$ if and only if there is an isomorphism $\Omega_a \to \Omega_b$ restricting to an isomorphism $\Delta_a \to \Delta_b$. The groups Ω_a and Ω_b can nevertheless be isomorphic even if $a \not\sim b$ (see Example 3 below). In this regard, we have the following result, which is a consequence of Proposition 2.

Theorem 1 ([6, Corollary 5.4]). *We have that $\Omega_a \cong \Omega_b$ if and only if there exists a $(\mathscr{U}_a, \mathscr{U}_b)$-continuous isomorphism $N_a \to N_b$.*

Example 1. Let p be a prime and assume $a = (\dots, p, p, p, \dots)$. Then $\Omega \cong \mathbb{Q}_p$ and $\Delta \cong \mathbb{Z}_p$, i.e. the usual p-adic numbers and p-adic integers.

Example 2. Let $a = (\dots, 4, 3, 2, 3, 4, \dots)$, i.e. $a_i = a_{-i} = i + 2$ for $i \geq 0$. Then $\Omega \cong \mathscr{A}_f$ and $\Delta \cong \hat{\mathbb{Z}}$, because every prime occurs infinitely often among both the positive and the negative tail of the sequence a (see the paragraph after Lemma 2).

Example 3. Let $a_i = 2$ for $i \neq 0$ and $a_0 = 3$, so that

$$N = \mathbb{Z}[\tfrac{1}{2}] \quad \text{and} \quad \mathscr{U} = \{2^i\mathbb{Z}, 2^i 3\mathbb{Z} : i \in \mathbb{Z}\}.$$

Then Ω contains torsion elements. Indeed, let

$$x = (\dots, 0, 1, 1, 0, 1, 0, 1, \dots), \qquad \text{so that} \quad 2x = (\dots, 0, 2, 0, 1, 0, 1, 0, \dots),$$

where the first nonzero entry is x_0. Then $3x = 0$ and $\{0, x, 2x\}$ forms a subgroup of Ω isomorphic with $\mathbb{Z}/3\mathbb{Z}$. Hence $\Omega \not\cong \mathbb{Q}_2$ since $\mathbb{Q}_2$ is a field.

Furthermore, let b be given by $b_i = a_{i+1}$, that is, $b_i = 2$ for $i \neq -1$ and $b_{-1} = 3$. Then

$$N_b = \tfrac{1}{3}\mathbb{Z}[\tfrac{1}{2}] \quad \text{and} \quad \mathscr{U}_b = \{2^i\mathbb{Z}, 2^i \tfrac{1}{3}\mathbb{Z} : i \in \mathbb{Z}\}.$$

We have $\Omega_a \cong \Omega_b$, but $a \not\sim b$ since $\Delta_a \not\cong \Delta_b$. Note also that the equation $3x = 1$ has no solution in Ω_a, but two solutions in Ω_b, and these are

$$\tfrac{1}{3} \in N_b \text{ and } y = (\dots, 0, 1, 1, 0, 1, 0, 1, \dots), \text{ where the first nonzero entry is } y_0.$$

11.3 The *a*-adic Algebras

We now want to define a multiplicative action on Ω, of some suitable subset of N, that is compatible with the natural multiplicative action of $\mathbb{Z}$ on Ω. Let S consist of all $s \in \mathbb{Q}_+^\times$ such that the map $\mathscr{U} \to \mathscr{U}$ given by $U \mapsto sU$ is well-defined and bijective. Clearly, the map $U \mapsto sU$ is injective if it is well-defined and it is surjective if the map $U \mapsto s^{-1}U$ is well-defined. Define a subset P of the prime numbers by

$$P = \{p \text{ prime} : p \text{ divides } a_k \text{ for infinitely many } k < 0 \text{ and infinitely many } k \geq 0\}.$$

It is not hard to see that S coincides with the subgroup $\langle P \rangle$ of $\mathbb{Q}_+^\times$ generated by P. Moreover, S is the largest subgroup of $\mathbb{Q}_+^\times$ that acts continuously on N. Indeed, the action is well-defined since all $q \in N$ belongs to some $U \in \mathscr{U}$. If $q + U$ is a basic open set in N, then its inverse image under multiplication by s, $s^{-1}(q+U) = s^{-1}q + s^{-1}U$, is also open in N as $s^{-1}U \in \mathscr{U}$. By letting S be discrete, it follows that the action is continuous.

