Towards a Categorical Theory of Creativity for Music, Discourse, and Cognition

Moreno Andreatta[1], Andrée Ehresmann[2], René Guitart[3], and Guerino Mazzola[4]

[1] IRCAM/CNRS/UPMC
Moreno.Andreatta@ircam.fr
[2] Université de Picardie
andree.ehresmann@u-picardie.fr
[3] Université Paris 7 Denis Diderot
rene.guitart@orange.fr
[4] School of Music, University of Minnesota
mazzola@umn.edu

Abstract. This article presents a first attempt at establishing a category-theoretical model of creative processes. The model, which is applied to musical creativity, discourse theory, and cognition, suggests the relevance of the notion of "colimit" as a unifying construction in the three domains as well as the central role played by the Yoneda Lemma in the categorical formalization of creative processes.

1 Historical Introduction to a Formal Theory of Creativity

Although the notion of *creativity* seems to be incompatible with formal and mathematical approaches, there have historically been many attempts to grasp the creative process using computational models. The history of algorithmic music composition, from information theory to algebraic models, exemplifies approaches that describe the computational component of creative process. For example, the use of *entropy* and *redundancy* as parameters to describe stylistic properties of artistic expression was one of the fundamental hypotheses of information theory; a theory which, according to Shannon and Weaver, is "so general that one does not need to say what kinds of symbols are being considered whether written letters or words, or musical notes, or spoken words, or symphonic music, or pictures. The theory is deep enough so that the relationships it reveals indiscriminately apply to all these and to other forms of communication" [29].

The underlying hypothesis, which also guided AI paradigms, was to simulate creative behavior by means of computer programs. In Douglas Hofstadter's words, "the notions of analogy and fluidity are fundamental to explain how the human mind solves problems and to create computer programs that show intelligent behavior" [18]. Within different computer-aided models of creative process, music and musical creativity occupy a distinguished place. According to David

J. Yust, J. Wild, and J.A. Burgoyne (Eds.): MCM 2013, LNAI 7937, pp. 19–37, 2013.

Cope, creativity is "the initialization of connections between two or more multifaceted things, ideas, or phenomena hitherto not otherwise considered actively connected. [...] It does not depend exclusively on human inspiration, but can originate from other sources, such as machine programs. [It] should not be confused with novelty. [It] does not originate from a vacuum, but rather synthesizes the work of others, no matter how original the results may seem" [4].

Despite the increasing number of studies on computer-aided models of creativity, many questions about its formal and conceptual character as well as its relationships with cognitive processes remain open. Clearly formal models of creativity do not reduce to algorithmic and computational ones. In Margaret Boden's influential model (as discussed, for example, in [3]), creativity occurs as a result of three different types of mental process: combinatorial, exploratory, and transformational. Although combinatorial creativity refers to unfamiliar combinations of familiar ideas, exploratory and transformational creativity arise within structured concept spaces. In conclusion, "if researchers can define those [conceptual] spaces and specify ways of navigating and even transforming them it will be possible not only to map the contents of the mind but also to understand how it is possible to generate novel, surprising, and valuable ideas" [3].

Interestingly, music offers a variety of concept spaces, particularly once geometrical models and algebraic methods are used to characterize the structural property of these concept spaces, as initially suggested by Gärdenfors [10] and recently discussed by Acotto and Andreatta [1]. Among different approaches that try to combine computational models of creative processes and concept spaces, one has to mention the notion of "conceptual blending", introduced in an informal way by Fauconnier and Turner [8] and further extended via algebraic and categorical methods by Goguen [11]. As observed by Pereira from a AI-oriented perspective, "Conceptual Blending is an elaboration of other works related to creativity, namely Bisociation, Metaphor and Conceptual Combination. As such, it attracts the attention of computational creativity modelers and, regardless of how Fauconnier and Turner describe its processes and principles, it is unquestionable that there is some kind of blending happening in the creative mind" [26]. In Goguen's algebraic semiotic approach to conceptual blending, Peirce's tripartite sign model is combined with categorical formalism, so that a structural component is added to the computational character of creativity. As claimed by the author, "the category of sign systems with semiotic morphisms has some additional structure over that of a category: it is an ordered category, because of the orderings by quality of representation that can be put on its morphisms. This extra structure gives a richer framework for considering blends; I believe this approach captures what Fauconnier and Turner have called 'emergent' structure, without needing any other machinery" [11]. This approach has been recently applied to style modeling (see [12]), providing an alternative to AI-oriented unifying models of conceptual spaces [9]. Our research is deeply related to this structural account of concept spaces and creative processes, as we will show by firstly focusing on music and then trying to make evident possible connections with the problem of a categorical analysis of the sense of discourse as well as explaining the underlying cognitive model. It also provides a first attempt at reactivating a mathematically-oriented tradition in developmental psychology, as inaugurated by Halford and Wilson in the Eighties [17] and discussed recently in [27].

This article is organized as follows. In section 2 to section 4 we introduce some constructions from category theory by focusing, in particular, on the Yoneda Lemma and its role in the constitution of a generic model for creative processes. This model is applied to music in section 5, by taking as a case study the creative process in Beethoven's six variations in the third movement of op. 109. In section 6 we develop further the previous notion of categorical modeling based upon a categorical shape theory of discourse. Applying the concept of a logical manifold, we suggest in section 7 how to grasp the notion of sense and ambiguity. In section 8 we show how the same categorical structures (and in particular the colimit construction) provide a hierarchical and evolutive model for cognitive systems. This model is finally restricted, in section 9, to the special case of neuro-cognitive systems by suggesting, in this way, a new approach to human creativity via retrospection, prospection and complexification processes.

The unity in the paper is grounded on the proposal of a single categorical approach for creativity, with Yoneda's Lemma, shape, limits and colimits.[1] Therefore this enables transductions between music, discourse, and cognition, our distinct areas of interest.

2 A Generic Model for Creative Processes

In [23], a generic model of human creativity is developed which can be summarized by the following seven-step sequence: (1) Exhibiting the open question, (2) Identifying the semiotic context, (3) Finding the question's critical sign or concept in the semiotic context, (4) Identifying the concept's walls, (5) Opening the walls, (6) Displaying extended wall perspectives, (7) Evaluating the extended walls.

In this model, creativity implies the solution of the open question stated in the initial step, and which must be tested in the last step. The contextual condition guarantees that creativity is not a formal procedure as suggested by David Cope in the aforementioned book [4], but generates new signs with respect to a given meaningful universe of signs. The critical action here is the identification of the critical sign's "walls", its boundaries which define the famous 'box', which creativity would open and extend.

