The Minkowski Geometry of Numbers Applied to the Theory of Tone Systems*

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Abstract. Euler's *speculum musicum* is a finite selection of tones from the two dimensional tone lattice known as the *Tonnetz*. The idea of representing larger or smaller collections of tones as finite subsets of the *Tonnetz* reappears in the scholarly discourse in various contexts. However, formal rules for such selections that would satisfactorily reflect musical reality are not known: those proposed in the past are either too restrictive (not allowing all musically relevant tone systems to enter the model) or too loose (not preventing musically irrelevant tone systems from entering the model). The paper offers a formal framework that yields selections satisfactorily reflecting the musical reality. The framework draws methods from the Minkowski geometry of numbers. It is shown that only *selection bodies* of very specific shapes called *(skewed) selection polygons* lead to relevant selections. Manifold music-theoretical examples include chromatic, superchromatic, and subchromatic tone systems.

Keywords: tone lattice, *Tonnetz*, comma lattice, generated tone system, selection body, selection polygon.

Euler's [7] *speculum musicum*, an arrangement of the twelve chromatic tones in three major-third related rows of four fifth-generated tones, is usually cited as an early precursor of the *Tonnetze* found in the writings of nineteenth-century German-speaking theorists such Oettingen, Riemann, or Hostinský. Yet, there is a significant difference between the *speculum musicum* and the *Tonnetz*, however trivial the observation may seem: the former is only a finite subset of the latter. Various powerful music theories rely on modeling tone systems as finite selections of tones from the *Tonnetz*. Works of Tanaka [19], Oettingen [18], Fokker [8,9], and, among the more recent ones, Erlich [6] are some of many examples that can be found in the field of the theory of just intonation. The idea of selecting subcollections from the *Tonnetz* has played an important role also outside that field as illustrated by work of Honingh and Bod [12,11], Wild [22], or the present author [23]. The key open question underlying many of these models is: How to define rules of selection so that the model reflects the musical reality as closely as possible? The correspondence with the musical reality means that the model

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includes tone systems – such as chords or scales – of musical relevance and does not contain musically irrelevant ones.

The paper offers a possible answer to this question. The formal framework around which the model is developed draws key ideas from the geometry of numbers. It turns out that questions surrounding the theory of tone selections from the *Tonnetz* find appropriate tools, and sometime even ready-made answers, in this mathematical discipline. Minkowski's work from the early twentieth century, especially his *Geometrie der Zahlen* [16] were foundational for the field, which later developed into a notable branch of modern mathematics. Important monographs related to the geometry of numbers include [10,2,14].

1 Mathematical Exploration of Selection Bodies

1.1 Preliminaries

Interior, Closure, Boundary. Let $\mathcal{S}$ be a set in $\mathbb{R}^2$. We say that $X \in \mathcal{S}$ is an *interior point* of $\mathcal{S}$ if some open circle centered at X is entirely contained in $\mathcal{S}$. The union of all interior points of $\mathcal{S}$ is called the *interior* of the set $\mathcal{S}$ and denoted $\mathsf{Int}(\mathcal{S})$. Further we say that $Y \in \mathbb{R}^2$ is a *point of closure* of $\mathcal{S}$ if any open circle centered at Y contains a point of $\mathcal{S}$. The union of all points of closure of $\mathcal{S}$ is called the *closure* of $\mathcal{S}$ and denoted $\mathsf{Cl}(\mathcal{S})$. The set of all points of closure of $\mathcal{S}$ not contained in the interior of $\mathcal{S}$ is called the *boundary* of the set $\mathcal{S}$ and denoted $\mathsf{Bd}(\mathcal{S})$, i.e. $\mathsf{Bd}(\mathcal{S}) = \mathsf{Cl}(\mathcal{S}) \setminus \mathsf{Int}(\mathcal{S})$.

Regular Open, Bounded, Star-Convex. The set $\mathcal{S}$ is *open* if it equals its interior, i.e. if $\mathcal{S} = \mathsf{Int}(\mathcal{S})$, and it is *regular open* if it equals the interior of its closure, i.e. if $\mathcal{S} = \mathsf{Int}(\mathsf{Cl}(\mathcal{S}))$. We say that a set $\mathcal{S} \subset \mathbb{R}^2$ is *bounded* if it is contained in some open circle with finite radius. Further, $\mathcal{S}$ is *convex* if for any pair of points $X, Y \in \mathcal{S}$ it contains all points on the straight line segment connecting X and Y. The set $\mathcal{S}$ is *star-convex* if there exists $A \in \mathcal{S}$ such that for any $X \in \mathcal{S}$ all points on the line segment connecting A and X are in $\mathcal{S}$. The star-convexity generalizes the notion of convexity as any non-empty convex set is also star-convex and the opposite is not true.

