# On the Extension Complexity of Combinatorial Polytopes*

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**Abstract.** In this paper we extend recent results of Fiorini et al. on the extension complexity of the cut polytope and related polyhedra. We first describe a lifting argument to show exponential extension complexity for a number of NP-complete problems including subset-sum and three dimensional matching. We then obtain a relationship between the extension complexity of the cut polytope of a graph and that of its graph minors. Using this we are able to show exponential extension complexity for the cut polytope of a large number of graphs, including those used in quantum information and suspensions of cubic planar graphs.

## 1 Introduction

In formulating optimization problems as linear programs (LP), adding extra variables can greatly reduce the size of the LP [5]. However, it has been shown recently that for some polytopes one cannot obtain polynomial size LPs by adding extra variables [8, 12]. In a recent paper [8], Fiorini et.al. proved such results for the cut polytope, the traveling salesman polytope, and the stable set polytope for the complete graph $K_n$. In this paper, we extend the results of Fiorini et. al. to several other interesting polytopes. We do not claim novelty of our techniques, in that they have been used - in particular - by Fiorini et. al. Our motivation arises from the fact that there is a strong indication that NP-hard problems require superpolynomial sized linear programs. We make a step in this direction by giving a simple technique that can be used to translate NP-completeness reductions into lower bounds for a number of interesting polytopes.

* Due to space constraints many proofs have been omitted here. A full version containing all proofs is available at http://arxiv.org/abs/1302.2340

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*Cut polytope and related polytopes.* The cut polytope arises in many application areas and has been extensively studied. Formal definitions of this polytope and its relatives are given in the next section. A comprehensive compilation of facts about the cut polytope is contained in the book by Deza and Laurent [7]. Optimization over the cut polytope is known as the max cut problem, and was included in Karp's original list of problems that he proved to be NP-hard. For the complete graph with $n$ nodes, a complete list of the facets of the cut polytope $\mathrm{CUT}_n^{\square}$ is known for $n \le 7$ (see Section 30.6 of [7]), as well as many classes of facet producing valid inequalities. The hypermetric inequalities (see Chapter 28 of [7]) are examples of such a class, and it is known that an exponential number of them are facet inducing. Less is known about classes of facets for the cut polytope of an arbitrary graph, $\mathrm{CUT}^{\square}(G)$. Interest in such polytopes arises because of their application to fundamental problems in physics.

In quantum information theory, the cut polytope arises in relation to Bell inequalities. These inequalities, a generalization of Bell's original inequality [4], were introduced to better understand the nonlocality of quantum physics. Bell inequalities for two parties are inequalities valid for the cut polytope of the complete tripartite graph $\mathrm{K}_{1,n,n}$. Avis, Imai, Ito and Sasaki [2] proposed an operation named *triangular elimination*, which is a combination of zero-lifting and Fourier-Motzkin elimination (see e.g. [14]) using the triangle inequality. They proved that triangular elimination maps facet inducing inequalities of the cut polytope of the complete graph to facet inducing inequalities of the cut polytope of $\mathrm{K}_{1,n,n}$. Therefore a standard description of such polyhedra contains an exponential number of facets.

In [1] the method was extended to obtain facets of $\mathrm{CUT}^{\square}(G)$ for an arbitrary graph $G$ from facets of $\mathrm{CUT}_n^{\square}$. For most, but not all classes of graphs, $\mathrm{CUT}^{\square}(G)$ has an exponential number of facets. An interesting exception are the graphs with no $K_5$ minor. Results of Seymour for the cut cone, extended by Barahona and Mahjoub to the cut polytope (see Section 27.3.2 of [7]), show that the facets in this case are just projections of triangle inequalities. It follows that the max cut problem for a graph $G$ on $n$ vertices with no $K_5$ minor can be solved in polynomial time by optimizing over the *semi-metric polytope*, which has $O(n^3)$ facets. Another way of expressing this is to say that in this case $\mathrm{CUT}^{\square}(G)$ has $O(n^3)$ *extension complexity*, a notion that will be discussed next.

*Extended formulations and extensions* Even for polynomially solvable problems, the associated polytope may have an exponential number of facets. By working in a higher dimensional space it is often possible to decrease the number of constraints. In some cases, a polynomial increase in dimension can yield an exponential decrease in the number of constraints. The previous paragraph contained an example of this.

