Incorporating Voice Permutations into the Theory of Neo-Riemannian Groups and Lewinian Duality

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Abstract. A familiar problem in neo-Riemannian theory is that the P, L, and R operations defined as contextual inversions on pitch-class segments do not produce parsimonious voice leading. We incorporate permutations into T/I–PLR duality to resolve this issue and simultaneously broaden the applicability of this duality. More precisely, we construct the dual group to the permutation group acting on n-tuples with distinct entries, and prove that the dual group to permutations adjoined with a group G of invertible affine maps $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$ is the internal direct product of the dual to permutations and the dual to G. Musical examples include Liszt, R. W. Venezia, S. 201 and Schoenberg, String Quartet Number 1, Opus 7. We also prove that the Fiore–Noll construction of the dual group in the finite case works, and clarify the relationship of permutations with the RICH transformation.

Keywords: dual group, duality, Lewin, neo-Riemannian group, PLR, permutation, RICH, retrograde inversion enchaining.

1 Introduction: Neo-Riemannian Groups and Voice Leading Parsimony

The motivation for this paper was a working session of the three authors on the article [1] back in September 2011. While we were discussing the task of

J. Yust, J. Wild, and J.A. Burgoyne (Eds.): MCM 2013, LNAI 7937, pp. 100–114, 2013.
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properly defining neo-Riemannian operations for triadic pitch-class segments, i.e. for triads with a concrete ordering of the three voices, we realized that there are in fact several alternatives, all of which are music-theoretically attractive.

(i) The mathematically straightforward definition presupposes that the classical neo-Riemannian operations are already defined for "Oettingen/ Riemann"-root position forms in accordance with the classical dualistic voice leading model. By *conjugation with voice permutations* one may then extend the known definitions to all triadic pitch-class segments. In this perspective the transformations are applied with respect to the characteristic tone roles within triads, regardless of their location in the voices.

(ii) A prominent competitor of the dualistic voice leading model is motivated by the parsimonious voice-leadings between P, L, and R-related triads. One may alternatively define these three transformations on ordered triads simply through the condition that they literally mimic the parsimonious voice leading. This definition is compliant with conjugation by voice permutations, and therefore it is closely related to the definition (i). In fact the definitions (i) and (ii) differ from each other by voice permutations.

(iii) A conceptual alternative are the *contextual inversions*, where two voices are exchanged and where the third voice is mirrored at the center between the two others. The three contextual inversions are always individually compatible with the dualist neo-Riemannian transformations P, L, and R on the underlying pitch class sets, but their roles are mixed up among the various orderings of the three voices. In this perspective the transformations are applied to the voices, regardless of the distribution of the tones of the triad over the voices

(iv) Also with respect to definition (iii) it may be attractive to concatenate it with particular voice permutations. For example, an adaption of Lewin's RI-chains from the transformational study of 12-tone series to that of triads, offers an attractive analytical potential, see [1].

Apart from the desire to balance these alternatives with respect to their musical interpretation, there is also an immediate theoretical challenge: Is it possible lift to the duality between the T/I and S/W groups to suitable groups acting on triadic pitch-class segments? Robert Peck's investigation [2] into generalized commuting groups lays a good basis for such a project. The main focus of the present paper is a combination of the cases (i) and (ii) in terms of a simply transitive group action, where the Lewinian duality still holds. Section 4 briefly demonstrates that the definitions in (iii) and (iv) lead to a quite different situation. The RICH transform is of order 24 and has powers with fixed points. This opens an interesting working domain with cross connections to other investigations, such as the joint paper [3] by Julian Hook and Jack Douthett.

With the combination of (i) and (ii) we wish to touch a sore spot at the very heart of neo-Riemannian theory. It concerns the remarkable solidarity between voice-leading parsimony on the one hand and triadic transformations on the other. How do the two aspects fit together, precisely? The study of voice leading requires the localization of chord tones within an ensemble of voices. The study of triadic transformations, and in particular the investigation of the duality between the T/I- and PLR-groups, seems either to require an abstraction of the triads from their concrete construction from tones or it leads to a dualistic voice leading behavior, which is in conflict with the principle of voice-leading parsimony (see Fig. 1).

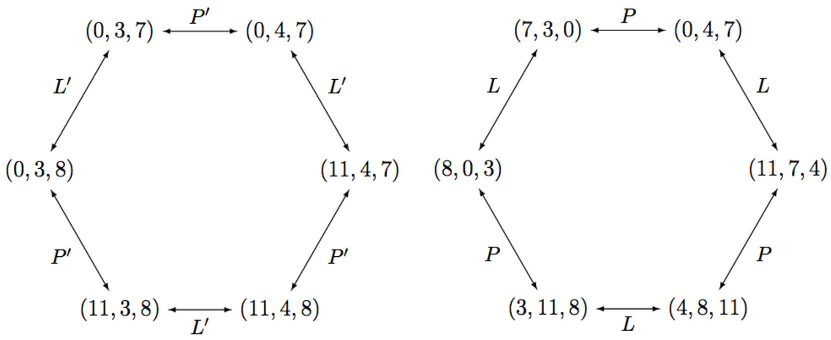


Fig. 1. Two "proto-transformational" networks representing different voice leadings for a hexatonic cycle (left: parsimonious voice leading, right: dualistic voice leading)

In the light of the impact of dialectics upon the development of music theoretical ideas in the writings of Moritz Hauptmann and Hugo Riemann it is remarkable that Nora Engebretsen portrays in [4] a main line of conceptual development in the second half of the 19th century within the garb of a dialectical triad:

(i) Hauptmann's focus on common-tone retention in (diatonic) triadic progressions (Thesis)
(ii) Von Oettingen's focus on the dualism between major and minor triads (Antithesis)
(iii) Riemann's attempts to integrate both view points in a chromatic context (Synthesis)

Despite of its historical attractiveness this dialectical metaphor remains euphemistic, until a successful neo-Riemannian synthesis of voice leading and Lewinian transformational theory has been achieved. The present paper takes a step in this direction and, in particular, attributes precise transformational meanings to the arrow labels in the networks of Fig. 1.

