Strong Connectivity of Wireless Sensor Networks with Double Directional Antennae in 3D

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Abstract. Using directional antennae in forming a wireless sensor network has many advantages over omnidirectional, including improved energy efficiency, reduced interference, increased security, and improved routing efficiency. We propose using double (Yagi) directional antennae in 3D space: for a given spherical angle such antennae transmit from their apex simultaneously directionally along two diametrically opposing cones in 3D. We study the resulting network formed by such directional sensors. We design a new algorithm to address strong connectivity of the resulting network and compare its hop-stretch factor with the three-dimensional omnidirectional model. We also obtain a lower bound on the minimum range required to ensure strong connectivity for sensors with double antennae. Further, we present simulation results comparing the diameter of a traditional sensor network using omnidirectional and one using directional antennae.

Keywords: Antennae, Diameter, Directional, Range, Sensor Network, Stretch Factor, Strong Connectivity, Yagi Antenna.

1 Introduction

Most studies of wireless sensor networks (WSNs) assume that sensors employ omnidirectional antennae to communicate. For sensors with identical transmission range, this has lead to the so-called UDG (Unit Disk Graph) model (also known as *protocol model*) whereby sensors are able to communicate with each other if and only if the distance between them is less than or equal to the transmission range of the two sensors. In this paper we consider sensors in 3D and adopt a different model, where the sensors use *directional antennae*. This offers many possible advantages such as: reduced energy consumption, lowered interference, tighter security, and improved routing efficiency. In 2D, the directional antenna model originates in the work of [2]. However our work is mainly motivated by the work in [4], which explored for the first time the use of single directional antennae in 3D space.

More specifically, we address *the orientation problem for strong connectivity* and the *stretch factor problem* for double directional antennae in three-dimensional space.

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Both problems are concerned with finding an orientation of the antennae. However, in the former case we merely want to ensure the resulting network is strongly connected, while in the latter for a given angle of the antenna we want to determine the minimum possible range (of the antenna) so as to guarantee that a directional c-hop-spanner is induced, for some constant $c > 0$. This guarantees that the length of shortest paths in the resulting directional graph are at most c times the length of shortest paths in the underlying sensor network of omnidirectional antennae.

Several communication models are possible depending on how sensors send (directional or omnidirectional transmissions) and receive (directional or omnidirectional receptions) messages. We adopt the model where transmissions occur directionally while receptions occur omnidirectionally. This is the simplest and most intuitive communication model but it has the additional benefit that it simplifies and illustrates the underlying complexity issues of the resulting directed graph.

Although our work presents mainly theoretical results, we note that directional antennae have proven in real-world applications and their use is emerging in various settings where energy, interference, security, etc, are of primary concern (see [2] and [5] for extensive discussions and references concerning these issues). We also note that papers such as [1,8] discuss electronic beam steering, which relies on and uses the advantages of directional antennae in its implementation.

1.1 Preliminaries and Notation

In this section we define several concepts and ideas that will be used in the main results. Throughout this paper, we assume that sensors are located at points in 3D space. The closed ball centered at the point p with radius r is denoted by $\mathbb{B}[p,r]$. Given a set S of sensors, define the Unit Ball Graph $U := UBG(S)$ as the graph whose set of vertices consists of S and two vertices u, v are connected by an edge if and only if $d(u,v) \leq 1$. We denote the hop distance between two sensors u and v in a graph G by $d_G(u,v)$ (When the graph is easily understood from the context we omit the subscript G so as to simplify notation.) Our sensor network is formed from directional antennae which replace corresponding omnidirectional antennae. We measure the quality of the resulting graph of directional antennae with the stretch factor which compares the length of shortest paths in the two kinds of graphs. If G is the graph resulting by orienting the antennae on the set S of points, the stretch factor can be defined by $\tau_G(S) = \max_{\forall u,v \in P} \frac{d_G(u,v)}{d_U(u,v)}$, where $d_G(u,v), d_U(u,v)$ is the hop-distance between u and v in the graphs G, U, respectively. To measure the stretch factor of the resulting directional graph it will be convenient to define the concept of k-coverage as follows (see [6]). Define a *k-orientation* over a set S of sensors as the orientation of the sensors of S such that $\forall s \in S$: 1) $\mathbb{B}[s,1]$ is covered by S, and 2) $\forall p \in \mathbb{B}[s,1]$, the shortest path from s to a sensor covering p has length at most $k-1$. This means that those sensors which are reachable by 1 hop in the $UBG(S)$ are still reachable by a constant number of hops k in the new orientation.