Lattice. Assume two linearly independent vectors λ_1, λ_2 in $\mathbb{R}^2$. The vector set $\Lambda = \{a_1\lambda_1 + a_2\lambda_2 \mid a_1, a_2 \in \mathbb{Z}\}$ is called the *(vector) lattice* with basis $\{\lambda_1, \lambda_2\}$. Elements of the lattice Λ are called Λ-*vectors*. A point set $\Pi \subset \mathbb{R}^2$ is called a *(point) lattice* if $\Pi = P + \Lambda$ for some point $P \in \mathbb{R}^2$ and a vector lattice Λ. A subset of a point lattice $\Pi = P + \Lambda$ is called a *set of* Λ-*points*. We omit the noun adjuncts "vector" and "point" when the full meaning is clear. The basis of a vector lattice is not unique: there are (infinitely) many bases generating the same lattice. For instance both $\{(1,0),(0,1)\}$ and $\{(0,1),(1,2)\}$ are bases of the vector lattice $\mathbb{Z}^2$. However, the absolute value of the determinant of the basis $\mathsf{det}(\lambda_1, \lambda_2)$ is invariant and, therefore, it is also denoted as $|\mathsf{det}(\Lambda)|$. The parallelogram demarcated by the vectors of the basis is called *fundamental parallelogram*. The area of the fundamental parallelogram and the number of integer points that its interior can contain equals $|\mathsf{det}(\Lambda)|$.

1.2 Definitions and Mathematical Results

I will define two key concepts: *selection body* and *selection polygon* with regard to a lattice Λ. The former is introduced first: a selection body is a maximal subset of $\mathcal{S}$ that does not include a Λ-vector and does not have a "bizarre" shape. The "bizarreness" is prevented by requiring the sets be bounded, regular open, and star-convex. Then the second key concept – selection polygon (a specific shape) – is introduced and investigated. The section culminates with two theorems showing that a set is a selection body if and only if it is a selection polygon. This provides an exhaustive geometrical characterization of selection bodies. Two concluding corollaries characterize selection bodies of more specific shapes: straight (to be defined) and convex. The theorems and along with the corollaries represent the main mathematical results of the paper. They are presented here without proofs, which could not be included due to space limitations. The main idea underlying the proofs is that selection bodies are prototiles of a lattice tiling.

Definition 1. *Consider a lattice Λ and a point set*[1] $\mathcal{S} \subset \mathbb{R}^2$. *We say that $\mathcal{S}$ is a* selection body *with respect to Λ if the following conditions hold:*

(i) $\mathcal{S}$ is bounded, regular open, and star-convex;
(ii) $\mathcal{S}$ does not contain a non-zero Λ-vector, i.e. for any $X, Y \in \mathcal{S}$ such that $X - Y \in \Lambda$ we have $X = Y$;
(iii) $\mathcal{S}$ is maximal with properties (i) and (ii), i.e. it has no proper superset satisfying both (i) and (ii).

Informally speaking, the first condition excludes weird properties (infinite parts, strange boundaries, or disconnectedness) of the selection bodies. The second condition is crucial as it reflects our music-theoretical considerations. We will be looking for selection bodies with respect to various comma lattices. The condition (ii) ensures that the selection bodies will not include commas. Finally, the maximality will warrant completeness of the generated tone systems determined by the selection bodies.

Consider a lattice Λ with the basis $\{\lambda_1, \lambda_2\}$ and a point $X_0 \in \mathbb{R}^2$. Denote $X_1 = X_0 + \lambda_1$, $X_2 = X_0 + \lambda_2$, and $X = X_0 + \lambda_1 + \lambda_2$ and construct two open (i.e. excluding the boundary) triangles: the triangle t_0 with vertices X_0, X_1, and X_2 and the triangle t with vertices X, X_1, and X_2. Denote p the half-open parallelogram obtained as a union of t_0, t, and the boundary of t, i.e. $p = t_0 \cup t \cup \mathsf{Bd}(t)$. Further, consider any point Y_0 of the half-open parallelogram p, i.e. $Y_0 \in p$, and put $Y_1 = Y_0 - \lambda_1$ and $Y_2 = Y_0 - \lambda_2$. Finally, denote $\mathcal{S}$ the interior of the polygon $X_0Y_1X_2Y_0X_1Y_2$. (See Figure 1.) Then $\mathcal{S}$ is a selection body with respect to Λ. Furthermore, $\{X_0, X_1, X_2\}$ and $\{Y_0, Y_1, Y_2\}$ are (the only) three-element sets of Λ-points of $\mathsf{Cl}(\mathcal{S})$.

[1] The term "point set" refers to a set of points, rather than vectors. It should not be confused by the notion of "set of Λ-points". A selection body $\mathcal{S}$ is typically a continuous set of points and not a discrete set of lattice points.

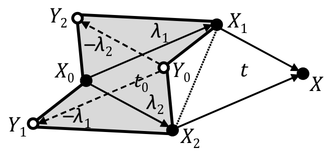

Fig. 1. Construction of a selection polygon

Definition 2. *Consider the notation and assumptions from previous paragraph. Then we say that $\mathcal{S}$ is a* selection polygon *with regard to lattice Λ (or, more precisely, with regard to the vector basis $\{\lambda_1, \lambda_2\}$). More specifically, we say that:*

(i) $\mathcal{S}$ *is a* fundamental parallelogram*, if* Y_0 *is a vertex of the triangle* t*, i.e.* $Y_0 \in \{X, X_1, X_2\}$*,*
(ii) $\mathcal{S}$ *is a* brick*, if* Y_0 *is an edge point of the triangle* t *but not its vertex, i.e.* $Y_0 \in \mathsf{Bd}(t) \setminus \{X, X_1, X_2\}$*,*
(iii) $\mathcal{S}$ *is a* honey-cell*, if* Y_0 *is an interior point of the triangle* t*, i.e.* $Y_0 \in t$*,*
(iv) $\mathcal{S}$ *is a* butterfly*, if* Y_0 *is an interior point of the triangle* t_0*.*

It is easy to see that the four conditions included in the definitions of the specific shapes are mutually disjoint and their union covers all possible cases. Therefore, any selection polygon corresponds to exactly one specific shape listed.