We will not always be interested in the action of the whole group S on N, but rather a subgroup of S. So henceforth, let H denote any subgroup of S. Furthermore, let G be the semidirect product of N by H, i.e. $G = N \rtimes H$ where H acts on N by multiplication. This means that there is a well-defined $ax + b$-action of G on N given by

$$(r, h) \cdot q = r + hq \quad \text{for } q, r \in N \text{ and } h \in H.$$

This action is continuous with respect to $\mathscr{U}$, and can therefore be extended to an action of G on Ω, by uniform continuity.

Proposition 1 ([6, Proposition 2.4]). *Assume $P \neq \varnothing$ and let H be a nontrivial subgroup of S. Then the action of $G = N \rtimes H$ on Ω is minimal, locally contractive and topologically free.*

Definition 1. Suppose $P \neq \varnothing$. If H is a nontrivial subgroup of S, we define the C^*-algebra $\overline{\mathscr{Q}} = \overline{\mathscr{Q}}(a, H)$ by

$$\overline{\mathscr{Q}} = C_0(\Omega) \rtimes_{\alpha^{\text{aff}}} G,$$

where

$$\alpha^{\text{aff}}{}_{(n,h)}(f)(x) = f(h^{-1} \cdot (x - n)).$$

Remark 1. The bar-notation on $\overline{\mathscr{Q}}$ is used so that it agrees with the notation for stabilized Cuntz-Li algebras in [1] and [9].

Theorem 2 ([6, Corollary 2.8]). *The C^*-algebra $\overline{\mathscr{Q}}$ is simple and purely infinite. Moreover, $\overline{\mathscr{Q}}$ is a nonunital Kirchberg algebra in the UCT class.*

Example 4. If $a = (\ldots, 2, 2, 2, \ldots)$ and $H = S = \langle 2 \rangle$, then $\overline{\mathscr{Q}}$ is the algebra $\overline{\mathscr{Q}}_2$ of Larsen and Li [9]. More generally, if p is a prime, $a = (\ldots, p, p, p, \ldots)$ and $H = S = \langle p \rangle$, we are in the setting of Example 1 and get algebras similar to $\overline{\mathscr{Q}}_2$.

If $a = (\ldots, 4, 3, 2, 3, 4, \ldots)$ and $H = S = \mathbb{Q}_+^\times$, then we are in the setting of Example 2. In this case $\overline{\mathscr{Q}}$ is the algebra $\overline{\mathscr{Q}}_\mathbb{N}$ of Cuntz [1].

Both these algebras are special cases of the most well-behaved situation, namely where $H = S$ and $a_i \in H$ for all $i \in \mathbb{Z}$. The algebras arising this way are completely determined by the set (finite or infinite) of primes P, and are precisely the kind of algebras that fit into the framework of [5]. The cases described above are the two extremes, where P consists of either one single prime or all primes.

If $a \sim b$, then $S_a = S_b$ and $\overline{\mathscr{Q}}(a, H) = \overline{\mathscr{Q}}(b, H)$ for all $H \subset S_a = S_b$. Suppose $\Omega_a \cong \Omega_b$. Then $S_a = S_b$ as well, and for all $H \subset S_a = S_b$, we have that $\overline{\mathscr{Q}}(a, H) \cong \overline{\mathscr{Q}}(b, H)$. Indeed, by Theorem 1 there exists an isomorphism $\varphi : \Omega_a \to \Omega_b$ restricting to an isomorphism $N_a \to N_b$. Therefore, the map

$$\varphi_* : C_c(N_a \rtimes H, C_0(\Omega_a)) \to C_c(N_b \rtimes H, C_0(\Omega_b))$$

given by

$$\varphi_*(f)(n, h)(x) = f(\varphi^{-1}(n), h)(\varphi^{-1}(x))$$

determines an isomorphism $\overline{\mathscr{Q}}(a, H) \cong \overline{\mathscr{Q}}(b, H)$.

Example 5. Let a and b be the sequences from Example 3. Then $\overline{\mathscr{Q}}(a, H) \cong \overline{\mathscr{Q}}(b, H)$ for all $H \subset S_a = S_b = \langle 2 \rangle$.