This model has been successfully discussed in [23] with respect to many examples, such as Einstein's *annus mirabilis* 1905 when he created the theory of special relativity, or Spencer Silver's discovery of 3M's ingenuous Post-It in 1968. Relating more specifically to musical creativity in composition, we shall discuss

[1] Colimits have been introduced by Kan in [19] under the name of "inductive limits", to distinguish them from the dual notion of "projective limits" as introduced by the author in the same article. Projective limits are normally referred to as "limits". In our article we will make use of both terminologies.

here the creative architecture of Ludwig van Beethoven's six variations in the third movement of op. 109 in the light of our model. This not only provides an excellent example of artistic creativity, but more specifically realizes a special case of our generic model: the creative process associated with Yoneda's famous lemma in category theory. It is remarkable that the Yoneda-based model relates to colimits in category theory, a construction which is also crucial in the shape theoretical approach to sense and ambiguity in sections 6 and 7, as well as in the neuro-cognitive model described in sections 8 and 9.

3 Categories, Functors, and the Yoneda Lemma

To understand the role of the Yoneda Lemma within a category-theory model of creativity, we first need to provide a short introduction to categories and functors. We will do it in a rather informal way, stressing the perspective of directed graph theory.[2]

A category $\mathcal{C}$ is a directed graph, possibly infinite, with possibly multiple arrows, whose vertices are called the *objects* of the category, while its arrows are called *morphisms*. An arrow is denoted by $f : X \to Y$, where X is its tail, called *domain* in category theory, and where Y is its head, called *codomain*. The set of morphisms from X to Y is denoted by $\mathcal{C}(X, Y)$, or sometimes by $X@Y$ if the underlying category is clear. Arrows admit an associative composition operation that is defined in the following cases: If $f : X \to Y$ and $g : Y \to Z$ are morphisms, then there is a morphism $g \circ f : X \to Z$ called the *composition of f with g*. There is also a special morphism $Id_X : X \to X$ for every object X, its *identity*, which is neutral, i.e. we have $Id_Y \circ f = f \circ Id_X = f$ for every morphism $f : X \to Y$.

The classical examples are these: (1) The category **Sets** of sets. Its objects are the sets, the morphisms are the Fregean maps between sets, the composition being the classical composition of set maps. (2) The category **Digraph** of directed graphs Γ. The objects are the directed graphs, while a morphism $f : \Gamma \to \Delta$ is a pair of maps $f = (f_{\mathrm{Vert}}, f_{\mathrm{Arr}})$ with $f_{\mathrm{Vert}} : \mathrm{Vert}_\Gamma \to \mathrm{Vert}_\Delta$ a map from the vertex set Vert_Γ of Γ to the vertex set Vert_Δ of Δ and $f_{\mathrm{Arr}} : \mathrm{Arr}_\Gamma \to \mathrm{Arr}_\Delta$ a map from the arrow set Arr_Γ of Γ to the arrow set Arr_Δ of Δ which are compatible with tails and heads of arrows. Composition of digraph morphisms goes componentwise for vertex and arrow maps. (3) The category **Top** of topological spaces. Its objects are the topological spaces, and the morphisms are the continuous maps between topological spaces.

The three previous examples also provide interesting category-theoretic frameworks in mathematical music theory. In fact, if set-theoretical approaches in music analysis can be easily described in terms of objects in the category **Sets** of sets, transformational music theory is elegantly formalized via the category **Digraph** of directed graphs. The third case, i.e. the category **Top** of topological

[2] Saunders MacLane's book [21] is the classical reference on category theory.

spaces, is by contrast the correct framework to approach musical gestures from a mathematical perspective.[3]

The Yoneda Lemma makes use of the so-called opposite category $\mathcal{C}^{op}$ of a category $\mathcal{C}$: Its objects are the same, but its arrows are the arrows of $\mathcal{C}$, however noted in reversed direction, while the composition of arrows is written in opposite order.

$$\begin{array}{ccc} F(Y) & \xrightarrow{h(Y)} & G(Y) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(X) & \xrightarrow{h(X)} & G(X) \end{array}$$

The composition of such functor morphisms is the evident composition of all morphisms of sets. Morphisms between functors are called *natural transformations*; their set, for functors F and G, is denoted by $\mathrm{Nat}(F, G)$. Yoneda's idea was to define a functor $\mathrm{Yon}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{@}$ by assigning to each object A of $\mathcal{C}$ a presheaf $@A : \mathcal{C}^{op} \to \mathbf{Sets}$ defined by $@A(X) = X@A$ and for each morphism $f : A \to B$ in $\mathcal{C}$ a natural transformation $@f : @A \to @B$ given by $@f(X) : X@A \to X@B : g \mapsto f \circ g$. Yoneda's lemma says that $\mathrm{Nat}(@A, F) \xrightarrow{\sim} F(A) =: A@F$, for every object A of $\mathcal{C}$ and every functor F in $\mathcal{C}^{@}$. This means in particular for $F = @B$ that A and B are isomorphic[4] if and only if their functors $@A$ and $@B$ are so. We may therefore replace the category $\mathcal{C}$ by its Yoneda-image in $\mathcal{C}^{@}$.

4 Creative Subcategories, a Yoneda-Based Colimit Model of Creativity, and Examples

Although as we have seen in the previous section Yoneda's lemma enables the replacement of a given category $\mathcal{C}$ by its Yoneda-image in $\mathcal{C}^{@}$, the functor $@A$ must be evaluated on the entire category $\mathcal{C}$ to yield the necessary information for its identity. The creative moment comes in here: could we not find a subcategory $\mathcal{A} \subset \mathcal{C}$ such that the functor $\mathrm{Yon}_{|\mathcal{A}} : \mathcal{C} \to \mathcal{A}^{@} : A \mapsto @A|_{\mathcal{A}^{op}}$ is still fully faithful? We call such a subcategory *creative*, and it is a major task in category theory to find creative categories which are as small as possible. One may even hope to find what we call an *objectively creative subcategory* for a given object A in $\mathcal{C}$, namely a creative subcategory $\mathcal{A}$ such that for this given object A in $\mathcal{C}$ there is a creative diagram D_A in $\mathcal{A}$ whose colimit C is isomorphic to A. Intuitively, a colimit of a diagram of spaces is obtained by gluing them along common subspaces; it is a generalized union operator. Taking a colimit is a natural condition since the functor $@A$ defines a big diagram whose arrows are the triples $(f : X \to Y, x \in X@A, y \in Y@A)$ with $y \circ f = x$. The colimit object C of such a diagram would ideally replace the functor $@A$ by a unique isomorphism from C to A.