The *solid angle* subtended by a surface is defined as the ratio between the area of the spherical cap and the square of the radius of the sphere of which it is part. It is usually denoted by Ω. The *apex angle* of a spherical cone, denoted by 2θ, is defined as the maximum planar angle between any two generatrices of the spherical cone.

The apex angle 2θ and the solid angle Ω are related by the following well-known relation due to Archimedes:

$$\Omega = 2\pi \cdot (1 - \cos\theta). \tag{1}$$

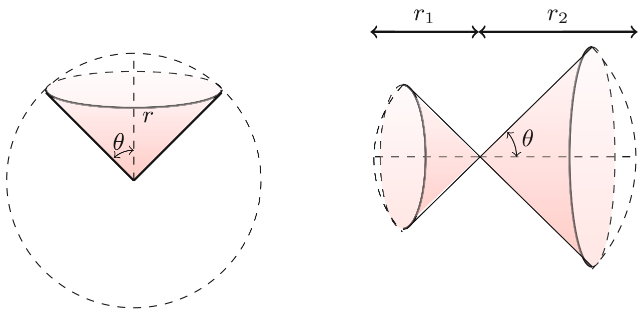


Fig. 1. Left: Illustration of a single antenna of apex angle 2θ and radius r. Right: Illustration of a Yagi double antenna of apex angle 2θ and radii r_1 and r_2.

A *single antenna* is modelled as a spherical cone characterized by its apex angle (which also determines the solid angle) and its range. A $(2\theta, r)$ *double antenna* is modelled as two spherical cones diametrically opposed, of apex angle 2θ and radii both equal to r. A more general model, is based on the *Yagi antenna*, in which the opposing spherical cones do not necessarily have the same range, formally denoted as $(2\theta, r_1, r_2)$ antenna, where r_1 and r_2 are the radii of the two cones of the antenna. A three-dimensional Yagi antenna is depicted in Figure 1. Given a set of points S in three-dimensional space that form a Unit Ball Graph U, the *optimal range* is defined as the length of the longest edge of the minimum spanning tree of U and is denoted by $r_{MST}(S)$. It is clear that any range lower than r_{MST} will fail to provide a strongly connected graph since the MST will be disconnected.

A packing problem that relates to the problems addressed in this paper is the *Tammes Radius* [10], which can be described as the maximum length r of the radius of n equal circles placed in the surface of a sphere, without overlapping and it is denoted by R_n. Another related problem is the *Kissing Number*, which can be described for the three-dimensional scenario as the maximum number of disjoint unit spheres such as every sphere is tangent to a given unit sphere [7].

1.2 Related Work

Many advantages of networks using directional antennae have been explored, and there are many works which examine the topological changes inherent to the use of directional antennae. A comprehensive survey can be found in [5].

A three-dimensional scenario was first adopted in [4] where a lower bound on the solid angle necessary to ensure strong connectivity with optimal range is calculated as $\frac{18\pi}{5}$. A proof is given showing that the problem of determining the existence of an orientation which achieves strong connectivity with optimal range is NP-complete for sensors of solid angle $\Omega < \pi$. An algorithm is also presented which finds such an orientation for antennae with beam width $\frac{18}{5}\pi \leq \Omega \leq 4\pi$. Finally, simulation results are provided which show how the stretch factor of the directional model compares with the omnidirectional model.

Other works closely related to our paper are [3] and [6]. The former presents algorithms to orient directional single antennae with constant stretch factor, while the latter deals with the connectivity problem and the stretch factor problem for double-antennae. Unlike our work, both of these papers refer to sensors in the plane.