2 Construction of the Dual Group in the Finite Case

In preparation for our treatment of permutations in neo-Riemannian groups, we briefly recall the well-known duality between the T/I-group and PLR-group, and present a new proof of the Fiore–Noll construction of the dual group in the finite case. The basic objects upon which the T/I-group and PLR-group act are pitch-class segments with three constituents. Recall that a *pitch-class segment* is an ordered subset of $\mathbb{Z}_{12}$, or more generally $\mathbb{Z}_m$. We use parentheses[1] to denote a pitch-class segment as an n-tuple $(x_1, \ldots, x_n)$. The sequential order of the pitch classes may, for example, relate to the temporal order of notes in a score, or to the distribution of pitches in different voices in a certain registral order. In connection with recent studies to voice leading, such as [5], one may wish to include voice permutations into the investigation of contextual transformations in non-trivial ways, as we do in Section 3.

2.1 Lewinian Duality between the T/I-Group and PLR-Group

The T/I-group consists of the 24 bijections $T_j, I_j \colon \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ with $T_j(k) = k+j$ and $I_j(k) = -k + j$, where $j \in \mathbb{Z}_{12}$. Via its componentwise action on 3-tuples, this dihedral group acts simply transitively on the set S of all the transposed and inverted forms of the root position C-major 3-tuple $(0, 4, 7)$. Note that the minor triads in S are in Oettingen/Riemann root position, e.g., a-minor is $(4, 0, 9)$. Like any group action, this action corresponds to a homomorphism from the group to the symmetric group on the set upon which it acts, namely a homomorphism $\lambda \colon T/I \to \mathrm{Sym}(S)$. The *symmetric group on* S, denoted $\mathrm{Sym}(S)$, consists of all bijections $S \to S$, while the group homomorphism $\lambda \colon T/I \to \mathrm{Sym}(S)$ is $g \mapsto (s \mapsto gs)$. Since the action is simply transitive, the homomorphism λ is an *embedding* (=injective group homomorphism), and we consider the T/I-group as a subgroup of $\mathrm{Sym}(S)$ via this embedding λ.

The other key character in this by now classical story is the neo-Riemannian PLR-group, which is the subgroup of $\mathrm{Sym}(S)$ generated by the bijections $P, L, R \colon S \to S$. These transformations, respectively called *parallel*, *leading-tone exchange*, and *relative*, are given on major chords in root position and minor chords in open second inversion by[2]

$$\begin{aligned} P(y_1, y_2, y_3) &:= I_{y_1+y_3}(y_1, y_2, y_3) &&= (y_3, -y_2 + y_1 + y_3, y_1) \\ L(y_1, y_2, y_3) &:= I_{y_2+y_3}(y_1, y_2, y_3) &&= (-y_1 + y_2 + y_3, y_3, y_2) \\ R(y_1, y_2, y_3) &:= I_{y_1+y_2}(y_1, y_2, y_3) &&= (y_2, y_1, -y_3 + y_1 + y_2). \end{aligned} \tag{1}$$

For instance,

$$P(0, 4, 7) = (7, 3, 0), \quad L(0, 4, 7) = (11, 7, 4), \quad R(0, 4, 7) = (4, 0, 9)$$

and

$$P(7, 3, 0) = (0, 4, 7), \quad L(11, 7, 4) = (0, 4, 7), \quad R(4, 0, 9) = (0, 4, 7).$$

[1] We do not use the traditional musical notation $\langle x_1, \ldots, x_n \rangle$ for pitch-class segments because it clashes with the mathematical notation for the subgroup generated by $x_1, \ldots, x_n$, which we will also need on occasion.

[2] Our usage of ordered n-tuples allows these root-free, mathematical formulations of musical operations. See also [6, Footnote 20].

These operations are sometimes called *contextual inversions* because the inversion in the definition depends on the input.[3] Note that input and output always have two pitch classes in common, though *their positions are reversed*. In Example 3.4, we will see how to use permutations to define variants $P', L', R' \colon\ S' \to S'$ which *retain the positions of the common tones*, and generate a dihedral group of order 24 we call the *Cohn group*. We will also see in Section 3 how permutations allow us to mathematically extend P, L, and R to triads in first inversion or second inversion. Note that the right-hand formulas in (1) correspond differently to P, L, and R when the input chords are not in Oettingen/Riemann root position. For instance on a first inversion C-major chord, the first right-hand formula yields R rather than P, namely $I_{4+0}(4,7,0) = (0,9,4)$ is a permuted a-minor chord.

The main properties of the PLR-group were observed by David Lewin: it acts simply transitively on S, and it consists precisely of those elements of $\mathrm{Sym}(S)$ which commute with the T/I-group. For instance $RT_7(0,4,7) = (11,7,4) = T_7R(0,4,7)$.