Example 6. If $a = (\ldots, 2, 2, 2, \ldots)$ and $b = (\ldots, 4, 4, 4, \ldots)$, then $a \sim b$. Hence, for all nontrivial $H \subset S = \langle 2 \rangle$ we have $\overline{\mathscr{Q}}(a, H) = \overline{\mathscr{Q}}(b, H)$. However, if $H = \langle 4 \rangle$, then $\overline{\mathscr{Q}}(a, S) \ncong \overline{\mathscr{Q}}(a, H)$, as remarked after Question 1.

In light of this example, it could also be interesting to investigate the $ax+b$-action on Ω of other subgroups G' of $N \rtimes S$. It follows from the proof of Proposition 1 that the action of G' on Ω is minimal, locally contractive and topologically free if and only if $G' = M \rtimes H$, where $M \subset N$ is dense in Ω and $H \subset S$ is nontrivial.

Moreover, it can be shown that a proper subgroup M of N is dense in Ω if and only if $M = qN$ for some $q \geq 2$ such that q and a_i are relatively prime for all $i \in \mathbb{Z}$. This property is also invariant under isomorphisms, i.e. if $\Omega_a \cong \Omega_b$ and $q \geq 2$, then qN_a is dense in Ω_a if and only if qN_b is dense in Ω_b (see Sect. 11.5). However, if M is such a subgroup of N that is dense in Ω and $H \subset S$, then

$$C_0(\Omega) \rtimes_{\alpha^{\text{aff}}} (N \rtimes H) \cong C_0(\Omega) \rtimes_{\alpha^{\text{aff}}} (M \rtimes H). \tag{11.2}$$

The reason for the isomorphism (11.2) is the following. If

$$Q = \{p \text{ prime} : p \text{ does not divide any } a_i\},$$

then multiplication by a prime p is an automorphism of Ω if and only if $p \in P \cup Q$. Indeed, if $p \in Q$, then $p\overline{U} = \overline{pU} = \overline{U}$ for all $U \in \mathscr{U}$. Thus, $\frac{1}{p} \in \Omega$ when $p \in Q$ and it is possible to embed the subgroup

$$N_Q = \{\tfrac{n}{q} : n \in N, q \in \langle Q \rangle\} \subset \mathbb{Q}$$

in Ω, where $\langle Q \rangle$ denotes the multiplicative subgroup of $\mathbb{Q}_+^\times$ generated by Q.

We complete this discussion by considering the $ax+b$-action on Ω of potentially larger groups than $N \rtimes S$. The largest subgroup of $\mathbb{Q} \rtimes \mathbb{Q}_+^\times$ that can act on Ω through an $ax+b$-action is $N_Q \rtimes \langle P \cup Q \rangle$. However, the only groups $N \subset M \subset N_Q$ that give rise to the duality theorems in the next section are of the form $M = \frac{1}{q}N$ for $q \in \langle Q \rangle$ (see Remark 2). Moreover, S is the largest subgroup of $\langle P \cup Q \rangle$ that acts on M, and of course, (11.2) also holds for all $H \subset S$ in this case.

Finally, we remark that one may also involve the roots of unity of $\mathbb{Q}^\times$ in the multiplicative action, that is, replace H with $\{\pm h : h \in H\} = \{\pm 1\} \times H$ as in [3]. The associated algebras will then be of the form $\overline{\mathscr{Q}} \rtimes \mathbb{Z}/2\mathbb{Z}$. However, we restrict to the action of the torsion-free part of $\mathbb{Q}^\times$ in this paper.

11.4 The *a*-Adic Duality Theorem

For any a, let a^* be the sequence given by $a_i^* = a_{-i}$. In particular, $(a^*)^* = a$. We now fix a and write Ω and Ω^* for the a-adic and a^*-adic numbers, respectively.

Let $x \in \Omega$ and $y \in \Omega^*$ and for $j \in \mathbb{N}$ put

$$z_j = e^{2\pi i x^{(j)} y^{(j)}/a_0},$$

where the sequences $x^{(j)}$ and $y^{(j)}$ are treated as their corresponding rational numbers in N. It can be checked that z_j is eventually constant. We now define the pairing $\Omega \times \Omega^* \to \mathbb{T}$ by

$$\langle x, y \rangle_\Omega = \lim_{j \to \infty} e^{2\pi i x^{(j)} y^{(j)}/a_0}.$$

The pairing is a continuous homomorphism in each variable separately and gives an isomorphism $\Omega^* \to \hat{\Omega}$. Indeed, this map coincides with the one in [4, 25.1].