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[3] We will come back to these three main examples of categories in section 4.

[4] This means that there is an isomorphism $f : A \xrightarrow{\sim} B$, i.e. a morphism such that there is an *inverse* $g : B \to A$, meaning that $g \circ f = Id_A$ and $f \circ g = Id_B$.

In the context of the Yoneda Lemma with its creative subcategories, the described generic model of creativity looks as follows: (1) Exhibiting the open question: understand the object A; (2) Identifying the semiotic context: this is the category $\mathcal{C}$ where A has been identified; (3) Finding the question's critical sign or concept in the semiotic context: this is A; (4) Identifying the concept's walls: this is the uncontrolled behavior of $@A$; (5) Opening the walls: finding an objectively creative subcategory $\mathcal{A}$; (6) Displaying extended wall perspectives: calculate the colimit C of a creative diagram; and (7) Evaluating the extended walls: try to understand A via the isomorphism $C \xrightarrow{\sim} A$. Let us look at some illustrative examples:

Example 1. For the category **Sets**,we may take the creative subcategory $\mathcal{A}$ with the singleton $1 = \{\emptyset\}$ as unique object. This subcategory is even objectively creative since the colimit of the discrete diagram defined by the elements of $1@A$ is isomorphic to A.

Example 2. For the category **Digraph**, we may take the full creative subcategory $\mathcal{A}$ defined by the two objects $\mathrm{Vert}, \mathrm{Arr}$ with $\mathrm{Vert} = (\{V\}, \emptyset)$ and $\mathrm{Arr} = (T, H, A : T \to H)$. The category $\mathcal{A}$ is also objectively creative.

Example 3. The third example, the category **Top** of topological spaces, is less simple. We do not know of any strictly smaller creative subcategory in this case. A number of workarounds for this unsolved problem is dealt with in algebraic topology [30]. One approximates the total understanding of a topological space T by the selection of subcategories **Simple** that are composed of "simple" topological spaces and continuous maps which one knows very well. It is then hoped that the Yoneda restriction $\mathrm{Yon}|_{\mathbf{Simple}} : \mathbf{Top} \to \mathbf{Simple}^{@}$ may reveal important information about topological spaces. Typically, algebraic topology takes the category **Simple** = **Simplex** of n-dimensional standard simplexes Δ_n with their face operators as morphisms, or the category $\mathbf{Simple}' = \mathbf{Cube}$ of n-dimensional unit cubes I^n with their face operators ($I = [0, 1]$ is the real closed unit interval). In order to understand Yoneda restrictions to **Simple** categories, it is useful to refer to homology theory (which also plays a crucial role for the solution of Weil's and Fermat's conjectures). We will see a crucial example of homological reasoning in section 6.2 of this paper. Homology plays a crucial role in the mathematical theory of musical hypergestures [24].

5 Interpreting the Six Variations in Beethoven's op. 109 as a Yoneda-Based Creative Process

Beethoven's six variations $V_1, V_2, \ldots V_6$ of the main theme X, entitled "Gesangvoll, mit innigster Empfindung"[5], define the third movement of his piano sonata op. 109. They offer an interesting interpretation in the sense of the above Yoneda-oriented colimit model of creativity. This interpretation is discussed in

[5] "Lyrical, with deepest sentiment."

detail in [23, ch. 26]; here, we want to summarize those results. This analysis is based on the detailed music-theoretical analysis by Jürgen Uhde [31].

The crucial point stems from Uhde's beautiful picture of a configuration of variable perspectives. Each perspective stresses a particular aspect of X. When the first five variations are over, he asks whether there is still an efficient position for the sixth, and adds: "Wasn't the theme illuminated from all sides from near and from far, and following sound and structure? The preceding variations 'danced' around the theme, and each was devoted to another thematic property."

We therefore interpret his comment in terms of Yoneda-inspired category theory as describing a set of six morphisms $f_1, f_2, ..., f_6$, each variational perspective being one morphism$f_i : V_i \rightarrow X$. This is a set of six elements of the functor $@X$, evaluated on arguments $V_1, V_2, \ldots V_6$. Let us be clear: There is no explicit mathematical category involved in this description, and it is a challenge for mathematical music theory to come up with a category where this setup becomes mathematically rigorous. But supposing that this category can be found, Uhde's discourse is astonishingly categorical. Saying that the first five perspectives encompass "all that can be said," means (for Uhde) that with the first five variations, *Beethoven has composed an objectively creative category.* The main theme X can completely be understood from the system of these five variational perspectives, including a melodic, a rhythmical, a contrapuntal, and two permutational variations.

It is now obvious what could be the role of the sixth variation. It could be that colimit object guaranteed for objectively creative categories. This means that it should be a gluing of a diagram deduced from the characteristic objects $V_1, V_2, \ldots V_5$ (see Figure 1).

Intuitively, knowing that a colimit is a gluing of the diagram's objects along with common subspaces, one would expect V_6 to be a patchwork of smaller units. It is fascinating to read Uhde's interpretation of the sixth variation. He views it as if it were itself a body of six micro-variations, and he describes this body as a "streamland with bridges," the bridges connecting the six micro-variations. This is very similar to the construction of a colimit, which is also essentially a landscape connecting its components by bridge functions. Looking at the sixth variation, it in fact contains six variational restatements of the theme, beginning with a short version of the original theme. We have six micro-variations in V_6, representing X and $V_1 \ldots V_5$. The dramatic convergence of the finale synthesizing all previous perspectives is described by Uhde as an "explosion of energy".

6 Categorical Modeling, Emergence of New Shapes

In the previous sections we have seen a first example of category-oriented analysis of a musical process. More generally, *categorical modeling* consists in descriptions and computations with signposts made of arrows and compositions of arrows organized in categories, functors (i.e. homomorphisms of categories), and natural