Definition 2.1 (Dual Groups in the Sense of Lewin, see page 253 of [8]). Let $\mathrm{Sym}(S)$ be the symmetric group on the set S. Two subgroups G and H of the symmetric group $\mathrm{Sym}(S)$ are *dual in the sense of Lewin* if their natural actions on S are simply transitive and each is the centralizer of the other, that is,

$$C_{\mathrm{Sym}(S)}(G) = H \quad \text{and} \quad C_{\mathrm{Sym}(S)}(H) = G.$$

For an exposition of T/I–PLR duality, see Crans–Fiore–Satyendra [9], and for its extension to length n pitch-class segments in $\mathbb{Z}_m$ satisfying a tritone condition, see Fiore–Satyendra [6]. Childs and Gollin both developed the relevant dihedral groups in the special case of the pitch-class segment $X = (0,4,7,10)$, i.e., for the set class of dominant seventh chords and half-diminished seventh chords (see [10] and [11]).

2.2 Construction of the Dual Group in the Finite Case After Fiore–Noll [12]

The dual group for a simply transitive action of a finite group always exists. This was pointed out in [12], though not proved there, so we present a proof now. Let S be a general finite set, as opposed to the specific set of pitch-class segments in Section 2.1.

Proposition 2.2 (Construction 2.3 of Fiore–Noll [12], Finite Case). *Suppose G is a finite group which acts simply transitively on a finite set S. Fix an element $s_0 \in S$ and consider the two embeddings*

$$\lambda\colon\ G \longrightarrow \mathrm{Sym}(S) \qquad\qquad \rho\colon\ G \longrightarrow \mathrm{Sym}(S)$$
$$g \longmapsto \Big(s \mapsto gs\Big) \qquad\qquad g \longmapsto \Big(hs_0 \mapsto hg^{-1}s_0\Big).$$

[3] For an approach to contextual inversions in terms of indexing functions and a choice of canonical representative, see Kochavi [7].

Then the images $\lambda(G)$ and $\rho(G)$ are dual groups in $\mathrm{Sym}(S)$. *The injection ρ depends on the choice of s_0, but the image $\rho(G)$ does not.*

Proof. If $j, k \in G$, then $\lambda(j)$ and $\rho(k)$ commute because

$$\lambda(j)\rho(k)(hs_0) = j(hk^{-1})s_0 = (jh)k^{-1}s_0 = \rho(k)\lambda(j)(hs_0)$$

for any $h \in G$. Simple transitivity of both $\lambda(G)$ and $\rho(G)$ is fairly clear. Thus, so far we have $\rho(G) \subseteq C_{\mathrm{Sym}(S)}\big(\lambda(G)\big)$ and $|\rho(G)| = |G| = |S|$. Recall from the Orbit-Stabilizer Theorem that a finite group acting on a finite set acts simply if and only if it acts transitively, and in this case the cardinality of the group is the same as the cardinality of the set.

We next claim that the centralizer $C_{\mathrm{Sym}(S)}(\lambda(G))$ acts simply on S. If $c, c' \in C_{\mathrm{Sym}(S)}\big(\lambda(G)\big)$ and $cs_1 = c's_1$ for some single $s_1 \in S$, then $chs_1 = c'hs_1$ for all $h \in G$, which means c and c' are equal as functions on S. Thus this centralizer acts simply and $|C_{\mathrm{Sym}(S)}\big(\lambda(G)\big)| = |S|$, and consequently the inclusion $\rho(G) \subseteq C_{\mathrm{Sym}(S)}\big(\lambda(G)\big)$ from above is actually an equality. A similar counting argument shows that $\lambda(G) = C_{\mathrm{Sym}(S)}\big(\rho(G)\big)$. □

We will use this construction several times in the following sections to find the dual group for the symmetric group Σ_n acting on n-tuples and to include permutations into T/I–PLR duality.

Two immediate corollaries to Proposition 2.2 are as follows.

Corollary 2.3. *If S is a finite set, and a subgroup G of* $\mathrm{Sym}(S)$ *acts simply transitively on S, then the centralizer of G in* $\mathrm{Sym}(S)$ *also acts simply transitively.*

Corollary 2.4. *If S is a finite set, and a subgroup G of* $\mathrm{Sym}(S)$ *acts simply transitively on S, then the centralizer of G is isomorphic to G.*

In connection with Corollary 2.4, we remark that Peck [2] has studied the structure of centralizers in non-simply transitive situations, with numerous examples in music theory.

3 Permutation Actions

We now turn to the main theorem of this paper, Theorem 3.2. Let Σ_3 denote the symmetric group on $\{1, 2, 3\}$. Its coordinate-permuting action on 3-tuples in $\mathbb{Z}_{12}$ commutes with transposition and inversion. When we consider all transpositions and inversions of all reorderings of $(0, 4, 7)$, the T/I-group and symmetric group Σ_3 form an internal direct product denoted $\Sigma_3(T/I)$. Recall that a group H is an *internal direct product of subgroups K and L* if K and L commute, $K \cap L = \{e\}$, and every element of H can be written as $k\ell$ for some $k \in K$ and $\ell \in L$. As a consequence, in such a direct product, each decomposition $h = k\ell$ with $k \in K$ and $\ell \in L$ is unique. See [13, Chapter 2, Section 9] for background and an equivalent definition. Another reference is [14].

Theorem 3.2 essentially says in the case $X = (0, 4, 7)$ that the dual group to $\Sigma_3(T/I)$ is the internal direct product of the dual group to Σ_3 and the PLR-group, where P, L, and R are defined on a reordering $\sigma(0, 4, 7)$ by $\sigma P \sigma^{-1}$, $\sigma L \sigma^{-1}$, and $\sigma R \sigma^{-1}$. Theorem 3.2 is formulated more generally for n-tuples and any group of invertible affine maps instead of just for 3-tuples and T/I. The method for constructing dual groups is always Proposition 2.2. For the case $n = 3$, we indicate in Section 3.2 specific generators of the group $\rho(\Sigma_3)$, which is the dual group to the standard permutation action recalled in Section 3.1.