1.3 Outline and Results of the Paper

In Section 2, we show how to orient a single antenna, when the solid angle Ω is greater than 2π. In Section 3, we show how to orient a double-antenna and show that the solid angle Ω for which it is possible to attain connectivity with optimal range is bounded below by $\Omega \geq \frac{25}{13}\pi$. In both sections, the orientations are achieved with constant stretch factor.

Table 1. Summary of results

Solid Angle	Antenna	Antenna Radius	Stretch Factor	Proof
$\Omega \geq 2\pi$	Single	$\max(1, 2\sin\theta)$	2	Theorem 1
$2\pi \leq \Omega < \frac{18\pi}{5}$	Single	$r_{MST}(S) \cdot \frac{\sqrt{\Omega\cdot(4\pi-\Omega)}}{\pi}$	N/A	[4]
$\Omega \geq \frac{18\pi}{5}$	Single	$r_{MST}(S)$	N/A	[4]
$\Omega \geq \frac{25\pi}{13}$	Double	r_{MST} (both)	N/A	Theorem 2
$\pi \leq \Omega < 2\pi$	Double	$4\cdot\sin(\frac{\pi}{4}+\theta)$ and 2	4	Theorem 3

In Section 4, we present the results of our simulation, which evaluated how the use of directional double-antenna impacts the diameter of the resulting graph, compared to the original UBG. Table 1 summarizes our results, along with other existing results.

2 Orienting Single Antennae with Constant Stretch Factor

In this section we show how to orient small groups of sensors with apex angle $2\theta \geq \pi$, so as to form a strongly connected directed graph with constant stretch factor.

Using [6][Lemmas 4 and 6] and the results presented in [4], it is possible to derive the following lemmas, for the three-dimensional case.

Lemma 1. *Given two sensors u, v, in three-dimensional space, with apex angle $2\theta > \pi$, if the Euclidean distance between them is δ, there exists a 2-orientation of u and v with transmission range* $\max\{1, \delta, \sqrt{1+\delta^2-2\delta\cos(2\theta)}\}$.

Proof. Orient each antenna in such a way that the straight line which connects u and v contains one edge of the apex angle of v, and vice-versa, one orientation being clockwise and the other one counterclockwise. The farthest point which lies in $\mathbb{B}[u, 1]$ must be covered by v is at a distance d equal to $\sqrt{1 + \delta^2 - 2\delta\cos(2\theta)}$ (by the law of cosines), or equal to δ, in the case of large values of 2θ. Furthermore, in the case of $\delta < 1$ and large values of 2θ, the distance d might be less than 1, but u and v need to cover points which are within a distance of 1 from itself and are not covered by the other node. Therefore, a range $r = \max\{1, \delta, \sqrt{1 + \delta^2 - 2\delta\cos(2\theta)}\}$ is required to ensure that $\mathbb{B}[u, 1]$ and $\mathbb{B}[v, 1]$ are covered. As u covers v directly, and vice-versa, this is a 2-orientation.

Lemma 2. *Given a set S of $n \geq 3$ sensors in three-dimensional space, with apex angle $2\theta > \pi$. Suppose there exists $s \in S$ such that the maximum distance between s and every other sensor in S is δ. If all the sensors in $S \setminus \{s\}$ are contained within a solid spherical sector centered at s with apex angle 2θ, then there is a 2-orientation of S with transmission range r =* $\max\{1, \delta, \sqrt{1 + \delta^2 - 2\delta\cos(2\theta)}\}$.

Proof. Consider sensor s and any sensor $t \in S \setminus \{s\}$. Due to Lemma 1, there is a 2-orientation of s and t with range $\max\{1, \delta, \sqrt{1 + \delta^2 - 2\delta\cos(2\theta)}\}$. Assume that t is on the edge of the apex angle of the spherical sector containing all the sensors $x \in S \setminus \{s\}$, i.e, t forms an angle of at least $2\pi - 2\theta$ with another sensor $u \in S \setminus \{s\}$, such that the spherical sector defined by t, u is empty. Since all the sensors $x \in S \setminus \{s\}$ exist in a spherical sector of apex angle 2θ, t certainly exists.