The injection $\iota : N \to \mathbb{R} \times \Omega$ given by $q \mapsto (q, q)$ has discrete range, and N may be considered as a closed subgroup of $\mathbb{R} \times \Omega$. Similarly, N^* may be considered as a closed subgroup of $\mathbb{R} \times \Omega^*$.

Remark 2. Subgroups M of $\mathbb{Q}$ such that $N \subset M \subset N_Q$ also embed densely into Ω. For example, $\mathbb{Q}$ itself can be embedded densely into $\mathbb{Q}_p$ for all primes p. On the other hand, it is not hard to see that the image of the diagonal map $\mathbb{Q} \to \mathbb{R} \times \mathbb{Q}_p$ is not closed in this case. More generally, a subgroup M of $\mathbb{Q}$ embeds densely into Ω

such that the image of the diagonal map $M \to \mathbb{R} \times \Omega$ is closed if and only if M is of the form $\frac{1}{q}N$ for $q \in \langle Q \rangle$.

By applying the facts about the pairing of Ω and Ω^* stated above, the pairing of $\mathbb{R} \times \Omega$ and $\mathbb{R} \times \Omega^*$ given by

$$\langle (u, x), (v, y) \rangle = e^{-2\pi i u v / a_0} \lim_{j \to \infty} e^{2\pi i x^{(j)} y^{(j)} / a_0} = \langle u, v \rangle_{\mathbb{R}} \langle x, y \rangle_{\Omega}$$

defines an isomorphism $\mathbb{R} \times \Omega^* \to \widehat{\mathbb{R} \times \Omega}$ that restricts to an isomorphism $\iota(N^*) \to \iota(N)^{\perp}$. Thus, we get the following theorem.

Theorem 3 ([6, Theorem 3.3]). *We have that*

$$(\mathbb{R} \times \Omega^*)/N^* \cong \widehat{\mathbb{R} \times \Omega}/N^{\perp} \cong \hat{N},$$

where the isomorphism $\omega : (\mathbb{R} \times \Omega^*)/N^* \to \hat{N}$ *is given by*

$$\omega\big((v, y) + N^*\big)(q) = \langle (q, q), (v, y) \rangle \quad \textit{for } (v, y) \in \mathbb{R} \times \Omega^* \textit{ and } q \in N.$$

Remark 3. In general, note that $P^* = P$ so $S^* = S$. Hence, every subgroup $H \subset S$ acting on N and Ω also acts on N^* and Ω^*. In particular $\overline{\mathcal{Q}}(a, H)$ is well-defined if and only if $\overline{\mathcal{Q}}(a^*, H)$ is.

Theorem 4 ([6, Theorem 4.1]). *Assume that* $P \neq \varnothing$ *and that* H *is a nontrivial subgroup of* S. *Set* $G = N \rtimes H$ *and* $G^* = N^* \rtimes H$. *Then there is a Morita equivalence*

$$C_0(\Omega) \rtimes_{\alpha^{\mathrm{aff}}} G \sim_M C_0(\mathbb{R}) \rtimes_{\alpha^{\mathrm{aff}}} G^*,$$

where the action on each side is the $ax + b$*-action.*

We give an outline of the proof that involves a few classical results in the theory of crossed products. To simplify the notation in the proof, we switch the stars, and seek a Morita equivalence between $C_0(\Omega^*) \rtimes_{\alpha^{\mathrm{aff}}} G^*$ and $C_0(\mathbb{R}) \rtimes_{\alpha^{\mathrm{aff}}} G$. Our strategy is to first find a Morita equivalence

$$C_0(T/\Omega) \rtimes_{\mathrm{lt}} N \sim_M C_0(N \backslash T) \rtimes_{\mathrm{rt}} \Omega,$$

where $T = \mathbb{R} \times \Omega$, that is equivariant for actions α and β of H on $C_0(T/\Omega) \rtimes_{\mathrm{lt}} N$ and $C_0(N \backslash T) \rtimes_{\mathrm{rt}} \Omega$, respectively, and then find isomorphisms

$$\big(C_0(T/\Omega) \rtimes_{\mathrm{lt}} N\big) \rtimes_{\alpha} H \cong C_0(\mathbb{R}) \rtimes_{\alpha^{\mathrm{aff}}} G,$$
$$\big(C_0(N \backslash T) \rtimes_{\mathrm{rt}} \Omega\big) \rtimes_{\beta} H \cong C_0(\Omega^*) \rtimes_{\alpha^{\mathrm{aff}}} G^*.$$