Of course, everything in this section works just as well for general $\mathbb{Z}_m$ beyond $\mathbb{Z}_{12}$, but we work with $\mathbb{Z}_{12}$ for concreteness.

Permutations have been considered in music theory before, for instance by Mazzola [15, I.2].

3.1 The Standard Permutation Action on n-Tuples and Its Dual Group

Let Σ_n denote the symmetric group on $\{1, \ldots, n\}$. Consider the standard left action of the symmetric group Σ_n on all n-tuples with $\mathbb{Z}_{12}$ entries,

$$\Sigma_n \times (\mathbb{Z}_{12})^n \longrightarrow (\mathbb{Z}_{12})^n$$

defined[4] by $\sigma(y_1, \ldots, y_n) := \big(y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(n)}\big)$. Let $X = (x_1, \ldots, x_n)$ denote a particular pitch-class segment with n *distinct* pitch classes, and consider its orbit

$$\Sigma_n X = \big\{ \big(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\big) \mid \sigma \in \Sigma_n \big\} .$$

This orbit $\Sigma_n X$ consists of all the reorderings of X, or all the *permutations* of X. The restricted left action on the orbit

$$\Sigma_n \times (\Sigma_n X) \longrightarrow \Sigma_n X$$

is clearly simply transitive, as the components of X are distinct. Consequently, we have an associated embedding

$$\lambda\colon\ \Sigma_n \longrightarrow \mathrm{Sym}\,(\Sigma_n X)\ ,$$

the image of which we call $\lambda(\Sigma_n)$.

As in Construction 2.3 of [12], recalled in Section 2.2 above, we now construct the dual group $\rho(\Sigma_n)$ for $\lambda(\Sigma_n)$ in the symmetric group $\mathrm{Sym}(\Sigma_n X)$. The fixed element s_0 is X. By simple transitivity, any element of $\Sigma_n X$ can be written as νX for some unique $\nu \in \Sigma_n$. On the set of X-permutations $\Sigma_n X$, we define in terms of the standard left action a second left action

$$\Sigma_n \times (\Sigma_n X) \overset{\cdot}{\longrightarrow} \Sigma_n X$$

[4] The inverses must be included because the first inclination to define $\sigma(y_1, \ldots, y_n) = \big(y_{\sigma(1)}, \ldots, y_{\sigma(n)}\big)$ is not a left action, since we would have $(\sigma\sigma')Y = \sigma'(\sigma Y)$.

by $\sigma \cdot (\nu X) := (\nu\sigma^{-1})X$. One can quickly check from the axioms for the standard left action that

$$(\sigma\tau) \cdot (\nu X) = \sigma \cdot (\tau \cdot (\nu X))$$

$$e \cdot (\nu X) = \nu X$$

and that this second left action is simply transitive. This second left action gives us a second embedding

$$\rho\colon\ \Sigma_n \longrightarrow \mathrm{Sym}\,(\Sigma_n X)\ ,$$

the image of which we call $\rho(\Sigma_n)$. The groups $\lambda(\Sigma_n)$ and $\rho(\Sigma_n)$ commute because

$$\sigma(\nu\tau^{-1})X = (\sigma\nu)\tau^{-1}X$$

for all $\sigma, \nu, \tau \in \Sigma_n$. We have sketched a proof of the following proposition (and by example also some details of Proposition 2.2).

Proposition 3.1. *The order* $n!$ *groups* $\lambda(\Sigma_n)$ *and* $\rho(\Sigma_n)$ *are dual subgroups of* $\mathrm{Sym}(\Sigma_n X)$*, which has order* $(n!)!$.

3.2 The Standard Permutation Action and Its Dual Group in the Case $n = 3$

The standard permutation action $\lambda(\Sigma_n)$ and its dual group $\rho(\Sigma_n)$ in the case $n = 3$ are of particular interest for our present paper. We now work out explicitly this special case of Section 3.1. Let $X = (x_1, x_2, x_3)$ denote the pitch-class segment of a trichord. The symmetric group on 3 letters in cycle notation[5] is

$$\Sigma_3 = \{\mathrm{id}, (123), (132), (23), (13), (12)\}.$$

We obtain

$$\Sigma_3 X = \left\{ \begin{array}{r} X = (x_1, x_2, x_3) \\ (123)X = (x_3, x_1, x_2) \\ (132)X = (x_2, x_3, x_1) \\ (23)X = (x_1, x_3, x_2) \\ (13)X = (x_3, x_2, x_1) \\ (12)X = (x_2, x_1, x_3) \end{array} \right\}.$$

[5] We follow the standard cycle notation *without commas*. For example, the cycle (123) is the map $1 \mapsto 2 \mapsto 3 \mapsto 1$. Cycles are composed as ordinary functions are. For example, $(123)(23) = (12)$ because we do (23) first and then (123).