Definition 3. *Let $\mathcal{S}$ be a selection polygon (fundamental parallelogram, brick, honey-cell, or butterfly) and consider the following construction. Take an edge of $\mathcal{S}$, replace it by a continuous path that has no self-intersections and connects its vertices and replace the opposite edge with a corresponding translate of the path. Then do the same with the other two pairs of opposite edges. If the initial and terminal points of the the six replacing paths are their only pair-wise intersections and the resulting shape is star-convex then it is called* skewed *selection polygon (fundamental parallelogram, brick, honey-cell, or butterfly, respectively).*

See Figure 2 for illustrations of various straight and skewed selection polygons. We are ready to formulate the main mathematical results of the paper. Formally, any straight selection polygon is also a skewed selection polygon. Skewed selection polygons inherit the key property of straight selection polygons: they also are selection bodies. The first theorem, which provides a sufficient condition for a point set to be a selection body, formalizes this feature and also specifies the number of integer points included in (skewed) selection polygons. The second theorem states that being a skewed selection polygon is also a necessary condition for sets to be selection bodies. This way we obtain a complete geometrical characterization of selection bodies: they are exactly the selection polygons.

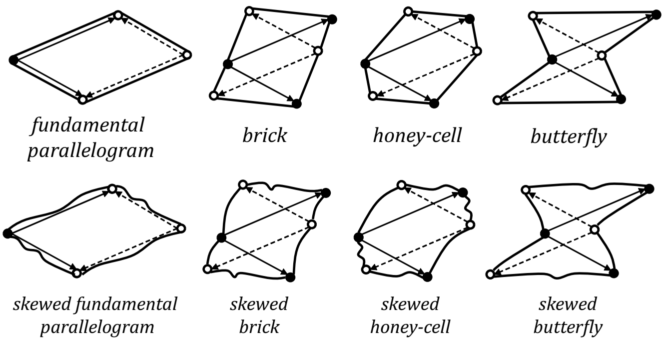

Fig. 2. (Straight) selection polygons and skewed selection polygons

Theorem 1 (Sufficient Condition). *Let $\mathcal{S} \subset \mathbb{R}^2$ be a skewed selection polygon with regard to lattice Λ. Then $\mathcal{S}$ is a selection body with regard to lattice Λ. Assume further that the boundary of $\mathcal{S}$ contains no integer points, i.e. $\mathsf{Bd}(\mathcal{S}) \cap \mathbb{Z}^2 = \emptyset$. Then the number of integer points included in $\mathcal{S}$ is exactly $|\mathsf{det}(\Lambda)|$.*

Theorem 2 (Necessary Condition). *Let $\mathcal{S} \subset \mathbb{R}^2$ be a selection body with regard to lattice Λ. Then $\mathcal{S}$ is a skewed selection polygon with regard to lattice Λ. Therefore, it has one of the following shapes: skewed fundamental parallelogram, skewed brick, skewed honey-cell, or skewed butterfly.*

Corollary 1. *Let $\mathcal{S} \subset \mathbb{R}^2$ be a selection body with respect to a lattice Λ. If $\mathcal{S}$ is straight then it has one of the following shapes: fundamental parallelogram, brick, honey-cell, or butterfly with respect to Λ.*

Corollary 2. *Let $\mathcal{S} \subset \mathbb{R}^2$ be a selection body with respect to a lattice Λ. If $\mathcal{S}$ is convex then it has one of the following shapes: fundamental parallelogram, brick, or honey-cell with respect to Λ.*

Convexity of a selection body implies that the selection body is straight. Therefore, the category of convex selection bodies is a subcategory of straight selection bodies, which in turn is a subcategory of (general) selection bodies. Corollary 2, which was derived here from a more general statement of Theorem 2, belongs to the folklore of the geometry of numbers.