Therefore, there is an orientation of s and t so that they form a 2-orientation, and so that s covers every sensor $x \in S \setminus \{s\}$. Each of the other sensors x can be oriented to cover s and the portion of $\mathbb{B}[x, 1]$ not covered by s. By the law of cosines, the farthest point from s which is covered by it is at a distance $\max\{\delta, \sqrt{1 + \delta^2 - 2\delta\cos(2\theta)}\}$. Therefore, this is the range required to cover $\mathbb{B}[x, 1]$. The resulting orientation is strongly connected and any sensor can be reached in 2 hops, which means it is a 2-orientation.

Following [6][Lemmas 4 and 6], paper [6] presents the theorem which creates a connected graph with stretch factor $\tau_G(S) \leq 2$. The three-dimensional version uses the same ideas, since the lemmas are adaptable to three dimensions, as well as the algorithm for finding the convex hull, which is used in the proof.

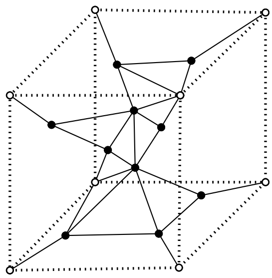
Theorem 1. *Given a a set of sensors S in three-dimensional space, each with one directional antenna of apex angle $2\theta \geq \pi$ and let $U(S) = UBG(S)$. Then there exists an antenna orientation of S with range* $\max\{1, 2\sin(\theta)\}$ *which creates a directed spanner G_θ with stretch factor $\tau_{G_\theta} \leq 2$.*

Proof. Define $C(G)$ as the set of the vertices from the union of the convex hulls of all connected components of a graph G. Let Q be a hierarchical structure defined as follows: $Q_0(V_0, E_0) = U(S)$ and $Q_{k+1}(V_{k+1}, E_{k+1}) = Q_k[V_k - C(Q_k)]$. In other words, at each iteration the components of the convex hull of each connected component are taken away, which means that every iteration is a proper subset of the previous one. Using this hierarchical structure, it is possible to prove by induction that an orientation can be found for the unit ball graph $U(S)$ on S. This is done by maintaining the invariant that in each iteration every sensor is either *locally convex* or *oriented*.

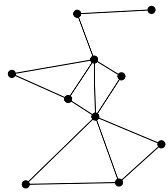
Definition 1 (Local convexity). *A sensor s is locally convex in a graph G if it is a member of the convex hull of the set of s and its 1-hop neighbours in G.*

Throughout the proof we will use the term *convex* interchangeably to mean locally convex. Since all sensors have apex angles $2\theta \geq \pi$, if a sensor s is convex, it is possible to orient s so that it covers all its neighbours in G. Consider now the iteration Q_i such that $Q_{i-1} \neq \emptyset$ and $Q_i = \emptyset$, i.e., the first iteration in which we get an empty set. Since Q_i is empty, Q_{i-1} must have only convex sensors - so the invariant holds immediately. The iteration Q_{i-2} is the first which may contain sensors which are not convex. We must orient these non-convex sensors in order to satisfy the invariant. We do so as follows: each non-convex sensor t requests to orient with one of its neighbours in $C(Q_{i-2})$ (t must have at least one such neighbour, otherwise it would have been non-convex in Q_{i-1})). Now, for each sensor in C(Q_{i-2}), there are three possibilities:

- No request to orient is received.
- A single request to orient is received: The convex sensor and the requestor orient themselves according to Lemma 1, forming a 2-orientation.
- Multiple requests to orient are received: The convex sensor and all requestors orient themselves according to Lemma 2, also forming a 2-orientation.



(a) $Q_0(V, E)$ and $C(Q_0)$



(b) After the convex hull is peeled off.

Fig. 2. One iteration of the construction of Q for a given $UBG(V)$. $C(Q_0)$ is denoted by hollow points.

Consider now any iteration Q_k. Assuming Q_{k+1} has a valid orientation, the sensors in Q_k either are convex, already oriented, or can be oriented as explained above, which creates a valid orientation for Q_k. As a valid orientation for the basis case Q_{i-1} was already shown, then by induction there is a valid orientation $G_\theta(S)$ for $Q_0 = U(S)$. The way this orientation was constructed always guaranteed that a sensor can reach one of its neighbours in $U(S)$ in at most 2 hops. As proved in [6][Lemma 1], when groups are merged, this property still holds. So, $G_\theta(S)$ is a connected graph, with $\tau_{G_\theta} \leq 2$. The detailed algorithm is as follows.