As generators for the actions $\lambda(\Sigma_3)$ and $\rho(\Sigma_3)$ we may choose $\lambda(123)$, $\lambda(23)$ and $\rho(123)$, $\rho(23)$, respectively, which have the following explicit form.

$$\lambda(123):\begin{array}{rcr} X & \mapsto & (123)X \\ (123)X & \mapsto & (132)X \\ (132)X & \mapsto & X \\ (23)X & \mapsto & (12)X \\ (13)X & \mapsto & (23)X \\ (12)X & \mapsto & (13)X \end{array}, \quad \rho(123):\begin{array}{rcr} X & \mapsto & (132)X \\ (123)X & \mapsto & X \\ (132)X & \mapsto & (123)X \\ (23)X & \mapsto & (12)X \\ (13)X & \mapsto & (23)X \\ (12)X & \mapsto & (13)X \end{array}$$

$$\lambda(23):\begin{array}{rcr} X & \mapsto & (23)X \\ (123)X & \mapsto & (13)X \\ (132)X & \mapsto & (12)X \\ (23)X & \mapsto & X \\ (13)X & \mapsto & (123)X \\ (12)X & \mapsto & (132)X \end{array}, \quad \rho(23):\begin{array}{rcr} X & \mapsto & (23)X \\ (123)X & \mapsto & (12)X \\ (132)X & \mapsto & (13)X \\ (23)X & \mapsto & X \\ (13)X & \mapsto & (132)X \\ (12)X & \mapsto & (123)X \end{array}$$

We may write these generators more compactly in cycle notation.

$$\begin{aligned} \lambda(123) &= \Big(X \quad (123)X \quad (132)X\Big)\Big((23)X \quad (12)X \quad (13)X\Big) \\ \lambda(23) &= \Big(X \quad (23)X\Big)\Big((123)X \quad (13)X\Big)\Big((132)X \quad (12)X\Big) \\ \rho(123) &= \Big(X \quad (132)X \quad (123)X\Big)\Big((23)X \quad (12)X \quad (13)X\Big) \\ \rho(23) &= \Big(X \quad (23)X\Big)\Big((123)X \quad (12)X\Big)\Big((132)X \quad (13)X\Big) \end{aligned}$$

3.3 Affine Groups with Permutations and Their Duals

Now consider a pitch-class segment $X = (x_1, \ldots, x_n)$ with n distinct pitch classes x_k and a group $G \subseteq \mathrm{Aff}^*(\mathbb{Z}_{12})$ of invertible affine transformations. We let G act componentwise on n-tuples, and consider the orbit GX of X. We assume, for the sake of simplicity, that the underlying set of X is not symmetric with respect to any element of G. That is, we require $f\{x_1, \ldots, x_n\} \neq \{x_1, \ldots, x_n\}$ for all $f \in G$. This condition guarantees that G acts simply transitively on GX and that none of the affine transformations $f \in G$, except the identity transformation, acts on X merely like a permutation. We now extend the action of Σ_n on $\Sigma_n X$ to an action on $\Sigma_n GX$.

The group $\Sigma_n G = G\Sigma_n$ is the subgroup of $\mathrm{Sym}\Big((\mathbb{Z}_{12})^n\Big)$ generated by Σ_n and G. Since Σ_n and $\mathrm{Aff}^*(\mathbb{Z}_{12})$ commute, the group $\Sigma_n G$ is an internal direct product of Σ_n and G, and every element of $\Sigma_n G$ can be written uniquely as σg with $\sigma \in \Sigma_n$ and $g \in G$.

The orbit of X under $\Sigma_n G$ decomposes as a disjoint union, which gives a principle Σ_n-bundle over the pitch-class sets underlying the G-orbit of X.

$$G\Sigma_n X = \coprod_{g \in G} \Sigma_n(gX) \longrightarrow G\{x_1, \ldots, x_n\}$$

As detailed in Section 3.1, on each set $\Sigma_n(gX)$ in the disjoint union we have dual groups $\lambda^g(\Sigma_n)$ and $\rho^g(\Sigma_n)$ in $\mathrm{Sym}(\Sigma_n(gX))$. In light of the disjoint union decomposition, these actions fit together to give commuting, but not dual,[6] subgroups of $\mathrm{Sym}(G\Sigma_n X)$. However, these commuting groups form part of dual groups as in the following theorem.

Theorem 3.2 (Affine Groups with Permutations and their Duals). *Let $X = (x_1, \ldots, x_n)$ be a pitch-class segment in $\mathbb{Z}_{12}$ with n distinct pitch-classes $x_1, \ldots, x_n$. Let G be a subgroup of the group $\mathrm{Aff}^*(\mathbb{Z}_{12})$ of all invertible affine transformations $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$, which acts componentwise on all n-tuples in $\mathbb{Z}_{12}$. Suppose $f\{x_1, \ldots, x_n\} \neq \{x_1, \ldots, x_n\}$ for all $f \in G$. Let Σ_n denote the symmetric group on n letters, which acts on n-tuples as in Section 3.1. As above, let $\lambda(\Sigma_n G)$ be the subgroup of $\mathrm{Sym}(\Sigma_n G X)$ determined by the action of the internal direct product $\Sigma_n G$ on the orbit $\Sigma_n G X$. Recall that the dual group $\rho(\Sigma_n G)$ has elements $\rho(\nu h)$ for $\nu \in \Sigma_n$ and $h \in G$ where*

$$\rho(\nu h)\sigma g X := \sigma g \big(\nu h\big)^{-1} X$$

for all $\sigma \in \Sigma_n$ and $g \in G$.