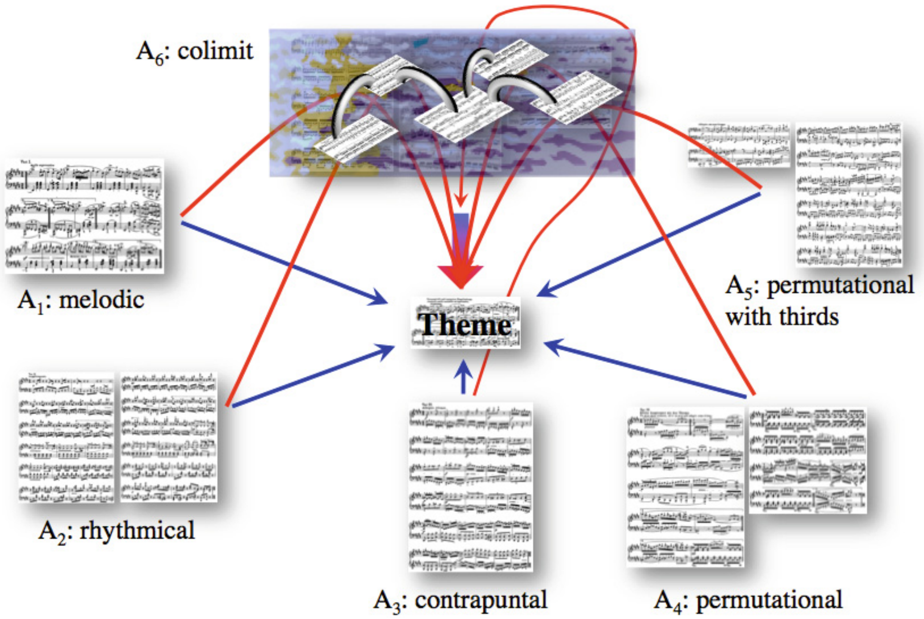

Fig. 1. The sixth and last variation is a colimit of variations one to five of the theme in the third movement of Beethoven's op. 109

transformations, via composition laws and universal properties such as inductive limits or glueings.

6.1 Signposts, Autographs, Categories, Colimits or Glueings

Firstly, according to a very general assertion of Charles Saunders Peirce we consider that semiosis is the living system of signs, and that each sign is a ternary datum where a representamen R is interpreted by an interpretant I as a representation of an object O. For us this is illustrated by an arrow $R \xrightarrow{I} O$. This could be read: "From the point of view I, R is an indicator of O. So we could also think of I as being a *difference* (a supplementary information) which when added to R produces O."

Secondly, for Peirce [25] any arrow $A : D \to C$ is a sign from a sign D as a source or domain toward another sign C as a target or codomain; then each "object" D or C is supposed to be again a sign, i.e. an arrow, from a source to a target, and so on. Each arrow is a difference between two other arrows. We consider that from the beginning, there are no real objects, only signs between other signs. Each sign takes its value only from its place in a net of signs, as shown in Figure 2.

A basic setting for a model of semiosis is an *autograph*, a set of signs $\mathcal{S}$ and a map $[d, c] : \mathcal{S} \to \mathcal{S}^2$. Any modeling starts with such an autograph of signs, where the value of each sign simply is its position in the system, i.e. the system of its relations to the other signs. The way in which this is made mathematically precise is the Yoneda Lemma, as follows.

We start with a *category*, i.e. an autograph in which at first some arrows are selected as "objects", and where, for consecutive arrows between such objects, we suppose an associative and unitary composition law. Given a category $\mathcal{C}$, Yoneda's Lemma says that the knowledge of $C \in \mathcal{C}_0$ is equivalent to the knowledge of $@C$. For objects F in $\mathcal{C}^@$ and A in $\mathcal{C}$ we have $F(C) = C@F$, and we denote by $\int_{\mathcal{C}} F$ the *category of elements of* F which is the category with objects the pairs (C, p) where $C \in \mathcal{C}_0$ and $p \in C@F$, a morphism from (C, p) to (C', p') being a $u : C \to C'$ in $\mathcal{C}$ such that $p'.u = p$. Then F is a glueing (inductive limit) as

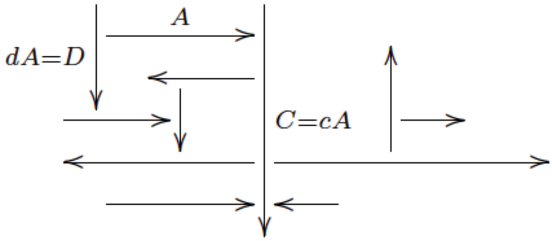

Fig. 2. Autograph as a net of signs between signs

$$F = \varinjlim_{[C;p\in C@F]\in(\int_{\mathcal{C}} F)_0} @C.$$

6.2 Shape with Respect to Models, Cohomology, Differentials

Now in the place of the $\mathrm{Yon}_{\mathcal{C}} = @? : \mathcal{C} \to \mathcal{C}^@$ we start with a functor $J : \mathcal{M} \longrightarrow \mathcal{X}$ where $\mathcal{M}$ is thought as the category of *known simple models* M, and $\mathcal{X}$ as the category of *unknown complex objects* X. Analogous to the category of elements $\int_{\mathcal{C}} F$, we consider the J*-shape of* X, which is the category $\int_{\mathcal{M}} X$ — more classically denoted by J/X — with objects the pairs (M, p) where $M \in \mathcal{M}_0$ and $p : J(M) \to X$, a morphism from (M, p) to (M', p') being a morphism $u : M \to M'$ in $\mathcal{M}$ such that $p'.J(u) = p$. Let $q_X : J/X \to \mathcal{M}$ be the forgetful functor $q_X(M, p) = M$, and, if it exists, X_J the inductive limit of $J.q_X$, and a comparison map $k_{X,J}$:

$$X_J = \varinjlim(J.q_X) = \varinjlim_{[M;p:J(M)\to X]} J(M), \qquad k_{X,J} : X_J \longrightarrow X.$$

If $k_{X,J}$ is not an isomorphism, then we consider that, with respect to J, X is an *absolute novelty*; otherwise we say that X is a J*-manifold*. Given a J-manifold X and a functor $H^* : \mathcal{X} \longrightarrow \mathcal{V}$ (e.g. cohomology), if the comparison or differential

$$dX = d_{(H^*,J)}X : \varinjlim_{[M;p:J(M)\longrightarrow X]} H^*J(M) \to H^*\Big(\varinjlim_{[M;p:J(M)\to X]} J(M)\Big)$$

is not an isomorphism, then we say that the J-manifold X has an H^**-emergent property*. The expression of emergence in this way is proposed in [14] as directly inspirated by [7]. The method of inspection and extension of a concept's walls, previously described in section 4, could be rephrased and extended in terms of analysis and perturbations of shapes : the initial moment of creativity (opening the walls) consists of choosing an inclusion functor $J_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}$, and then the analysis of A in $\mathcal{C}$ with respect to $J_{\mathcal{A}}$ is — second step of creativity —the introduction of a diagram $D_{\mathcal{A}} : \Delta_{\mathcal{A}} \subset \int_{\mathcal{A}} A \overset{q_A}{\to} \mathcal{A}$ (this introduces a perturbation of the $J_{\mathcal{A}}$- shapes towards $(J_{\mathcal{A}}.D_A)$- shapes) and the final step consists in displaying extended wall perspectives accordind to D_A, i.e. in examining if A is a $(J_{\mathcal{A}}.D_A)$-manifold.