Recall that N and Ω sit inside T as closed subgroups. All the groups are abelian, and therefore, by “Green’s symmetric imprimitivity theorem” (for example [12, Corollary 4.11]) we get a Morita equivalence

$$C_0(T/\Omega) \rtimes_{\mathrm{lt}} N \sim_M C_0(N\backslash T) \rtimes_{\mathrm{rt}} \Omega \tag{11.3}$$

via an imprimitivity bimodule X that is a completion of $C_c(T)$. Here N acts on the left of T/Ω by $n \cdot ((t, y) \cdot \Omega) = (n + t, n + y) \cdot \Omega$ and Ω acts on the right of $N\backslash T$ by $(N \cdot (t, y)) \cdot x = N \cdot (t, y + x)$, and the induced actions on C_0-functions are given by

$$\mathrm{lt}_n(f)(p \cdot \Omega) = f\big(-n \cdot (p \cdot \Omega)\big)$$
$$\mathrm{rt}_x(g)(N \cdot p) = g\big((N \cdot p) \cdot x\big)$$

for $n \in N$, $f \in C_0(T/\Omega)$, $p \in T$, $x \in \Omega$, and $g \in C_0(N\backslash T)$.

Moreover, H acts by multiplication on N, hence on Ω, and also on $\mathbb{R}$. Thus H acts diagonally on $T = \mathbb{R} \times \Omega$ by $h \cdot (t, x) = (ht, h \cdot x)$.

One can then show that the Morita equivalence (11.3) is equivariant for the actions α, β, and γ of H on $C_c(N, C_0(T/\Omega))$, $C_c(\Omega, C_0(T\backslash N)$, and $C_c(T)$ given by

$$\begin{aligned}
\alpha_h(f)(n)\big((t, y) \cdot \Omega\big) &= f(hn)\big((ht, h \cdot y) \cdot \Omega\big),\\
\beta_h(g)(x)\big(N \cdot (t, y)\big) &= \delta(h)g(h \cdot x)\big(N \cdot (ht, h \cdot y)\big),\\
\gamma_h(\xi)(t, y) &= \delta(h)^{\frac{1}{2}}\xi(ht, h \cdot y),
\end{aligned}$$

where δ is the modular function for the multiplicative action of H on Ω.

The next step is now to show that

$$\begin{aligned}
\big(C_0(T/\Omega) \rtimes_{\mathrm{lt}} N\big) \rtimes_\alpha H &\cong \big(C_0(\mathbb{R}) \rtimes_{\mathrm{lt}} N\big) \rtimes_{\alpha'} H\\
&\cong \big(C_0(\mathbb{R}) \rtimes_{\mathrm{lt}} N\big) \rtimes_{\alpha''} H\\
&\cong C_0(\mathbb{R}) \rtimes_{\alpha^{\mathrm{aff}}} (N \rtimes H).
\end{aligned}$$

The first isomorphism is induced from $T/\Omega \xrightarrow{\cong} \mathbb{R}$ and then we get the correct α'' by composing α' with the automorphism $h \mapsto h^{-1}$ of H. The last isomorphism is a consequence of a result regarding decomposition of iterated crossed products (see [12, Corollary 3.11]).

The other part requires more work, and the aim is to get through the steps

$$\begin{aligned}
\big(C_0(N\backslash T) \rtimes_{\mathrm{rt}} \Omega\big) \rtimes_\beta H &\cong \big(C_0(\widehat{N^*}) \rtimes_{\mathrm{rt}} \Omega\big) \rtimes_{\beta'} H\\
&\cong \big(C_0(\Omega^*) \rtimes_{\mathrm{rt}} N^*\big) \rtimes_{\beta''} H\\
&\cong C_0(\Omega^*) \rtimes_{\alpha^{\mathrm{aff}}} (N^* \rtimes H).
\end{aligned}$$

Here, the first isomorphism is induced from the ω in Theorem 3. For the second isomorphism, we need the "subgroup of dual group theorem" (see [6, Appendix A]). Finally, the third isomorphism is, similarly as above, a consequence of the "iterated crossed products decomposition".