Algorithm 1. Orienting sensors with single antenna when apex angle is greater than π.

```
Q_0 ← UBG(S)
i ← 0
while Q_i ≠ Q_{i+1} do                         ▷ Will actually happen when they are empty sets
    Q_{i+1} ← Q_i[V_k − C(Q_k)]                ▷ Peel off the convex hull of previous iteration
end while
for all q ∈ Q do
    for all sensors ∈ q do
        if sensor is convex then
            continue
        else
            request_to_orient(sensor)      ▷ Sensor requests to orient with one of its neighbours
        end if
    end for
    for all sensors ∈ q do
        if sensor was requested to orient then
            orient(sensor)                     ▷ Orient according to Lemmas (1) or (2)
        end if
    end for
end for
```

The pseudocode for this orientation is given in Algorithm 1. The proof of Theorem 1 is now complete.

3 Orienting Double Antennae

In this section, we will show a lower bound on the solid angle of the antennae for which connectivity can be achieved with optimal range. An algorithm will also be presented for orienting double antennae with constant stretch factor.

Theorem 2. *Given a set S of points in three-dimensional space and a spherical angle $\Omega \geq \frac{25\pi}{13}$, there exists a polynomial time algorithm that computes a strong orientation of three-dimensional double-antennae of spherical sector with solid angle Ω and having optimal range.*

Proof. Consider a Euclidean minimum spanning tree (MST) T of S, in which $r_{MST}(S)$ is its longest edge. It will be determined how to orient the antennae at each point $p \in S$.

Let B_p be the sphere centered at each point p, with the minimum radius r_p such that all the neighbours of p in T are covered. This implies that $r_p \leq r_{MST}(S)$. For each neighbour u of p in T, let u' and u'' be the intersection points of B_p with the straight line containing the segment defined by u and p. Let $N_{B_p}(p)$ be the set of points projected on the surface of B_p. The maximum degree of a Euclidean MST is in general bounded by the *Kissing Number*, which means that in three dimensions it is bounded by 12. Since p has maximum degree 12 in T, $|N_{B_p}(p)| \leq 24$.

Let DT_p be the Delaunay Triangulation of $N_{B_p}(p)$ (in 3D) on the surface of B_p. The number of triangles of a complete triangulation of n points on a sphere is $2n - 4$. This

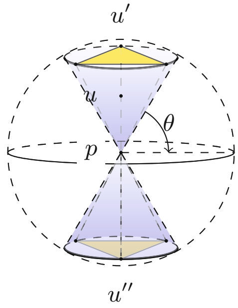


Fig. 3. Projections of three of the neighbours of p that form the greatest triangle. The projections of neighbours u, u' and u'', are highlighted.

can be proved easily by induction on the number n of points. Since $n \leq 24$ this gives at most $2 \cdot 24 - 4 = 44$ triangles. By the way the points were projected on B_p, we know that for each triangle there is a symmetric and identical triangle diammetrically opposed to it. Consider the two largest opposing triangles, t_p and t'_p (a tie can be broken arbitrarily). Orient the antennae at p with range r_p in such a way that the circles circumscribed to t_p and to t'_p are the only part not covered. This can be easily achieved by orienting each antenna towards the normal line to the straight line containing the centers of t_p and t'_p, that passes through the center of B_p.

To prove the lower bound on the solid angle at each point p, observe that, by the pigeonhole principle, the radius of the circumscribed circles of the greatest triangles will have length at least equal to the Tammes' Radius R_{44} [10] and the planar angle θ at the center of the sphere B_p is at least $\arcsin(R_{44})$. From [9], we find the optimal value for 2θ: $2\theta = 31.9834230°$. Therefore, the solid angle of the antennae can be calculated, using Archimedes' Equation as follows: $\Omega = 2\pi - 2\pi(1 - \cos(\theta)) = 2\pi(1 - (1 - \cos(\theta))) = 2\pi \cdot \cos(\theta) < \frac{25}{13}\pi$, where the last inequality is obtained after numerical calculation. It is easy to see that the resulting transmission graph is strongly connected, since T is connected and all edges of T are covered by exactly two antennae at opposite endpoints. This completes the proof of Theorem 2.