2 Application to the Theory of Tone Systems

The theory of tone system selections from the *Tonnetz* presented here is, in a certain sense, a reconciliation between two models: Fokker's [8] theory of extended just-intonation systems based on his concept of "periodic meshes"[2] and Honingh's [11] empirical study of convexity and star-convexity of various tone systems.[3] The present model addresses different limitations of either of these models. Fokker's "periodic mesh" is a special case in my model: it is what I call bellow fundamental generated tone system (GTS). Fundamental GTS adheres to the restrictions imposed by the comma lattice the most. As a result, many tone systems,[4] especially those that are not convex (although they still are star-convex), cannot be modeled as "periodic meshes" and Fokker's model remains limited in its applicability. By introducing a more general concept, the present model removes this restriction. On the other hand, Honingh's empirical conclusions considered in isolation may be seen as too loose. She correctly observes and documents that many tone systems encountered in music are convex or star-convex on the *Tonnetz*. However, many star-convex and even convex selections from the *Tonnetz* bear little musical relevance. For instance, the collection of tones {C, C♯, F♯, G♭, C♭} is a star-convex selection. What is its musical-theoretical relevance? Even the strong condition of convexity does not bring us much further: both the seven-tone selection {C, E, F, F♯♯, G♯, A♭, B♯} and the twelve-tone selection {C, D♭, E♭, E, F♭, F, G, A♭, A, B♭♭, B, C♭} are convex. One can easily construct many other such "weird" (star-)convex selections. By incorporating the restrictions imposed by the comma lattice the present framework addresses this issue. Thus, it is capable of modeling a wider range of musically relevant systems than Fokker's "periodic meshes" can while it still prevents irrelevant convex or star-convex selections from entering the model.

Tones (intervals) are modeled here as the point (vector) lattice $\mathbb{Z}^2$: The integer point $[0,0]$ and the vectors $(1,0)$ and $(0,1)$ are interpreted as the tone D and the intervals of perfect fifth and major third, respectively. Therefore the point lattice $\mathbb{Z}^2$ is a model of the *Tonnetz* or *tone lattice*.[5]

[2] Fokker [9] extended his theory of two-dimensional periodic meshes, which are fundamental selections of the two-dimensional (5-limit) tone lattice, to three dimensional "period blocks", which are fundamental selections of the three-dimensional 7-limit tone lattice. The model presented in this paper sticks with the two-dimensional tone lattice although it could be naturally extended to higher-dimensional tone lattices. Erlich's text published on web [6] provides a more recent treatment of Fokker's theory. Interestingly, Erlich discusses also non-fundamental GTS's: selection bodies of honey-cell shapes with respect to the chromatic and diatonic comma lattices. To my knowledge, this is the only example of explicit use of selection polygons different from the fundamental parallelogram.

[3] Both Fokker's model and Honingh's empirical observations, and especially the former, provided much inspiration for the present model. All critical remarks that follow are intended as constructive criticism aiming at improving our scholarly understanding of musical reality.

[4] Any of the brick, honey-cell, or butterfly systems mentioned below, which are not fundamental, are examples of these.

[5] On the most general level, all musical intervals are modeled here as elements of a free commutative algebra freely generated by perfect fifth and major third. In other words, perfect fifth and major third are linearly independent even over $\mathbb{R}$. The model is not preoccupied by the actual tuning of those intervals. Rather, it focuses on the structural features underlying the systems of tones and intervals.

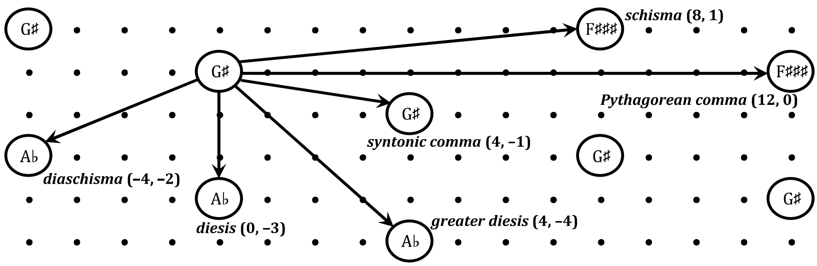

Fig. 3. Chromatic comma lattice X and some of its commas

The backbone of the model is formed by the *comma lattices*. They are special sublattices of the vector lattice $\mathbb{Z}^2$, generated by two vectors called *commas*. The term "comma" has been used in the theories of tunings since millennia referring to various "small" intervals obtained as a difference between two ways of tuning the "same" tone. Figure 3 depicts some of frequently theorized commas in the context of tuning; the Pythagorean comma (PC) and the syntonic comma (SC) being the most well-known. Various combinations of two of the six commas are bases of a lattice highlighted on Figure 3. We will call it the *chromatic comma lattice* X (the uppercase Greek letter "chi").[6] For instance, PC and SC form one of the bases of the lattice X.

In the previous paragraph we introduced the concept of comma vaguely as a kind of a "small" interval. In fact, there is an objective way of identifying appropriate bases of comma lattices. If the vertices of the parallelogram demarcated by a pair of vectors anchored at the tone lattice form a cluster of four tones that, in the pitch domain, is not disturbed by other tones inside the parallelogram or on its edges (we say that the pair of commas is *tight*) then the resulting selections of tones are tone systems of very special structural properties [23]. This topic will not be addressed any deeper in this paper. However, all comma lattice bases considered below are tight.