In the next examples we illustrate what is interesting when $C \to A$ is not an isomorphism, which is the situation of emergent property and absolute novelty.

Example 4. Let $\mathcal{X} = \mathbf{Top}$ be the category of continuous maps between topological spaces (as already discussed in Example 3 of section 4), $X = S^2$ the 2-dimensional sphere, and $\mathcal{M}$ the full-subcategory of **Top** generated by the open disk $D_2 = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$. Then S^2 is a manifold, and it has the emergent property that $\pi_2(S^2) \neq 1$. Of course S^3 is an *absolute novelty.*

Example 5. Let $\mathbb{A}$ be a ring and $\mathcal{X}$ be the category Fac$[\mathbb{A}[X]]$ whose objects are elements in the ring of polynomials $\mathbb{A}[X]$, with arrows $Q : B \to A$ given by elements in $\mathbb{A}[X]$ such that $BQ = A$. Let $\mathcal{M}$ be the full-subcategory generated by powers of polynomials of degree ≤ 1. Then if $\mathbb{A} = \mathbb{R}$, the polynomial $X^2 + 1$ is an *absolute novelty*, whereas if $\mathbb{A} = \mathbb{C}$ every polynomial is a manifold (this is the fundamental theorem of algebra).

7 Sense and Ambiguities in Logical Manifolds

In order to evaluate the sense and ambiguity of discourses we first have to characterize the notion of a logical manifold in terms of Lawvere Theory [20].

Definition: Let $N = \{0, 1, ..., N - 1\}$ be a natural number. We define the *theory of N-valuated propositional logic* as the Lawvere theory which is the full-subcategory of **Sets** with objects all the finite powers of N; we denote this by $\mathbb{P}_N$ (in memory of Post's algebras [28]).

So the theory of a Boolean algebra is $\mathbb{P}_2$: this follows from the fact that any map $\{0, 1\}^m \to \{0, 1\}$ can be obtained by composition of projections and the usual logical maps $\& : \{0, 1\}^2 \to \{0, 1\}$ and $\neg : \{0, 1\} \to \{0, 1\}$.

Definition: Given a I_2-manifold Θ in the situation $I_2 : \mathcal{P}_2 \longrightarrow \mathcal{T}$ where $\mathcal{T}$ is the category of Lawvere's theories, and $\mathcal{P}_2$ the full subcategory of $\mathcal{T}$ generated by the object $\mathbb{P}_2$, a *model* (or an *algebra*) of Θ is named a *classical logical manifold* (of type Θ).

The following result shows that the two previous concepts are deeply related.

Theorem. *For every integer N the theory $\mathbb{P}_N$ is an I_2-manifold.*

The case $N = 4$ is more precisely an example of *Borromean logic* [15].

For the purpose of discourse analysis, we consider that a discourse is made of propositions (logical islets) bound by non-logical connectors (such as "but", "of course", "what else?", etc.), and therefore consists of a kind of *logical manifold*. Each of its propositions may be logically evaluated, but the discourse will get only "sense;" the sense expresses to what extent the various logical meanings of the propositional components are compatible in the discourse. The decisive point here is precisely the structural ambiguity and the game of equivocations, the paradoxical way according to which the sense is logically impossible: this will be revealed, by the structure of the logical manifold, as an emergent cohomological property.

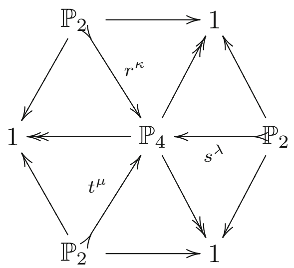

Fig. 3. $\mathbb{P}_4$ as a I_2-manifold, with the shape of a Borromean glueing of 3 copies of $\mathbb{P}_2$

For example, let us consider the following naive answer to the question: "Do you like this music?": "*It is great, but I don't like it.*" This answer is not a proposition, it is a discourse, with the shape: "G but $\neg L$." If, by mistake, we interprete "but" as an "and", we get "$G \wedge \neg L$", which is an antilogy, because of $G \Rightarrow L$. So we reach the paradoxical character of the answer. To solve this paradox we just have to realize that perhaps G is said from a point of view V_1 and $\neg L$ from a point of view V_2: it is precisely the work of the interpreter to construct and make precise these points of view by constructing a *logical speculation* [13], and such a construction is a sense of the answer "G but $\neg L$". Eventually in this case, a sense is an evaluation in an algebra that is a glueing of two boolean algebras, one for V_1 and one for V_2. In fact "G but $\neg L$" could be evaluated in several non-trivial ways in the classical logical manifold $\mathbb{F}_4$ (the field of cardinality 4), which is a model of $\mathbb{P}_4$. In this example, informally, one could see the logical conflict as a wall, and the colimit glueing of two boolean algebras, as an opening of the walls and extending the original box. So *creativity* (and invention of new objects) could be understood as the open development of new discourses, algebraic tools, or geometrical shapes — and so from a general point of view as the development of new J-shapes, for variable J, under a control of modifications of senses or meanings, solutions, geometrical invariants — i.e. from a general point of view under the control of cohomological information given by the differentials $d_{(H^*,J)}X$.

8 Memory Evolutive Systems: A Model for Cognitive Systems

In the next sections, we shall study creativity in a cognitive system that is able to learn from its experiences and to develop an integrative, robust though flexible memory. This topic is studied in the frame of the theory of Memory Evolutive

Systems (MES) [7], a bio-inspired mathematical model, based on category theory, for self-organized, multi-scale, and multi-agent dynamic cognitive systems.