Next we study the necessary range to orient the network with constant stretch factor. The algorithm consists of maximally partitioning the vertices of the graph into triples, in such a way that at least one vertex is a neighbour of the other two in the UBG. After this step, it is necessary to determine the necessary range to orient the vertices in the triples to cover each other, as well as to cover vertices that are not part of any triple.

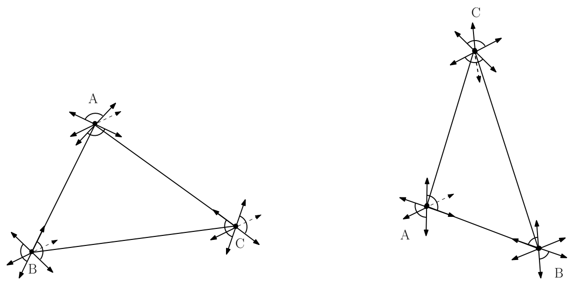
Lemma 3. *When maximally partiotioning the set of vertices of the UBG, the vertices which are not part of any triple are at distance at most 2 of the closest triple.*

Proof. Assume there is a vertex v which is at distance greater than 2. As the UBG is connected, there must be two more unmatched vertices, w and x that would connect v to the closest triple in the UBG. But if those exist, they could form a triple and the partition would not be maximal. Then, by contradiction, it is evident that v must be at distance of at most 2.

Next it will be shown that with infinite range, it is possible to orient antennae in each triple to cover the whole space.

Lemma 4. *Consider three points A, B, C in the space, forming a triangle. Three identical double antennae of apex angle* $2\theta \geq \pi/2$ *and infinite range can be oriented so as to cover the whole space.*

Proof. Consider the greatest angle of the triangle to be α. The orientation will depend on the value of α.



(a) $\alpha \leq 2\theta$ (b) $\alpha > 2\theta$

Fig. 4. Orientation of three double antennae with infinite range

(i) $\alpha \leq 2\theta$: Without loss of generality assume that BC is horizontal and A is above BC. Orient the antennae at as depicted in Figure 4(a) so that the antennae cover all three-dimensional subspace delimited by the triangle and its projections which are normal to the paper. It is easy to see that all the 3-dimensional space is covered.

(ii) $\alpha > 2\theta$: Without loss of generality, assume that AB is the second smallest edge in the triangle, AB is horizontal and C is above AB. Orient the antennae as depicted in Figure 4(b), so that the one antenna wedge of the apex angle of C is vertical and the wedge of the apex angle of the antennae at A and B are on AB. To prove the orientation covers the "whole space", observe that the antennae at A and B only leave uncovered the 3-dimensional subspace below the plane containing AB, that is normal to the paper. However, the antenna at C covers this subspace.

Lemma 5. *Let* A, B, C *be three points in the 3-dimensional space, such that* $d(A, B) \leq 1$ *and* $d(A, C) \leq 1$. *Assume that* $\frac{\pi}{2} \leq 2\theta \leq \pi$. *We can orient three* $(2\theta, r, 2)$*-double antennae of apex angle* 2θ *at* A, B, C *so that every point at distance at most two from one of these points is covered by at least one of the three antennae, where* $r \leq 4 \cdot sin(\frac{\pi}{4} + \theta)$.