Remark 4. The C^*-algebras $C_0(\Omega) \rtimes_{\alpha^{\mathrm{aff}}} G$ and $C_0(\mathbb{R}) \rtimes_{\alpha^{\mathrm{aff}}} G^*$ will actually be isomorphic by Zhang's dichotomy: a separable, simple, purely infinite C^*-algebra is either unital or stable.

If a is defined by $a_i = 2$ for all i and $H = \langle 2 \rangle$, then this result coincides with [9, Theorem 7.5], and if a is the sequence described in Example 2, it coincides with [1, Theorem 6.5].

11.5 Invariants and Isomorphism Results

Let $\mathbb{P}$ be the set of prime numbers. A supernatural number is a function

$$\lambda : \mathbb{P} \to \mathbb{N} \cup \{0, \infty\}$$

such that $\sum_{p \in \mathbb{P}} \lambda(p) = \infty$. Denote the set of supernatural numbers by $\mathbb{S}$. It may sometimes be useful to consider a supernatural number as an infinite formal product

$$\lambda = 2^{\lambda(2)} 3^{\lambda(3)} 5^{\lambda(5)} 7^{\lambda(7)} \cdots .$$

If λ is a supernatural number and p is a prime, let $p\lambda$ denote the supernatural number given by $(p\lambda)(p) = \lambda(p) + 1$ (with the convention that $\infty + 1 = \infty$) and $(p\lambda)(q) = \lambda(q)$ if $p \neq q$. The definition of $p\lambda$ extends to all natural numbers p by prime factorization.

Let λ and ϱ be two supernatural numbers associated with the sequence a in the following way:

$$\lambda(p) = \sup\{i : p^i \text{ divides } a_0 \ldots a_j \text{ for some } j \geq 0\} \in \mathbb{N} \cup \{0, \infty\}$$

$$\varrho(p) = \sup\{i : p^i \text{ divides } a_{-1} \ldots a_{-k} \text{ for some } k \geq 1\} \in \mathbb{N} \cup \{0, \infty\}$$

Lemma 2. *Let a and b be two sequences. The following hold:*

1. $\Delta_a \cong \Delta_b$ *if and only if* $\lambda_a = \lambda_b$.
2. $N_a = N_b$ *if and only if* $\varrho_a = \varrho_b$.
3. $\mathscr{U}_a = \mathscr{U}_b$ *if and only if both* $\lambda_a = \lambda_b$ *and* $\varrho_a = \varrho_b$.

Indeed, from [4, Theorem 25.16] we have

$$\Delta \cong \prod_{p \in \lambda^{-1}(\infty)} \mathbb{Z}_p \times \prod_{p \in \lambda^{-1}(\mathbb{N})} \mathbb{Z}/p^{\lambda(p)}\mathbb{Z}$$

and hence (1) holds. It is not difficult to see that condition (2) and (3) also hold.

This means that there is a one-to-one correspondence between supernatural numbers and noncyclic subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$, and also between supernatural numbers and Hausdorff completions of $\mathbb{Z}$.

Condition (3) is equivalent to $a \sim b$, and more generally, the following result clarifies when Ω_a and Ω_b are isomorphic.

Proposition 2 ([6, Proposition 5.2]). *Let a and b be two sequences. Then $\Omega_a \cong \Omega_b$ if and only if there are natural numbers p and q such that*

$$(\ldots, a_{-2}, qa_{-1}, pa_0, a_1, \ldots) \sim (\ldots, b_{-2}, pb_{-1}, qb_0, b_1, \ldots).$$

That is, $\Omega_a \cong \Omega_b$ if and only if there are $p, q \in \mathbb{N}$ such that $p\lambda_a = q\lambda_b$ and $q\varrho_a = p\varrho_b$.

Hence, if $\Omega_a \cong \Omega_b$, then $N_a \cong N_b$, $P_a = P_b$ so $S_a = S_b$ and $Q_a = Q_b$.

Corollary 1 ([6, Proposition 5.7]). *The group of a-adic numbers Ω is self-dual if and only if there are natural numbers p and q such that $p\lambda = q\varrho$.*

For two pairs of supernatural numbers (λ_1, ϱ_1) and (λ_2, ϱ_2), we write $(\lambda_1, \varrho_1) \sim (\lambda_2, \varrho_2)$ if there exist natural numbers p and q such that $p\lambda_1 = q\lambda_2$ and $q\varrho_1 = p\varrho_2$. Then the set of isomorphism classes of a-adic numbers coincides with $\mathbb{S}\times\mathbb{S}/\sim$ and the self-dual ones coincide with the diagonal, i.e. are of the form $[(\lambda, \lambda)]$.