8.1 Hierarchical Evolutive Systems (HES)

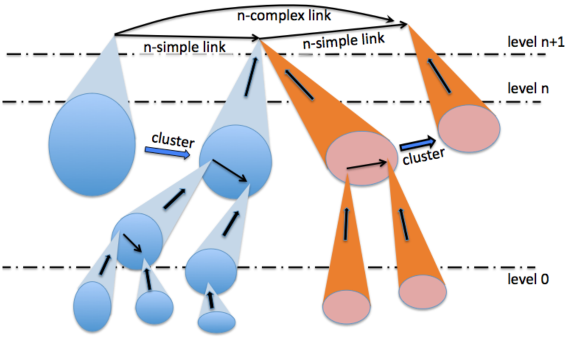

Fig. 4. Two ramifications of M, with simple and complex links

In an *Evolutive System* (ES), the configuration of the system at a given time t of the timescale T is represented by a category K_t: its objects are the states M_t of the components existing at t. A morphism from M_t to M'_t corresponds to a channel through which M_t can send information to M'_t; it is labeled by a propagation delay and a strength (both positive real numbers), and by an index of activity 0 (if passive) or 1 (meaning that information is sent) at t. The change of state from t to a later time t' is modeled by a transition functor from a sub-category of K_t to $K_{t'}$. The transition functors satisfy the transitivity condition: if M_t has a new state $M_{t'}$ at t', then M_t has a state $M_{t''}$ at t'' if and only if $M_{t'}$ has a state at t'', and this state is $M_{t''}$. A *component* M of the ES is a maximal family $(M_t)_{t \in TM}$ of objects of the K_t satisfying:

(i) TM is an interval of T which has a first element t_0 ('birth' of M);
(ii) all the successive states of M_{t_0} (i.e. its images by transitions) are in M.

A *link* s from M to M' is similarly defined as a maximal family $(s_t)_{t \in Ts}$ of morphisms s_t of K_t related by transitions, with Ts included in both, TM and TM'. To any interval I of T we associate the category K_I whose objects are the components M for which I is included in TM and the morphisms are the links between them.

The system is organized so that the components of a level are obtained by combination of (patterns of) lower levels. A musical example of such an "atomistic hierarchy" is provided by the metric organization which has been described by Zbikowski in [32]. Our presentation is formally described as follows.

A *pattern* (or *diagram*) in K_t is a homomorphism of a directed graph to K_t: we denote by P_i the image by P of a vertex i of the graph. The category K_t is *hierarchical* if the class $|K_t|$ of its objects is partitioned into a finite number of parts called *levels*, numbered $0 < 1 < ... < m$, verifying the following property: each object M_t into level n+1 admits at least one decomposition in a pattern P of lower levels, meaning that M is the colimit of P and each P_i pertains to a level $< n$+1 (meaning that P takes its values in the full subcategory of K_t whose objects are elements of one of the levels 0,1, ..., n). Intuitively, we think that the objects in a level $n+1$ are 'more complex' than the objects contained in the levels $\leqslant n$. The Evolutive System is called a *Hierarchical Evolutive System* (HES) if all its configuration categories are hierarchical and the transitions preserve the levels. Then M_t has a *ramification* obtained by taking a decomposition P of M_t of lower levels, then a decomposition of lower levels of each P_i and so on down to level 0 (Figure 4).

The *complexity order* of a component M is defined as the smallest length of one of its ramifications in one of the categories K_t. We suppose that the system satisfies a kind of 'flexible redundancy', called the *Multiplicity Principle* (MP) which extends the degeneracy property of the neural code emphasized by Edelman [5]: there are *multiform* components M which are the colimit of at least two patterns of lower levels which are not well-connected by a cluster of links between their components (see [7] for a technical presentation of these concepts in terms of morphisms of Ind-objects); the number of such patterns is called the *entropy* of M. A multiform component is adaptative: at a given time it can operate through any of its decompositions and switch between them depending on the context, though keeping its complex identity over time. In particular the existence of multiform components allows for the emergence of complex links (Figure 4) which give some flexibility to the system.

8.2 The Complexification Process

In a HES the transition from t to t' results from changes of the following types: 'adding' some external elements, 'suppressing' or 'decomposing' some components; adding a colimit or a limit to some given patterns. This is modeled by the *complexification* process: a *procedure* Pr on K_t consists of the data (E, A, U, U', V, V'), where E is a sub-graph of K_t, A is a graph not included in K_t, U is a set of colimit-cones in K_t, U' a set of limit-cones and V (resp. V') a set of patterns without a colimit (resp. a limit) in K_t. The *complexification* of K_t for a procedure Pr is a universal solution of the problem of constructing a category K' and a functor F from the full sub-category of K_t with objects not in E to K', such that: A is a sub-graph of K', the images by F of the cones in U and U' are colimit-cones and limit-cones in K' respectively and the image by F of a pattern P in V admits a colimit cP in K', and the image of a pattern P' in V' admits a limit in K'.[6] The complexification leads to the notion of emergence, which is central to any complex system, and which is characterized by the following result:

Emergence Theorem. *MP is necessary for the existence of components of complexity order > 1. It is preserved by complexification and it allows for the emergence of components of increasing orders through iterated complexifications.*

The Complexification describes not only new objects but also morphisms between them. It also provides a categorical formalization of the conceptual

[6] Formally a *sketch* is associated to the procedure, and the *complexification* with respect to Pr is the prototype associated to this sketch, which has been explicitly constructed in [2].

blending construction as described by Fauconnier and Turner [8] and systematized by Goguen [11].

8.3 The Local and Global Dynamics

A Memory Evolutive System (MES) is a HES which is self-organized by a net of functional sub-systems, the *co-regulators*. These modulate the global dynamics through their competitive interactions, and help develop a flexible central *Memory*. Formally, a MES consists of these data: a HES **K**, a sub-HES of **K** called the *Memory*, and a family of sub-ES with discrete timescales called *co-regulators*. The Memory is robust, though flexible and adaptative: its components can acquire new decompositions over time and later be recalled through any of them. In the Memory we distinguish a sub-ES, the *procedural memory* **Proc**; to a component S of **Proc** is associated a pattern EffS (the 'effectors' commanded by S) admitting S as its (projective) limit. Here is a rough outline of the two-part dynamics. We refer to [7] for details.

(i) The 'function' of a co-regulator CR is accounted for by the data of the set of its *admissible procedures*, which are components of **Proc** with links to some components of CR, memorizing the actions it can perform. Each CR has its own discrete timescale and acts accordingly in steps, a step extending between two consecutive instants. During the step from t to t', a temporal model of the system as perceived from the point of view of CR is formed; it is a category L_t called the *landscape* of CR at t defined as follows, where $\mathrm{I}=]t,t'[$ is the open interval between t and t' It is the full sub-category of the comma category $\mathrm{K}_I \mid \mathrm{CR}_I$ having as objects the links arriving at a component of CR and which are active during the step. An admissible procedure S of CR is selected on it, using the Memory, thus commanding the effectors of S. It starts a dynamical process carried on during the step, directed by differential equations specifying the links' propagation delays and strenghts. The result should lead to a complexification of L_t (an attractor for the dynamic corresponding to the formation of a colimit). The result is evaluated at t'; if the objectives are not attained, we speak of a *fracture* for CR.