Proof. Consider the balls $\mathbb{B}[A, 2], \mathbb{B}[B, 2], \mathbb{B}[C, 2]$, of radius 2 centered at A, B and C, respectively. Let $\mathcal{D} = \mathbb{B}[A, 2] \cup \mathbb{B}[B, 2] \cup \mathbb{B}[C, 2]$. Observe that each antenna covers one spherical sector of angle $\pi \leq \Omega \leq 2\pi$, relating to θ as described in Archimedes' Equation, with range two. It remains to prove that a range $r \leq 4 \cdot \sin(\frac{\pi}{4} + \theta)$ is sufficient to cover the remaining area. Two cases are to be considered:

(i) $\alpha \leq 2\theta$: Observe that the area of $\mathcal{D}$ that A covers is at most at distance three. Let us assume, without loss of generality, that $|AB| \leq |AC|$. Therefore, $\angle(BCA) \leq \angle(CBA) \leq \pi - 2\theta$. Hence, we only need to consider the case for C. Let P the farthest point of C in the coverage area $\mathcal{D}$. Observe that $\angle(PBC) = (\pi - 2\theta) + \angle(CBA)$. Therefore, a range of $4 \sin(\frac{\angle(PBC)}{2})$ is always sufficient to cover $\mathcal{D}$, since $|BC| \leq 2$ and $|BP| = 2$.

(ii) $\alpha > 2\theta$: From the orientation of Lemma 4, $\alpha = \angle(BAC)$ is the largest angle and $\beta = \angle(CBA)$ is the smallest angle in the triangle. Therefore, $\beta \leq \frac{\pi - 2\theta}{2}$. Observe that the farthest point of $\mathcal{D}$ that A covers is at distance three. Then, consider the farthest points P and Q in $\mathcal{D}$ from B and the farthest point R in $\mathcal{D}$ from C. Since the wedge of the apex angle of the antenna in C is vertical, $\angle(BCP) = \pi/2 + \beta \leq \pi - \theta$. Moreover, $\angle(QCB) = 2\pi - ((2\theta + \angle(BCP)) \leq 3\pi/2 - 2\theta$ and $\angle(CBR) = (\pi - 2\theta) + \beta \leq \frac{3}{2}(\pi - 2\theta)$. It is possible to determine algebraically that $\angle(CBR) \leq \angle(BCP) \leq \angle(QCB)$. Also, $|BQ| \leq 4 \cdot \sin(\frac{3\pi}{4} - \theta) = 4 \cdot \sin(\frac{\pi}{4} + \theta)$, since $|BC| = 2$ and $|CQ| = 2$. The lemma follows, since $4 \cdot \sin(\pi - 2\theta) \leq 4 \cdot \sin(\frac{\pi}{4} + \theta)$, for $\pi/4 \leq \theta \leq \pi/2$.

It is possible to orient the set of points that form a connected $UBG(P)$.

Theorem 3. *Given $\frac{\pi}{2} \leq 2\theta \leq \pi$, there is an algorithm which for any connected $UBG(P)$ on a set P of points in the space, orients $(2\theta, 4 \cdot sin(\frac{\pi}{4} + \theta), 2)$-double antenna so that the resulting graph is also connected and has stretch factor four. Furthermore, it can be done in linear time.*

Proof. Let $\mathcal{T}$ be any partition of the UBG with a maximal number of triples in such a way that the triangle defined by the three sensors has at least two edges of length at most one. This partition can be constructed in linear time, by selecting a node which is not yet in the partition and then trying to select two of its neighbours in a such a way that the criteria mentioned before is met. For any triangle T in $\mathcal{T}$, the antennae is oriented as shown in Lemma 5. For each sensor, the antenna covering an internal angle of the triangle have radius $4 \cdot \sin(\frac{\pi}{4} + \theta)$, while the opposite antenna have range 2. The remaining sensors, which are not in a triple, must be oriented towards its nearest triangle, which will be at distance at most two.

Let G be the directed spanner induced by the antennae. It is easy to see that it will be strongly connected, by the way the orientations are done. It remains to prove that for each edge u, v there is a directed path P from u to v and also a directed path from v to u of hop-length no more than 4 hops. Let T and T' be two differente triangles in the partition $\mathcal{T}$. We consider three cases, depending on the location of sensors u and v:

(i) $u, v \in T$: Then $|P| \leq 2$ and $|P'| \leq 2$;

(ii) $u \in T$ and $v \in T'$: Since $d(u, v) \leq 1$, v is in the coverage area of the triangle T. Therefore, v is reachable by u in at most three hops, which means $|P| \leq 3$. An analogous argument shows that $|P'| \leq 3$;

(iii) At least one of u and v is not unmatched, i.e., is not in any triangle of the partition. Assume without loss of genreality that u is unmatched. Observe that there exists a triangle $T \in \mathcal{T}$ at distance at most two from u. Otherwise, $\mathcal{T}$ would not be maximal. Therefore, u can reach v through T in at most four hops, i.e., $|P| \leq 4$. Similarly, we can prove that $|P'| \leq 4$.