Set $\mathscr{U}_P = \{\frac{m}{n}\mathbb{Z} \in \mathscr{U} : n \in S\} = \{U \in \mathscr{U} : U \subset \mathbb{Z}[\{\frac{1}{p} : p \in P\}]\}$. Then the open subgroup

$$R = \overline{\mathbb{Z}[\{\tfrac{1}{p} : p \in P\}]} = \overline{\bigcup_{U \in \mathscr{U}_P} U}$$

in Ω is the maximal open (and closed) ring contained in Ω. In particular, the a-adic numbers Ω can be given the structure of a topological (commutative) ring with multiplication inherited from $N \subset \mathbb{Q}$ if and only if [11, E. Herman, 12.3.35]

$$N = \bigcup_{h \in S} h\mathbb{Z} \quad \left(= \mathbb{Z}\left[\left\{\tfrac{1}{p} : p \in P\right\}\right]\right) \tag{11.4}$$

i.e. if and only if $\Omega = R$.

Moreover, by Theorem 4 and Remark 4, it should be clear that $\overline{\mathscr{Q}}(a, H) \cong \overline{\mathscr{Q}}(b, K)$ if $N_a^* \cong N_b^*$ and $H = K$, although the isomorphism is in general not canonical. Hence, for every sequence a, there is a sequence b such that Ω_b is a ring and $\overline{\mathscr{Q}}(a, H) \cong \overline{\mathscr{Q}}(b, H)$, since one can always pick b so that $\Omega_b = R_a$. (Warning: $\overline{\mathscr{Q}}(b, H)$ is still not a ring algebra in the sense of [10].) If both Ω_a and Ω_b are rings, then $\Omega_a \cong \Omega_b$ as topological rings if and only if $a \sim b$.

Example 7. Let a and b be the sequences of Examples 3 and 5, and let $H = \langle 2 \rangle$. Then $\overline{\mathscr{Q}}(a, H) \cong \overline{\mathscr{Q}}(b, H)$ and these algebras are also isomorphic to $\overline{\mathscr{Q}}_2$, but the isomorphisms are not canonical.

Question 1. Given two sequences a and b and subgroups $H \subset S_a$ and $K \subset S_b$. When is $\overline{\mathscr{Q}}(a, H) \ncong \overline{\mathscr{Q}}(b, K)$?

To enlighten the question, consider the following situation. Let $a = (n, n, n, \dots)$ and $H = \langle n \rangle$, and note that $H = S$ if and only if n is prime. Then $\mathscr{Q}(a, H) = C(\Delta) \rtimes_{\alpha^{\mathrm{aff}}} G_{\mathbb{Z}}$ (see next section) is the $\mathscr{O}(E_{n,1})$ of [7, Example A.6]. Thus

$$(K_0(\mathscr{Q}(a, H)), [1], K_1(\mathscr{Q}(a, H))) \cong (\mathbb{Z} \oplus \mathbb{Z}/(n-1)\mathbb{Z}, (0, 1), \mathbb{Z}).$$

Moreover, since all $\mathscr{Q}(a, H)$ are Kirchberg algebras in the UCT class, they are classifiable by K-theory.

In future work we hope to be able to compute the K-theory of $\overline{\mathscr{Q}}(a, H)$ using the following strategy. Since $C_0(\Omega) \rtimes N$ is stably isomorphic to the Bunce-Deddens algebra $C(\Delta) \rtimes \mathbb{Z}$, its K-theory is well-known, in fact

$$(K_0(C(\Delta) \rtimes \mathbb{Z}), [1], K_1(C(\Delta) \rtimes \mathbb{Z})) \cong (N^*, 1, \mathbb{Z}).$$

As H is a free abelian group, we can apply the Pimsner-Voiculescu six-term exact sequence by adding the action of one generator of H at a time. For this to work out, we will need to apply Theorem 4 and use homotopy arguments to compute the action of H on the K-groups (see also [2, Remark 3.16]).