Given a comma lattice, we introduce the concept of a *generated tone system* (*GTS*): it is a selection of tones from the *Tonnetz* determined by a selection body. In line with the mathematical section, selection body is required to be star-convex (i.e. some point connects to all selection points), contain no comma interval (i.e. no linear combination of the comma basis), and be maximal with these properties (i.e. it cannot be extended without some comma interval entering the set). To avoid singular cases, in addition it is required that there be no tones on the boundary of the selection body. In that case the number of tones in the selection body equals the determinant of the comma lattice (second statement of Theorem 1). In the mathematical section we also learnt what are all the possible shapes of such

[6] Chromatic comma lattice and construction of chromatic GTS's below are directly related to Noll's notions of "Kommamodul" and "enharmonische Projektion" [17, chapter III.4]. His results also appear in the Chapter 24 of Mazzola's *opus magnum* [15] where the aforementioned concepts appear as "*CommaZModule*" and "enharmonic projection".

selection bodies. They are skewed selection polygons, of which there are four types: fundamental parallelogram, brick, honey-cell, and butterfly. Two subcategories of selection bodies were also introduced: straight and convex. The straight selection bodies are: (straight) fundamental parallelogram, brick, honey-cell, and butterfly. The convex selection bodies are the first three of them.

The following subsections provide selective illustrations of generated tone systems with regard to various comma lattices. I will call the comma lattices with a determinant greater/lesser then twelve *superchromatic/subchromatic*. Only straight selection bodies will be considered.

2.1 Chromatic Systems

Any selection body with regard to the chromatic comma lattice X is a just-intonation (JI) system of twelve chromatic tones. Honingh [11, p. 69] investigated to what extent early JI systems exhibit the properties of convexity and star-convexity. Among her test sets were the twelve-tone JI systems mentioned by Barbour in his classical text on the history of tunings [1]. She demonstrated that all these systems are star-convex and all but three are convex. Above, I have argued that the model presented here improves Honingh's approach by introducing more restrictions. Here a question arises whether such restrictions do not result in losing the ability to model all relevant systems. The answer is "no": even the more restrictive model of GTS's accommodates all of them.

System	**Selection body**	**Comma basis**
Ramis' monochord	fundamental	{SC, diaschisma}
The Erlangen monochord	brick	{SC, diaschisma}
Erlangen monochord revised	brick	{SC, diaschisma}
Fogliano's monochord, no. 1	fundamental	{SC, diesis}
Fogliano's monochord, no. 2	brick	{SC, –diesis}
Agricola's monochord	brick	{SC, diaschisma}
De Caus's monochord	fundamental	{SC, diesis}
Kepler's monochord, no. 1	brick	{SC, diesis}
Kepler's monochord, no. 2	brick	{SC, diesis}
Mersenne's spinet tuning, no. 1	fundamental	{SC, diesis}
Mersenne's spinet tuning, no. 2	butterfly	{SC, –diaschisma}
Mersenne's lute tuning, no. 1	butterfly	{SC, –diaschisma}
Mersenne's lute tuning, no. 2	brick	{SC, diesis}
Marpurg's monochord, no. 1	fundamental	{SC, diesis}
Marpurg's monochord, no. 3	butterfly	{SC, greater diesis}
Marpurg's monochord, no. 4	honey-cell	{SC, –diesis}
Malcolm's monochord	honey-cell	{SC, diesis}
Rousseau's monochord	brick	{SC, –diesis}
Euler's *speculum musicum*	fundamental	{SC, diesis}
Montvallon's monochord	brick	{SC, diesis}
Romieu's monochord	brick	{SC, diesis}

Fig. 4. Twelve-tone JI systems from Barbour [1] modeled as GTS's

The table shown in Figure 4 lists all JI systems from Barbour/Honingh's test set. As we see all the systems are not only (star-)convex but they are also straight GTS's (i.e. they have the property of not including any comma from the chromatic lattice and can be selected by straight selection polygons). Five comma bases repeat in the table and all of them contain the syntonic comma: combined with diesis (eleven systems), diaschisma (four systems), negative diesis (three systems), negative diaschisma (two systems), and greater diesis (one system). We see all types of selection polygons: six fundamental polygons, ten bricks, two honey-cells, and three butterflies (which are the three non-convex systems). One of the selections repeats in the table three times: De Caus's monochord, Mersenne's spinet tuning no.1, and Euler's *speculum musicum* are equivalent. Similarly, Mersenne's second spinet and first lute tuning are also structurally equivalent. As an illustration of the geometrical details of the construction, Figure 5 shows detailed diagrams of selection bodies and commas for some of the systems from Figure 4.

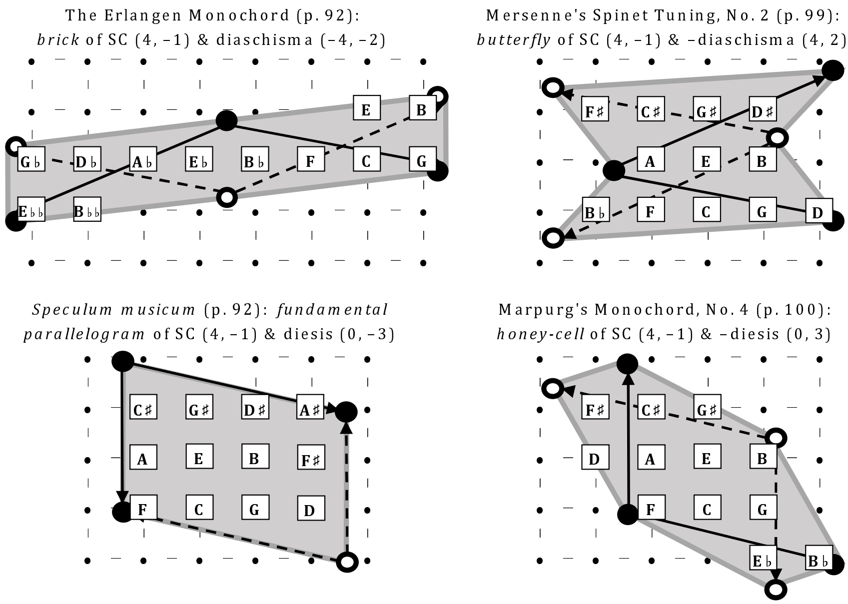

Fig. 5. Selection bodies of some JI systems mentioned by Barbour [1]