Then:

- *(i) The restriction of the subgroup $\rho(\Sigma_n)$ to $\Sigma_n X$ is the dual group for $\lambda(\Sigma_n)$ in $\mathrm{Sym}(\Sigma_n X)$, and similarly the restriction of the subgroup $\rho(G)$ to GX is the dual group for $\lambda(G)$ in $\mathrm{Sym}(GX)$.*
- *(ii) The subgroups $\rho(\Sigma_n)$ and $\rho(G)$ of $\mathrm{Sym}(\Sigma_n G X)$ commute, that is $\rho(\nu)\rho(h) = \rho(h)\rho(\nu)$ for all $\nu \in \Sigma_n$ and $h \in G$.*
- *(iii) The group $\rho(\Sigma_n G)$ is the internal direct product of $\rho(\Sigma_n)$ and $\rho(G)$, as defined in the introduction to Section 3.*
- *(iv) If $Y \in \sigma G X$ and $h \in G$, then $\rho(h)Y = \sigma\rho(h)\sigma^{-1}Y$.*

Proof. Statement (i) follows directly from the construction of the dual group in Section 2.2. Statements (ii) and (iii) follow from the analogous facts about Σ_n, G, and $\Sigma_n G$ because ρ is an embedding (and consequently an isomorphism onto its image). Alternatively, we may prove Statement (ii) as follows. For $\nu \in \Sigma_n$ and $h \in G$ we have

$$\begin{aligned}
\rho(\nu)\rho(h)\sigma g X &\overset{\text{def}}{=} \sigma g h^{-1}\nu^{-1} X \\
&= \sigma g \nu^{-1} h^{-1} X \\
&\overset{\text{def}}{=} \rho(h)\rho(\nu)\sigma g X,
\end{aligned}$$

where the unlabeled equality follows from the fact that ν^{-1} and h^{-1} commute because Σ_n and G commute as remarked above. Statement (iv) follows from the fact that $\rho(h)$ commutes with σ and σ^{-1} by duality. □

[6] These two groups cannot be dual, because they do not act simply transitively: their cardinalities are $n!$ while the set upon which they act has cardinality $|G| \cdot n!$.

Example 3.3 (Permutations with T/I and PLR Duality). If in Theorem 3.2 we take X to be $(0, 4, 7)$ and G to be the T/I-group, then we have the incorporation of permutations into T/I and PLR-duality. In particular, $\Sigma_3(T/I)(0, 4, 7)$ is the set of all possible orderings of major and minor triads, and $\rho(\Sigma_3(T/I))$ is the internal direct product of $\rho(\Sigma_3)$ and the extended PLR-group. By part (iv) any operation h of the PLR-group is extended to act on $Y = \sigma T_j(0, 4, 7)$ or $Y = \sigma I_j(0, 4, 7)$ by first "translating to Oettingen/ Riemann root position, then operating, and then "translating back", namely $hY := \sigma h \sigma^{-1} Y$. For instance,

$$R(7, 0, 4) = (123)R(321)(123)(0, 4, 7) = (123)(4, 0, 9) = (9, 4, 0).$$

Another way to justify this is that the extended R operation commutes with permutations, so

$$R(7, 0, 4) = R(123)(0, 4, 7) = (123)R(0, 4, 7) = (123)(4, 0, 9) = (9, 4, 0).$$

Thus, Theorem 3.2, in combination with the Sub Dual Group Theorem of Fiore–Noll [12, Theorem 3.1], gives a theoretical justification for the constructions at the end of [1, Section 5] concerning an analysis of Schoenberg, String Quartet Number 1, Opus 7.

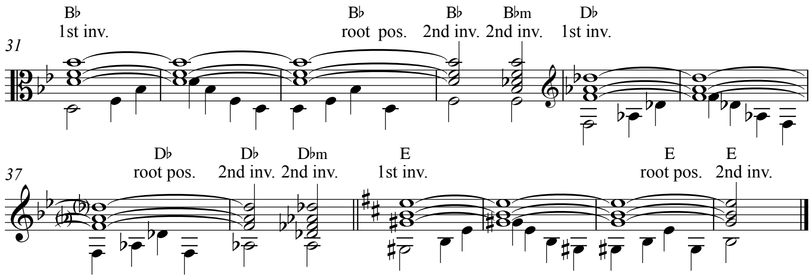
Example 3.4 (Cohn Group). We may now define new versions of P, L, and R which retain the positions of common tones in the ordering of any triad. Let $P' := \rho(13)P$, $L' := \rho(23)L$, and $R' := \rho(12)R$. Then we have for instance

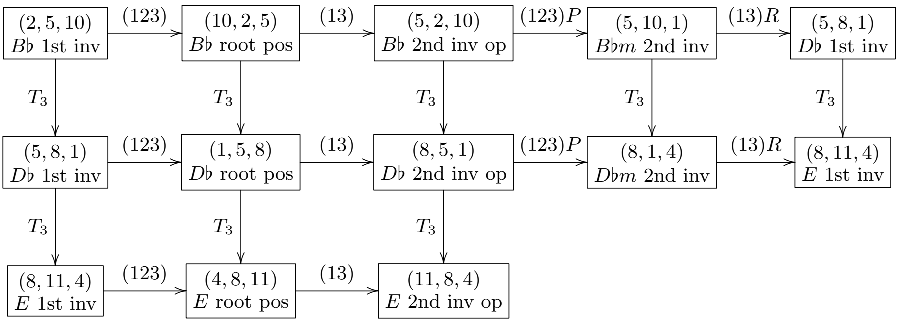
$$L'(4, 7, 0) = \rho(23)L(4, 7, 0) = L\rho(23)(321)(0, 4, 7) =$$

$$L(13)(0, 4, 7) = (13)L(0, 4, 7) = (13)(11, 7, 4) = (4, 7, 11)$$

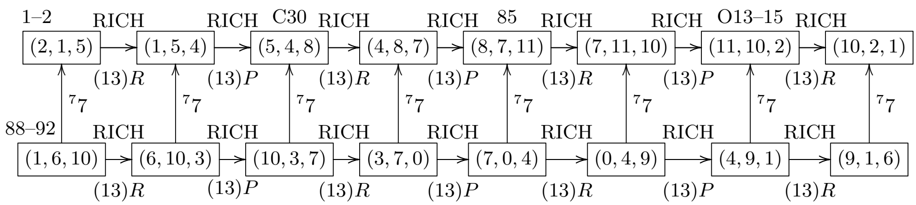
by the table for $\rho(23)$ in Section 3.2. See Fig. 1 for further examples. We call the group generated by P', L', R' the *Cohn group*. It is dihedral of order 24 (the relations can be checked directly using those of the PLR-group and the commutativity of $\rho(\Sigma_3)$ with the PLR-group).