For NP-hard problems the notion of extended formulations also comes into play. Even though a natural LP formulation of such a problem has exponential size, this does not rule out a polynomial size formulation in higher dimensions.

In a groundbreaking paper, Yannakakis [13] proved that every symmetric LP for the Travelling Salesman Problem (TSP) has exponential size. Here, an LP is called *symmetric* if every permutation of the cities can be extended to a permutation of all the variables of the LP that preserves the constraints of the LP. This result refuted various claimed proofs of a polynomial time algorithm for the TSP. In 2012 Fiorini et al. [8] proved that the max cut problem also requires exponential size if it is to be solved as an LP. Using this result, they were able to drop the symmetric condition, required by Yannakakis, to get a general super polynomial bound for LP formulations of the TSP.

## 2 Preliminaries

We briefly review basic notions about the cut polytope and extension complexity used in later sections. Definitions, theorems and other results for the cut polytope stated in this section are from [7], which readers are referred to for more information. We assume that readers are familiar with basic notions in convex polytope theory such as convex polytope, facet, projection and Fourier-Motzkin elimination. Readers are referred to a textbook [14] for details.

Throughout this paper, we use the following notation. For a graph $G = (V, E)$ we denote the edge between two vertices $u$ and $v$ by $uv$, and the neighbourhood of a vertex $v$ by $\mathrm{N}_G(v)$. We let $[n]$ denote the integers $\{1, 2, ..., n\}$.

### 2.1 Cut Polytope and Its Relatives

The *cut polytope* of a graph $G = (V, E)$, denoted $\mathrm{CUT}^{\square}(G)$, is the convex hull of the cut vectors $\boldsymbol{\delta}_G(S)$ of $G$ defined by all the subsets $S \subseteq V$ in the $|E|$-dimensional vector space $\mathbb{R}^E$. The cut vector $\boldsymbol{\delta}_G(S)$ of $G$ defined by $S \subseteq V$ is a vector in $\mathbb{R}^E$ whose $uv$-coordinate is defined as follows: $\delta_{uv}(S) = 1$ if $|S \cap \{u, v\}| = 1$, and $\delta_{uv}(S) = 0$ otherwise, for $uv \in E$. If $G$ is the complete graph $\mathrm{K}_n$, we simply denote $\mathrm{CUT}^{\square}(\mathrm{K}_n)$ by $\mathrm{CUT}^{\square}_n$.

We now describe an important well known general class of valid inequalities for $\mathrm{CUT}^{\square}_n$ (see, e.g. [7], Ch. 28).

**Lemma 1.** *For any $n \geq 2$, let $b_1, b_2, ..., b_n$ be any set of $n$ integers. The following inequality is valid for $\mathrm{CUT}^{\square}_n$:*

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq \left\lfloor \frac{(\sum_{i=1}^{n} b_i)^2}{4} \right\rfloor \tag{1}$$

The inequality (1) is called *hypermetric* (respectively, of *negative type*) if the integers $b_i$ can be partitioned into two subsets whose sum differs by one (respectively, zero). A simple example of hypermetric inequalities are the triangle inequalities, obtained by setting three of the $b_i$ to be +/- 1 and the others to be zero. The most basic negative type inequality is non-negativity, obtained by setting one $b_i$ to 1, another one to -1, and the others to zero.

For any fixed $n$ there are an infinite number of hypermetric inequalities, but all but a finite number are redundant. This non-trivial fact was proved by Deza, Grishukhin and Laurent (see [7] Section 14.2) and allows us to define the *hypermetric polytope*, which we will refer to again later.

### 2.2 Extended Formulations and Extensions

An *extended formulation* (EF) of a polytope $P \subseteq \mathbb{R}^d$ is a linear system

$$Ex + Fy = g, \ y \geqslant \mathbf{0} \tag{2}$$

in variables $(x, y) \in \mathbb{R}^{d+r}$, where $E, F$ are real matrices with $d, r$ columns respectively, and $g$ is a column vector, such that $x \in P$ if and only if there exists $y$ such that (2) holds. The *size* of an EF is defined as its number of *inequalities* in the system.