Obviously, comma basis is not unique for a given selection body. In general, any selection body $\mathcal{S}$ with regard to the comma basis $\{\kappa_1, \kappa_2\}$ is also a selection body with regard to the opposite comma basis $\{-\kappa_1, -\kappa_2\}$. Furthermore, in the case of a fundamental parallelogram, $\mathcal{S}$ is also a fundamental parallelogram with regard to $\{e_1\kappa_1, e_2\kappa_2\}$ where e_1 and e_2 are any combination of 1 and -1. In the case of a brick or a honey-cell, one can replace the basis $\{\kappa_1, \kappa_2\}$ by either of the two other bases delimiting the same triangle: $\{-\kappa_1, \kappa_2 - \kappa_1\}$ or

$\{\kappa_1 - \kappa_2, -\kappa_2\}$. Therefore, for instance, all bricks of {SC, diesis}, of which there are five in Figure 4, are also bricks of {–SC, diaschisma} or {SC, –diaschisma} as diaschisma equals diesis minus SC.

It is also obvious that selection bodies are not uniquely determined by a given selection of tones. Moreover, selection bodies of different types (fundamental parallelograms, bricks, honey-cells or butterflies) can select the same collection of tones from the *Tonnetz*. However, the four types of shapes can be ordered from (geometrically) the most restrictive fundamental parallelogram, through less restrictive brick, through even less restrictive honey-cell to the least restrictive butterfly. This ordering is given by the following reasoning. A fundamental parallelogram can easily be turned into brick or honey-cell without changing the selection of tones and, similarly, any selection given by a brick can also be selected by a honey-cell and a butterfly. At the same time, butterflies are the only non-convex types of selection bodies and this makes them (geometrically) least favorable. Therefore, in the table in Figure 4 the systems are marked as butterflies if they cannot be selected by any other shape, as honey-cells if they cannot be selected by bricks or fundamental parallelograms, and as bricks if they cannot be selected by fundamental parallelograms. Finally, whenever a selection can be made by a fundamental parallelogram this selection body is preferred as it is the most restrictive of all four types.

2.2 Superchromatic Systems

The theories of extended JI systems provide ample examples of GTS's with regard to superchromatic comma lattices. Consider the comma lattice K_{53} with the basis consisting of schisma $(8, 1)$ and kleisma $(-5, 6)$. Kleisma is a comma advocated by Tanaka [19] and it is not present in the chromatic lattice shown in Figure 3. The determinant of this comma lattice is $\mathsf{det}(\mathsf{K}_{53}) = 53$. Kleisma and schisma is also a "tight" pair of commas, in the sense described above. Tanaka's JI system of 53 tones is a GTS selected by a fundamental parallelogram with regard to this pair of commas. Oettingen's [18] 53-tone system is a related GTS with brick-shaped selection body. Both Tanaka's and Oettingen's elaborate analyses bear a deeper conceptual relation to the model presented here. See for instance Tanaka's diagram of lattice tiling by a straight fundamental parallelogram [19, p. 13] or Oettingen's depictions of lattice tiling by skewed (interpretable also as straight) brick [18, pp. 187, 195], both with regard to the comma lattice K_{53}.

Fokker's [8] theory of JI systems based on the concept of "periodic mesh" generalized Tanaka's model (without an explicit reference). Fokker described 12-, 19-, 22-, 31-, 41-, and 53-tone systems. These systems are all GTS's: one of them is selected by a honey-cell and all others are selected by fundamental parallelograms. Table in Figure 6 lists all Fokker's systems and also Tanaka and Oettingen's 53-tone systems. Oettingen's system is an example of a non-fundamental GTS with regard to a superchromatic comma lattice. For completeness, Fokker's two chromatic systems are also included in the table: the no. 1 is equivalent to Malcolm's and the no. 2 to Ramis' monochords from Figure 4. As we see, all

System	Selection body	Comma basis
Fokker's 12-tone no. 1, p. 255	honey-cell	{SC, diesis}
Fokker's 12-tone no. 2, p. 256	fundamental	{SC, diaschisma}
Fokker's 19-tone no. 1, p. 256	fundamental	{SC, kleisma + SC}
Fokker's 19-tone no. 2, p. 257	fundamental	{SC, kleisma}
Fokker's 22-tone no. 1, p. 258	fundamental	{diaschisma, kleisma + SC}
Fokker's 22-tone no. 2, p. 259	fundamental	{diaschisma, kleisma + schisma}
Fokker's 31-tone no. 1, p. 260	fundamental	{SC, kleisma + schisma}
Fokker's 31-tone no. 2, p. 260	fundamental	{SC, diaschisma – kleisma}
Fokker's 41-tone no. 1, p. 261	fundamental	{schisma, kleisma + SC}
Fokker's 41-tone no. 2, p. 262	fundamental	{schisma, kleisma + PC}
Fokker's 53-tone no. 1, p. 263	fundamental	{schisma, kleisma}
Fokker's 53-tone no. 2, p. 264	fundamental	{schisma, kleisma + schisma}
Tanaka's 53-tone, p. 13	fundamental	{schisma, kleisma}
Oettingen's 53-tone, p. 176	brick	{schisma, kleisma + schisma}