4 Simulation Results

In this section we use simulation results to analyze how replacing omnidirectional antennae with directional antennae in three-dimensional space impacts the diameter of the graph. The diameter $D(G)$ of a graph G is defined as the length of the maximum shortest path between any two nodes of the graph. For each simulation, a random set of points S was generated and the corresponding UBG was constructed. If the UBG was not connected, the set of points was discarded and a new one was generated, until a connected UBG was obtained. A directed spanner of S was constructed using the algorithm from Theorem 3. The construction of the triples was executed in a greedy and random manner.

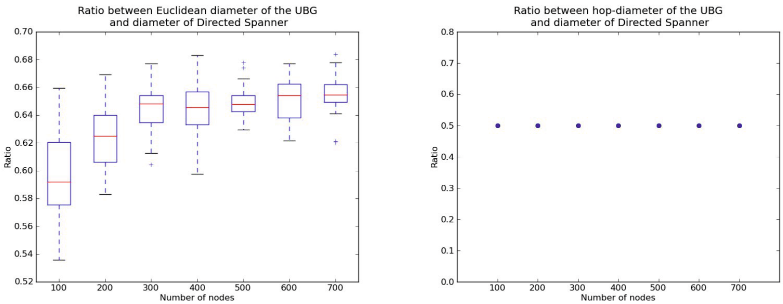


Fig. 5. Left: Boxplot comparing the Euclidean diameter of the UBG and the directed spanner, when varying the number of nodes. Right: Comparison of the hop diameter of the UBG and the directed spanner, when varying the number of nodes.

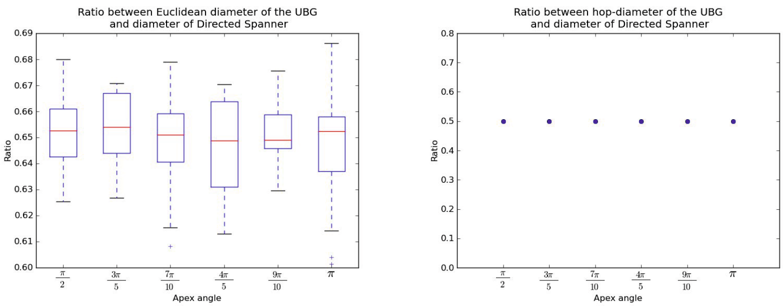


Fig. 6. Left: Boxplot comparing the Euclidean diameter of the UBG and the directed spanner, when varying the number of nodes. Right: Comparison of the hop diameter of the UBG and the directed spanner, when varying the apex angle.

We compared the hop-diameter of both graphs, as well as the Euclidean diameter. In the first simulation, the apex angle 2θ was fixed to $\frac{\pi}{2}$ and the number of nodes n

varied from 400 to 1000, in increments of 100. For each n, the simulation ran 30 times. Figure 5 shows the results. In the second simulation, the number of nodes was fixed to 500 and the apex angle varied from $\frac{\pi}{2}$ to π in increments of $\frac{\pi}{10}$. The simulation was run 30 times for each angle. Figure 6 shows the results. Both simulations show that the diameter of the directed spanner is in general smaller than the one of the UBG, with the hop-diameter of the directed spanner being half the diameter of the UBG. This advantage is most probably due to the increased range of communication present in the directed spanner.

5 Conclusion

We discussed how to orient single and doube antennae in three-dimensional space, and also observed with the simulations that the diameter of the directed spanner resulting by the use of directional antenna is in general smaller then the one of the original UBG. Several questions remain open. In addition to improving our results for single as well as double antennae (Table 1), another interesting question is concerned with how to orient sensors in three-dimensional space when each sensor is equipped with k antennae, $1 < k \leq 12$, as well as what the trade-offs between angle and range would be in these cases.

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