Example 3.5 (Venezia). Below we have a rhythmic reduction of Liszt, R. W. Venezia, S. 201, measures 31–42. For our analysis we identify the first strong-beat bass arrival of $B\flat$ in measure 33 as a relatively well articulated root position chord, since in measures 31–32 the weak-beat instances of $B\flat$ in the bass do not overturn the impression of a first-inversion position. Subsequent root position chords in the analysis were chosen similarly. The transformations in each of the three phrases are permutations, P, and R operations, as pictured in the rows of the subsequent network. The vertical arrows of the network indicate that the three phrases are related by transposition by 3 semitones. All the squares commute by Theorem 3.2, since the four groups $\lambda(\Sigma_n)$, $\lambda(T/I)$, $\rho(\Sigma_n)$, and $\rho(T/I) = PLR$ commute.





Example 3.6 (Schoenberg, String Quartet in D Minor, Op. 7). One of the main motivations of the present paper was our discussion [1] of Schoenberg's String Quartet in D Minor. We excerpt below the first two rows of Figure 15 of that paper. The first row (pictured below) is a piece-wide narrative constructed from the opening motivic cell of measures 1–2. The second row (also pictured below) is the triadic melody from measures 88–92. The up arrows ${}^{7}7$ are the affine transformation $x \mapsto 7x + 7$. The horizontal arrows can be labelled as the composite of (13) with R or P, or as RICH (discussed in the next section). The notes for these two rows are in [1, Figures 1 and 3]. Taking G to be the 48-element affine group generated by T_1, I_0, and multiplication by 7, we see that all the squares commute by Theorem 3.2, since the four groups $\lambda(\Sigma_n)$, $\lambda(G)$, $\rho(\Sigma_n)$, and $\rho(G) \supseteq PLR$ commute.



4 Properties of Other Contextual Transformations on Pitch-Class Segments Not Contained in $\rho(\Sigma_3(T/I)) = \rho(\Sigma_3)PLR$

The remainder of this paper illustrates some properties of *contextual inversion enchaining transformations.* These are certain transformations on pitch-class segments not contained in the dual group $\rho(\Sigma_n(T/I))$. In particular, we will discuss the RICH transformation, which goes beyond the scope of simply transitive actions as well as beyond the orbifold construction via voice permutation.

Consider the situation and notation of Theorem 3.2, and for $1 \leq q, r \leq n$ consider the globally defined *contextual inversion*[7]

$$J^{q,r}(Y) := I_{y_q+y_r}Y. \tag{2}$$

Composites of contextual inversions with permutations yield instances of *contextual inversion enchaining transformations.* Within the symmetric group Σ_n, consider the order 2 cycle[8] $(r\ s)$. On pitch-class segments $(y_1, \ldots, y_n)$, the permutation $(r\ s)$ acts through voice exchange by mutually exchanging the pitch classes y_r and y_s at their respective positions in $(y_1, \ldots, y_n)$.

$$(r\ s)\colon\ (y_1, \ldots, y_r, \ldots, y_s, \ldots, y_n) \mapsto (y_1, \ldots, y_s, \ldots, y_r, \ldots, y_n)$$

Definition 4.1. Consider a pitch-class segment $X = (x_1, \ldots, x_n)$ and select three distinct indices $1 \leq q, r, s \leq n$. A *contextual inversion enchaining transformation* is any composite

$$(r\ s) \circ J^{q,r}\colon\ \Sigma_n(T/I)X \to \Sigma_n(T/I)X$$

of a contextual inversion $J^{q,r}$ and a voice exchange $(r\ s)$ sharing the common index r.

The effect of enchaining will be illustrated by example. For $n = 3$ the cycle $(1\ 3)$ behaves like a retrograde, which motivates Lewin's notation RICH in [8] for the transformation $(1\ 3) \circ J^{2,3}$, meaning *retrograde inversion enchaining.* If Y is a pitch-class segment, then RICH(Y) is that retrograde inversion of Y which has the first two notes y_2 and y_3, in that order. This transformation was used in our analysis of Schoenberg in [1]. See Straus [16] for some recent analyses using RICH transformations. See also Catanzaro [17] for a classification of the trichord *Tonnetz* spaces in the unordered case, and also Fiore–Satyendra [6] for the group theory of contextual inversions and an analysis of Hindemith, *Ludus Tonalis*, Fugue in E.

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7 As we remarked earlier, the formulas in equation (1) for P, L, and R are only valid for major triads in root position, or minor triads in the ordering $I_n(0, 4, 7)$. For other orderings of consonant triads, conjugation must be used, as in Example 3.3. Thus, $J^{1,3}$, $J^{2,3}$, and $J^{1,2}$ do not agree with the respective extended functions P, L, and R beyond the T/I-class of $(0, 4, 7)$, and the name "contextual inversion" for $J^{q,r}$ is not optimal.

8 Of course, an order 2 cycle is more commonly called a "transposition" in the mathematics literature, but we avoid using that term here because "transposition" already has other meanings in this article.