11.6 The "Unstabilized" a-Adic Algebras

Fix a sequence a and a nontrivial subgroup $H \subset S$ and set $\overline{\mathscr{Q}} = \overline{\mathscr{Q}}(a, H)$. Let H_+ be the semigroup $H \cap \mathbb{N}^\times$ and for each $U \in \mathscr{U}$, let G_U denote the semigroup $U \rtimes H_+$ with multiplication inherited from G. Moreover, for $n \in N$ let p_{n+U} be the projection in $\overline{\mathscr{Q}}$ corresponding to the projection $\chi_{n+\overline{U}}$ in $C_0(\Omega)$.

Assume $U, V \in \mathscr{U}$ and $V \subset U$, so $U = r\mathbb{Z}$ for some r and set $k = |U/V|$. Then

$$U = \bigsqcup_{j=0}^{k-1} jr + V \quad \text{so that} \quad p_U = \sum_{j=0}^{k-1} p_{jr+V}. \tag{11.5}$$

Proposition 3. *The following hold:*

1. *p_U is a full projection in $\overline{\mathscr{Q}}$.*
2. *The full corner $p_U \overline{\mathscr{Q}} p_U$ is isomorphic to the semigroup crossed product*

$$C(\overline{U}) \rtimes_{\alpha^{\mathrm{aff}}} G_U, \quad {\alpha^{\mathrm{aff}}}_{(n,h)} f(x) = \begin{cases} f(h^{-1} \cdot (x - n)) & \text{if } x \in n + \overline{hU}, \\ 0 & \text{else.} \end{cases}$$

Proof. Note first that if $p_V \in \overline{\mathcal{Q}} p_U \overline{\mathcal{Q}}$ for some $V \in \mathcal{U}$, then $g p_{n+hV} \in \overline{\mathcal{Q}} p_U \overline{\mathcal{Q}}$ for all g and $(n,h) \in G$. Therefore, it suffices to check that $p_V \in \overline{\mathcal{Q}} p_U \overline{\mathcal{Q}}$ for all $V \in \mathcal{U}$.

Pick $V = r\mathbb{Z} \in \mathcal{U}$ and choose $W \in \mathcal{U}$ with $W \subset U \cap V$ (for example $W = U \cap V$). Let $k = |V/W|$, then by (11.5)

$$\begin{aligned} p_V = \sum_{j=0}^{k-1} p_{jr+W} &= \sum_{j=0}^{k-1} (jr,1) p_W (-jr,1) \\ &\in \operatorname{span}\{gpg' : g, g' \in G, p \text{ projection in } \overline{\mathcal{Q}} \text{ with } p \le p_U\} \\ &\subset \operatorname{span} \overline{\mathcal{Q}} p_U \overline{\mathcal{Q}} \end{aligned}$$

as $p p_U p = p$ if $p \le p_U$.

For the second part, we just remark that for $f \in C_0(\Omega)$ and $(n,h) \in G$,

$$p_U f(n,h) p_U = p_{U \cap (n+hU)} f(n,h) = f_{|\overline{U} \cap (n+h\overline{U})}(n,h),$$

which is nonzero only if $n \in U \cup hU$. □

The minimal automorphic dilation of $C(\overline{U}) \rtimes_{\alpha^{\text{aff}}} G_U$ does not necessarily take us back to $\overline{\mathcal{Q}}$. In fact, it gives

$$C_0(\overline{H_+^{-1}U}) \rtimes_{\alpha^{\text{aff}}} (H_+^{-1}U \rtimes H)$$

where

$$H_+^{-1}U = \{\tfrac{n}{h} : n \in U, h \in H_+\} = \bigcup_{h \in H_+} h^{-1}U = \bigcup_{h \in H} hU = \{hn : n \in U, h \in H\}.$$

Therefore, one gets $\overline{\mathcal{Q}}$ back precisely when $N = H_+^{-1}U$. For example, if $U = \mathbb{Z}$ one gets $\overline{\mathcal{Q}}$ back in the settings of Larsen and Li and also Cuntz, since $H = S$ and (11.4) holds in these cases.

In general, however, we get that

$$\overline{\mathcal{Q}} \sim_M C_0(\overline{H_+^{-1}U}) \rtimes_{\alpha^{\text{aff}}} (H_+^{-1}U \rtimes H)$$

which due to Remark 4 means that these are noncanonically isomorphic as well.

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