Fig. 6. Tanaka's [19], Oettingen's [18], and Fokker's [8] extended JI systems as superchromatic GTS's

superchromatic systems have bases consisting of one chromatic comma and one linear combination of kleisma with a chromatic comma.

Fokker's 53-tone system no. 1 has the same basis as Tanaka's system and both are selected by the fundamental parallelogram. However, they are not equivalent. Fokker's system is point-symmetric while Tanaka's is not. In fact, Fokker constructs all systems as point-symmetric at the tone D. For the systems where point-symmetry is not possible (systems of even cardinality) he either gives two alternatives for one tone (the 12-tone no. 1 and both 22-tone systems) or refrains from centering the system around a tone (a single case: the 12-tone no. 2). The number of GTS's selected by a fundamental parallelogram equals the cardinality of the system (i.e. the lattice determinant) if the coordinates of the commas in the basis are relatively prime.[7] Thus, Fokker's 53-tone system no. 1 and Tanaka's system are only two out of a total of 53-tone systems selected by the same fundamental parallelogram with regard to the basis consisting of schisma and kleisma.

2.3 Subchromatic Systems

Previous subsections clearly demonstrate the importance of selection bodies with regard to the chromatic comma lattice and superchromatic comma lattices, especially for the theory of microtonality. However, the present model has a wider field of applicability. GTS's of subchromatic comma lattices lead to analytical models applicable to various repertoires of music based on various subcollections of the standard system of 12 chromatic tones. Although it is not possible

[7] More precisely, the number of possible non-equivalent selections by the fundamental parallelogram with regard to the basis $\{(\kappa_1, \kappa_2), (\lambda_1, \lambda_2)\}$ equals $\frac{|\det((\kappa_1,\kappa_2),(\lambda_1,\lambda_2))|}{\gcd(\kappa_1,\kappa_2)\gcd(\lambda_1,\lambda_2)}$.

to demonstrate this point due to space limitations here fully, a small number of illustrations are given to indicate the potential of the theory in this area.

The basis of the *triadic comma lattice* T_3 consists of chromatic semitone $(-1, 2)$ and diatonic semitone $(-1, -1)$. The fundamental parallelogram with regard to this particular basis selects three types of chords: minor triad, major triad, and augmented triad. It means that if we move the fundamental parallelogram to any position against the *Tonnetz* it always selects only one of these three basic types of chords.

As presented elsewhere [24], the graph depicting all GTS's obtained by translating the fundamental parallelogram alongside the commas, called the *fundamental graph*,[8] is an extension of Douthett's [5] famous parsimonious graph of triads called *Cube Dance*.[9] The fundamental graph is infinite as it does not invoke the enharmonic equivalence (for instance the G major triad is adjacent to the G augmented triad but it is not directly connected to the E♭ augmented triad). If, in a subsequent step, the enharmonic equivalence is imposed on the graph it becomes isomorphic with Douthett's *Cube Dance*.[10]

A similar situation is encountered with the *tetradic comma lattice* T_4 and its basis consisting of chromatic semitone $(-1, 2)$ and greater diatonic semitone $(3, -2)$. In this case, the fundamental parallelogram selects four types of seventh-chords: dominant, minor, half-diminished, and diminished seventh-chords. The fundamental graph is a non-enharmonic extension of Douthett's *Power Towers*, the famous parsimonious graph of seventh-chords.

There are other (tight) bases of both T_3 and T_4 and GTS's determined by them and their relations reflect interesting properties of structures found in Western tonal music. Now, let us consider the *diatonic comma lattice* Δ. Table in Figure 7 lists three different bases of Δ, which I call *hiatal*, *octatonic*, and *whole-tone* comma bases. There exist also other (tight) bases of Δ (e.g. chromatic semitone and syntonic comma) but these three involve the following variability: even when the enharmonic equivalence is imposed they contain non-equivalent fundamental GTS's. Their fundamental GTS's are shown in the table; in total they encompass eleven heptatonic scales. It means that if we take a fundamental parallelogram demarcated by the vectors of one of the three bases and put it anywhere against the *Tonnetz* it will select one of the eleven heptatonic scales as listed in the table.

In the twelve-tone universe there are eleven heptatonic scales exhibiting the property of *quasi maximal evenness* (*QME*): the spectrum of any generic interval

[8] More precisely, the fundamental graph is obtained in the following way. We move the fundamental parallelogram alongside the commas in the basis. Selected GTS's (in this case triads) are the nodes of the graph and the edges connect GTS's that immediately follow one another in one of the comma directions.

[9] Waller's [21] graph-theoretical approach to representing relations among triads is an early harbinger of Douthett's work.