The explicit cycle notation of the RICH transformation on consonant triads is displayed in Table 1. More specifically, in Theorem 3.2, we take X to be $(0,4,7)$ and G to be the T/I-group, so that $\Sigma_3(T/I)(0,4,7)$ is the $144 = 6 \times 24$ possible orderings of major and minor triads, and $\rho(\Sigma_3(T/I))$ is the internal direct product of $\rho(\Sigma_3)$ and the PLR-group. The group $\rho(\Sigma_3(T/I))$ is also the subgroup of $\mathrm{Sym}(\Sigma_3(T/I))$ generated by $\rho(\Sigma_3)$ and the PLR-group. But RICH is not in the simply transitive group $\rho(\Sigma_3(T/I))$ as we now explain.

Table 1. Cycle decomposition of RICH action on all 144 permutations of the major and minor triads

Type	**Consonant Triad Cycles for** RICH
RL	(4, 7, 11) (7, 11, 2) (11, 2, 6) (2, 6, 9) (6, 9, 1) (9, 1, 4) (1, 4, 8) (4, 8, 11) (8, 11, 3) (11, 3, 6) (3, 6, 10) (6, 10, 1) (10, 1, 5) (1, 5, 8) (5, 8, 0) (8, 0, 3) (0, 3, 7) (3, 7, 10) (7, 10, 2) (10, 2, 5) (2, 5, 9) (5, 9, 0) (9, 0, 4) (0, 4, 7)
RL	(4, 0, 9) (0, 9, 5) (9, 5, 2) (5, 2, 10) (2, 10, 7) (10, 7, 3) (7, 3, 0) (3, 0, 8) (0, 8, 5) (8, 5, 1) (5, 1, 10) (1, 10, 6) (10, 6, 3) (6, 3, 11) (3, 11, 8) (11, 8, 4) (8, 4, 1) (4, 1, 9) (1, 9, 6) (9, 6, 2) (6, 2, 11) (2, 11, 7) (11, 7, 4) (7, 4, 0)
PR	(0, 7, 3) (7, 3, 10) (3, 10, 6) (10, 6, 1) (6, 1, 9) (1, 9, 4) (9, 4, 0) (4, 0, 7)
PR	(0, 4, 9) (4, 9, 1) (9, 1, 6) (1, 6, 10) (6, 10, 3) (10, 3, 7) (3, 7, 0) (7, 0, 4)
PR	(1, 8, 4) (8, 4, 11) (4, 11, 7) (11, 7, 2) (7, 2, 10) (2, 10, 5) (10, 5, 1) (5, 1, 8)
PR	(1, 5, 10) (5, 10, 2) (10, 2, 7) (2, 7, 11) (7, 11, 4) (11, 4, 8) (4, 8, 1) (8, 1, 5)
PR	(2, 9, 5) (9, 5, 0) (5, 0, 8) (0, 8, 3) (8, 3, 11) (3, 11, 6) (11, 6, 2) (6, 2, 9)
PR	(2, 6, 11) (6, 11, 3) (11, 3, 8) (3, 8, 0) (8, 0, 5) (0, 5, 9) (5, 9, 2) (9, 2, 6)
PL	(7, 4, 11) (4, 11, 8) (11, 8, 3) (8, 3, 0) (3, 0, 7) (0, 7, 4)
PL	(7, 0, 3) (0, 3, 8) (3, 8, 11) (8, 11, 4) (11, 4, 7) (4, 7, 0)
PL	(8, 5, 0) (5, 0, 9) (0, 9, 4) (9, 4, 1) (4, 1, 8) (1, 8, 5)
PL	(8, 1, 4) (1, 4, 9) (4, 9, 0) (9, 0, 5) (0, 5, 8) (5, 8, 1)
PL	(9, 6, 1) (6, 1, 10) (1, 10, 5) (10, 5, 2) (5, 2, 9) (2, 9, 6)
PL	(9, 2, 5) (2, 5, 10) (5, 10, 1) (10, 1, 6) (1, 6, 9) (6, 9, 2)
PL	(10, 7, 2) (7, 2, 11) (2, 11, 6) (11, 6, 3) (6, 3, 10) (3, 10, 7)
PL	(10, 3, 6) (3, 6, 11) (6, 11, 2) (11, 2, 7) (2, 7, 10) (7, 10, 3)

A close look at the cycle decomposition of RICH shows that there are cycles of length 24, behaving like RL-cycles, cycles of length 8, behaving like PR-cycles, and cycles of length 6, behaving like PL-cycles. Consequently the sixth and eighth powers RICH^6 and RICH^8 have fixed points, and RICH cannot be part of a simply transitive group action on all 144 ordered triads. In application to suitable subsets of $\Sigma_3(T/I)X$, e.g., to selected pitch-class segments in an octatonic cycle, the fixed-point effect disappears, and RICH can be part of a simply transitive group action on those.

For instance, each of first two PR-cycles in Table 1 gives rise to a simply transitive group action. These triadic pitch-class segments are over the octatonic scale $\{0,2,3,4,6,7,9,10\}$. The second PR-cycle is precisely the PR-cycle in measures 88–92 of Schoenberg, String Quartet Number 1, Opus 7 pictured in [1, Figures 1 and 2]. This octatonically restricted RICH-transformation involves two (and only two) Flip-Flop Cycles of length 8 in the sense of John Clough [18].

Analogous orbits can be obtained for pitch-class segments of jet and shark triads in [1]. The last PR-cycle in Table 1 contains the cello motive in measures 8–10, which is pictured in [1, Figures 13 and 14], and located in the octatonic scale $\{2, 3, 5, 6, 8, 9, 11, 0\}$. See also the Summary Network in [1, Figure 15].

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