(ii) As CR acts through its landscape which is only a partial view on the system, the commands to effectors sent by the different co-regulators at a given time may be conflicting. Thus their 'local' dynamics must be coordinated by their interplay, made flexible by the possibility to switch between ramifications of complex commands. It may cause fractures to some co-regulators, in particular if their temporal constraints (synchronicity laws) cannot be respected.

9 Modeling Creative Processes in MENS

9.1 Model MENS for a Neuro-Cognitive System

The 'hybrid' model **MENS** is a MES whose level 0, called **Neur**, models the 'physical' neuronal system while higher levels model the mental and cognitive system. **Neur** is an ES whose configuration category at t is the category of paths of the *directed graph of neurons* at t. A vertex of this graph models the state N_t of a neuron N existing at t and labeled by its activity (firing rate) around t; an *arrow* f from N_t to N'_t models a synapse from N to N' labeled by its propagation delay and its strength at t. According to Hebb's cognitive model, it is known that a mental object activates a more or less complex and distributed assembly of neurons operating synchronously; such an assembly is not necessarily unique because of the degeneracy of the neural code [5]. This property is used to construct **MENS** from **Neur** by iterated complexification processes: higher level components, called *cat(egory)-neurons*, are 'conceptual' objects which represent a mental object M as the common colimit cP = cP' of the synchronous assemblies of (cat-)neurons P, P' which can activate them. Because of the propagation delays, the activation of the colimit cP comes after that of P.

MENS admits a *semantic memory* SEM which is a sub-ES of the Memory developing over time. Its components, called *concepts* (in the sense of [5]) are obtained by categorization of cat-neurons of the Memory with respect to some attributes, followed by iterated complexifications (cf. [6]); the cat-neurons 'instances' of such a concept have different degrees of typicality. The 'cognitive' concepts used by Zbikowski [32] to define musical concepts (such as the concept of a motive), can be interpreted as concepts in SEM; his conceptual models and theories would also figure as concepts contained in higher levels of SEM.

9.2 The Archetypal Core and the Global Landscapes

The graph of neurons contains a central sub-graph, called the *structural core* which has many strongly connected hubs ([16]). The *archetypal core* **AC** is a subsystem of the Memory formed by higher order cat-neurons integrating significant memories, with many ramifications down to the structural core; for instance, the memory of a music associated to an event with emotional contents. Their strong and fast links form *archetypal loops* self-maintaining their activation. **AC** embodies the complex identity of the system ('Self'), and acts as a *flexible internal model*. Activation of part of **AC** diffuses through self-maintained archetypal loops. It propagates to a decomposition P of some A, then, via a switch, to another decomposition Q of A and through a ramification down to the neural level.

All this activation allows for more communication between different parts of **MENS**, and in particular increases the information received by higher level co-regulators directly linked to **AC**. Thus **AC** acts as a driving force for constructing a *global landscape* **GL** uniting and extending spatially and temporally the landscapes of these co-regulators; **GL** can be compared to the "Global Workspace" of different authors. Successive global spaces overlap, emphasizing the unity of the Self; they give a setting where higher level information can be 'consciously' processed, while keeping traces of the operations of lower level co-regulators.

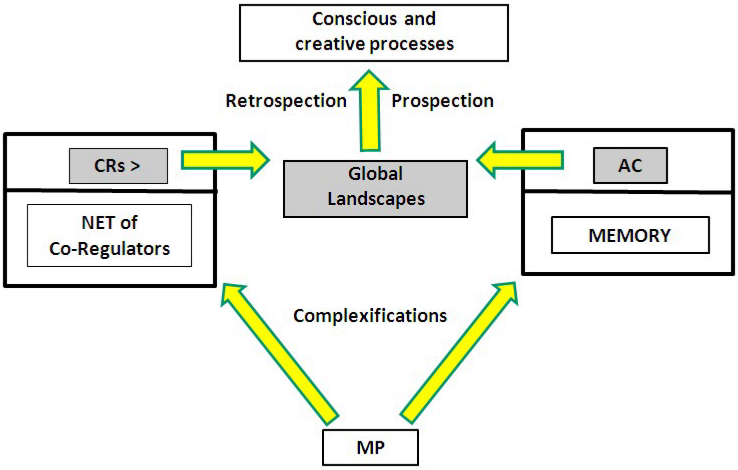

Fig. 5. General scheme of the RPC construction

9.3 The RPC Model of Creativity

Creative processes start after a striking, surprising or intriguing event S which increases the attention, translated by the activation of part of **AC** and leading to the formation of a long term global landscape **GL**. They take place through a sequence of intermingled retrospection and prospection processes in overlapping **GL**s as follows:

(i) *Retrospection*: **GL** receives information from components A of **AC** related to S. Since A is activated at t, it must have at least one ramification which has been activated before t and through which **GL** receives information about the past activation of lower levels. Thus it enables an analysis of the situation at different levels with a recall of the near past for "making sense" of S (by 'abduction'). This would correspond to the identification of the "critical concept" and of its "walls" (cf. sections 2, 4) as well as to the "exploratory creativity" of Boden [3].

(ii) A *prospection* process then develops within **GL**. The activation of A being maintained via archetypal loops, it also maintains that of its ramifications which transmit information to **GL** and anticipates the future, whence the possibility of *prospection* by search of adequate procedures and elaboration of "scenarios". A procedure Pr is selected on (a sub-system) of **GL** (playing the role of a "mental space") for adding or suppressing elements or combining patterns. The corresponding *complexification* for Pr is a 'virtual landscape' V in which Pr can be evaluated. Examples of prospection processes include "extension of walls" (sections 2, 4), "combinatory creativity" of Boden [3], "conceptual blending" of Fauconnier & Turner [8] (the blend is obtained by push-out).

More ‘innovative’ scenarios, corresponding to the “transformational creativity” of Boden [3], are obtained by iterated complexifications of virtual landscapes: a new procedure Pr’ is selected on V and the complexification for Pr’ is a new ‘virtual landscape’ V’ which is not directly deducible from **GL** because of the following theorem:

Iterated Complexification Theorem. *A double complexification of a category satisfying MP is generally not reducible to a unique complexification.*

The Retrospection, Prospection, Complexification (RPC) model of creativity consists therefore of an iteration of intermingled processes: formation of a global landscape, retrospection, prospection, complexification of virtual landscapes, evaluation. This RPC model can be developed in any MES satisfying MP and with a central Archetypal Core consisting of strongly linked higher order components, with many interactions between their ramifications. It points to the following measures of creativity: complexity order and ‘entropy’ of the components, connectivity, and centrality orders of the archetypal core.