[10] In other words, *Cube Dance* is a homomorphic image of the fundamental graph of the triadic comma lattice T_3.

Comma basis	Fundamental GTS's – heptatonic scales
hiatal: $\{(3,1),(-1,2)\}$	diatonic, harmonic minor, harmonic major, Neapolitan minor (harm. minor with ♭2), Neapolitan major (major with ♭2), Hungarian
octatonic: $\{(-1,2),(-5,3)\}$	diatonic, harmonic minor, harmonic major, acoustic, sub-octatonic major (acoustic with ♭2), sub-octatonic minor (harm. minor with ♭5)
whole-tone: $\{(-5,3),(-9,4)\}$	diatonic, acoustic, super-whole-tone (whole-tone with an added tone), pseudo whole-tone (whole-tone with an enharmonically duplicated tone)

Fig. 7. Three bases of the diatonic comma lattice Δ (first column). The fundamental parallelograms with regard to these bases select the heptatonic scales listed in the second column.

equals or is a subset of a set of three consecutive integers.[11] Interestingly, there is an exact overlap between the eleven trivalent scales and eleven fundamental GTS's of Δ. A very close overlap is found also with Hook's [13] selection of those "spelled heptachords" for which he decided to provide a specific name.[12] The intersection of the three collections of the fundamental GTS's contains a single element: the diatonic scale. Except for the diatonic one, the pair-wise intersections contain the following scales: harmonic minor and harmonic major (the hiatal and the octatonic systems) and acoustic (the octatonic and the whole-tone systems). These scales (diatonic, harmonic minor, harmonic major, and acoustic) are exactly the seven-note "Pressing scales" that lie at the core of Tymoczko's [20] scale theories. The fundamental graphs of the three bases of the diatonic comma lattice provide a generalization of Douthett's parsimonious graphs to seven-tone collections and are powerful analytical tools applicable to scale based repertoires.

The final example illustrates non-fundamental GTS's in the tetradic comma lattice T_4. Figure 8 shows that both the dominant seventh chord with augmented fifth $\mathrm{G}^{7/5\sharp}$ and the dominant seventh chord with diminished fifth $\mathrm{G}^{7/5\flat}$ (or the French augmented sixth chord on D♭) are butterfly GTS's of T_4 with regard to the bases consisting of chromatic semitone $(-1,2)$ combined with whole tone $(2,0)$ and hiatus $(1,2)$, respectively.As we see, rarer musical structures are

[11] In their seminal 1991 paper, Clough and Douthett defined maximal evenness through the following property: "the spectrum of each dlen is either a single integer or two consecutive integers" [4, p. 96]. In this context, my definition of QME is a natural generalization of Clough and Douthett's original notion of maximal evenness. QME is also related to the property of "trivalence" as defined by Clampitt [3] as it also limits the number of specific sizes for generic intervals to three.

[12] He did not provide specific reasons for his decision to give a specific name to only 12 out of a total 66 translation classes of "spelled heptachords". The selection presented here reflects the overall formal framework. – The overlap between Hook's named spelled heptachords and my diatonic fundamental GTS's is not perfect, though: Hook does not consider the pseudo whole-tone scale while my framework does not include Hook's super-hexatonic scales.

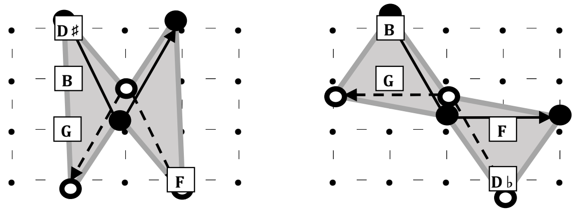

Fig. 8. $G^{7/5\sharp}$ and $G^{7/5\flat}$ as butterflies of the tetradic comma lattice T_4 with the bases $\{(-1,2),(1,2)\}$ and $\{(-1,2),(2,0)\}$, respectively

modeled through less regular selection polygons. Oddly, these two chords are not similarly accommodated in Honingh's model. While the 7/5♯ chord is categorized expectedly as non-convex but still star-convex (line 9 of the Table 4.4 on p. 91 of [11]) the 7/5♭ (and so the French augmented sixth) are the only exceptions to the star-convexity hypothesis: Honingh lists them as non-star-convex (lines 11 and 5, respectively). How is it possible that Honingh finds the 7/5♭ non-star-convex while it is selected by a butterfly, i.e. a star-convex selection polygon, in my model? The reason is that instead of the star-convexity in $\mathbb{R}^2$, which is considered here, Honingh introduced a discrete version of star-convexity: a set of integer points in $\mathbb{Z}^2$ is (discrete) star-convex if it contains an integer point such that all integer points on the lines connecting this point with any point of the set are included in the set [11, pp. 81–82]. In this sense, there is a huge difference between $G^{7/5\sharp}$ and $G^{7/5\flat}$ in Honingh's approach: G connects to all other tones in the former while it (or any of the other tones) does not in the latter (B♭ is on the line connecting G and D♭). This way, the (discrete) star-convexity causes odd conclusions of Honingh's model for certain kinds of tone systems. The model of selection based on selection bodies as presented here remedies such weak points.

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