10 Conclusion

This paper makes evident that categorical colimits are a unifying construction for understanding creativity in music, categorical shape theory of discourse, and cognitive processes. A first hint at this unification may be understood through Yoneda’s Lemma, which enables the construction of general presheaves as canonical colimits of representable presheaves, at the crucial moment when instead of canonical general colimits we decide to consider special colimits: this is the root of our analysis of creativity. For musical creativity Yoneda’s Lemma applies when representable presheaves are restricted to small, fully faithful “creative” subcategories, generated in Beethoven’s op. 109 by the the five variations of the main theme. In the categorical shape theory of discourse, the Yoneda construction of general presheaves as colimits of representable presheaves is generalized to powerful shapes and manifolds, which are shape colimits, and the production of sense relies on constructions of special colimits. In the context of Lawvere’s theory, logical manifolds provide a creative synthesis of multiple logical perspectives and help create new sense. Finally, for cognitive neuroscience, hierarchical systems of neuronal networks (diagrams) generate in a creative way higher levels of mental objects, namely category-neurons, as colimits of lower level networks.These three perspectives unite the arts, discursive logic, and neuroscience on the common ground of a central device, the colimit idea, in category theory. The question of experimental verification or falsification of such a theoretical unification is important, but the very generality of our result poses fundamental problems which transcend standard empirical methods and ask for more in-depth investigations into the nature and limits of an experimental approach in music and, more generally, in the humanities.

References

1. Acotto, E., Andreatta, M.: Between Mind and Mathematics. Different Kinds of Computational Representations of Music. Mathematics and Social Sciences 199(50e année), 9–26 (2012)
2. Bastiani(-Ehresmann), A., Ehresmann, C.: Categories of sketched structures, Cahiers Top. et Géom. Dif. XIII-2 (1972), http://archive.numdam.org
3. Boden, M.A.: Conceptual Spaces. In: Meusburger, P., et al. (eds.) Milieus of Creativity. Springer (2009)
4. Cope, D.: Computer Models of Musical Creativity. MIT Press, Cambridge (2005)
5. Edelman, G.M.: The remembered Present. Basic Books, New York (1989)
6. Ehresmann, A., Vanbremeersch, J.-P.: Semantics and Communication for Memory Evolutive Systems. In: Lasker (ed.) Proc. 6th Intern. Conf. on Systems Research. International Institute for Advanced Studies in Systems Research and Cybernetics, University of Windsor (1992), http://ehres.pagesperso-orange.fr
7. Ehresmann, A., Vanbremeersch, J.-P.: Memory Evolutive Systems: Hierarchy, Emergence, Cognition. Elsevier, Amsterdam (2007)
8. Fauconnier, G., Turner, M.: The way we think. Basic Books (2002) (reprint)
9. Forth, J., Wiggins, G.A., McLean, A.: Unifying Conceptual Spaces: Concept Formation in Musical Creative Systems. Minds & Machines 20, 503–532 (2010)
10. Gärdenfors, P.: Conceptual Spaces: On the Geometry of Thought. MIT Press, Cambridge (2000)
11. Goguen, J.: An Introduction to Algebraic Semiotics, with Application to User Interface Design. In: Nehaniv, C.L. (ed.) CMAA 1998. LNCS (LNAI), vol. 1562, pp. 242–291. Springer, Heidelberg (1999)
12. Goguen, J., Harrell, D.F.: Style: A Computational and Conceptual Blending-Based Approach. In: Dubnov, S., et al. (eds.) The Structure of Style: Algorithmic Approaches to Understanding Manner and Meaning. Springer (2009)
13. Guitart, R.: L'idée de Logique Spéculaire. Journées Catégories, Algèbres, Esquisses, Néo-esquisses, Caen, Septembre 27-30, p. 6 (1994)
14. Guitart, R.: Cohomological Emergence of Sense in Discourses (As Living Systems Following Ehresmann and Vanbremeersch). Axiomathes 19(3), 245–270 (2009)
15. Guitart, R.: A Hexagonal Framework of the Field $\mathbb{F}_4$ and the Associated Borromean Logic. Log. Univers. 6, 119–147 (2012)
16. Hagmann, P., Cammoun, L., Gigandet, X., Meuli, R., Honey, C.J., Wedeen, V.J., Sporns, O.: Mapping the Structural Core of Human Cerebral Cortex. PLoS Biology 6(7), 1479–1493 (2008)
17. Halford, G.S., Wilson, W.H.: A Category-Theory approach to cognitive development. Cognitive Psychology 12, 356–411 (1980)
18. Hofstadter, D.: Fluid Concepts and Creative Analogies: Computer Models of the Fundamental Mechanisms of Thoughts. Basic Books (1995)
19. Kan, D.M.: Adjoint Functors. Transactions of the American Mathematical Society 87, 294–329 (1958)
20. Lawvere, W.F.: Functorial Semantics of Algebraic Theories. Ph.D. Thesis, Columbia University (1963)
21. Mac Lane, S.: Categories for the Working Mathematician. Springer, New York (1971)
22. Mazzola, G., et al.: The Topos of Music—Geometric Logic of Concepts, Theory, and Performance. Birkhäuser, Basel (2002)

23. Mazzola, G., Park, J., Thalmann, F.: Musical Creativity, Heidelberg. Springer Series Computational Music Science (2011)
24. Mazzola, G.: Singular Homology on Hypergestures. Journal of Mathematics and Music 6(1), 49–60 (2012)
25. Peirce, C.S.: Collected Papers, vol. I-VI (1931-1935), par Hartshorne, C., Weiss, P.: vol. VII-VIII (1958), par Burks, W.: Harvard University Press. Harvard
26. Pereira, F.C.: Creativity and Artificial Intelligence - A Conceptual Blending Approach (2007)
27. Phillips, S., Wilson, W.H.: Categorical Compositionality: A Category Theory Explanation for the Systematicity of Human Cognition. PLoS Computational Biology 6(7), 1–14 (2010)
28. Post, E.: Introduction to a General Theory of Elementary Propositions. American Journal of Mathematics 43, 163–185 (1921)
29. Shannon, C., Weaver, W.: The Mathematical Theory of Communication (1949)
30. Spanier, E.: Algebraic Topology. McGraw Hill, New York (1966)
31. Uhde, J.: Beethovens Klaviermusik III. Reclam, Stuttgart (1974)
32. Zbikowski, L.M.: Conceptualizing Music: Cognitive Structures, Theory, and Analysis. Oxford University Press (2002)