An *extension* of the polytope $P$ is another polytope $Q \subseteq \mathbb{R}^e$ such that $P$ is the image of $Q$ under a linear map. Define the *size* of an extension $Q$ as the number of facets of $Q$. Furthermore, define the *extension complexity* of $P$, denoted by $\mathrm{xc}\,(P)$, as the minimum size of any extension of $P$.

In this paper we make use of the machinery developed and described in Fiorini et al. [8]. The reader is referred to the original paper for more details and proofs. The main result of Fiorini et al. [8] that we are interested in is the following

**Theorem 1 (Lower Bound Theorem).** $\mathrm{xc}(\mathrm{CUT}_n^{\square}) \geqslant 2^{\Omega(n)}$.

### 2.3 Proving Lower Bounds for Extension Complexity

We now note two observations that are useful in translating results from one polytope to another. Let $P$ and $Q$ be two polytopes. Then,

**Proposition 1.** *If $P$ is a projection of $Q$ then* $\mathrm{xc}\,(P) \leqslant \mathrm{xc}(Q)$.

**Proposition 2.** *If $P$ is a face of $Q$ then* $\mathrm{xc}\,(P) \leqslant \mathrm{xc}(Q)$.

Naturally there are many other cases where the conditions of neither of these propositions apply and yet a lower bounding argument for one polytope can be derived from another. However we would like to point out that these two propositions already seem to be very powerful. In fact, out of the three lower bounds proved by Fiorini et. al. [8] two (for TSP($n$) and STAB($n$)) use these propositions, while the lower bound on the cut polytope is obtained by showing a direct embedding of a matrix with high nonnegative rank in the slack matrix of $\mathrm{CUT}_n^{\square}$.

In the next section we will use these propositions to show superpolynomial lower bounds on the extension complexities of polytopes associated with four NP-hard problems.

## 3 Polytopes for Some NP-Hard Problems

In this section we use the method of Section 2.3 to show super polynomial extension complexity for polytopes related to the following problems: subset sum, 3-dimensional matching and stable set for cubic planar graphs. These proofs are derived by applying this method to standard reductions from 3SAT, which is our starting point.

***3SAT.*** For any given 3SAT formula $\Phi$ with $n$ variables in conjunctive normal form define the polytope $\mathrm{SAT}(\Phi)$ as the convex hull of all satisfying assignments. That is, $\mathrm{SAT}(\Phi) := \mathrm{conv}(\{x \in [0,1]^n \mid \Phi(x) = 1\})$. The following theorem and its proof are implicit in [8], who make use of the correlation polytope.

**Theorem 2.** *For every $n$ there exists a 3SAT formula $\Phi$ with $O(n)$ variables and $O(n)$ clauses such that* $\mathrm{xc}(\mathrm{SAT}(\Phi)) \geqslant 2^{\Omega(\sqrt{n})}$.

***Subset Sum.*** The subset sum problem is a special case of the knapsack problem. Given a set of $n$ integers $A = \{a_1, \ldots, a_n\}$ and another integer $b$, the subset sum problems asks whether any subset of $A$ sums exactly to $b$. Define the subset sum polytope $\mathrm{SUBSETSUM}(A, b)$ as the convex hull of all characteristic vectors of the subsets of $A$ whose sum is exactly $b$. That is, $\mathrm{SUBSETSUM}(A, b) := \mathrm{conv}\left(\{x \in [0,1]^n \mid \sum_{i=1}^n a_i x_i = b\}\right)$

The subset sum problem then is asking whether $\mathrm{SUBSETSUM}(A, b)$ is empty for a given set $A$ and integer $b$. Note that this polytope is a face of the knapsack polytope $\mathrm{KNAPSACK}(A, b) := \mathrm{conv}\left(\{x \in [0,1]^n \mid \sum_{i=1}^n a_i x_i \leqslant b\}\right)$

In this subsection we prove that the subset sum polytope (and hence the knapsack polytope) can have superpolynomial extension complexity.

**Theorem 3.** *For every 3SAT formula $\Phi$ with $n$ variables and $m$ clauses, there exists a set of integers $A(\Phi)$ and integer $b$ with $|A| = 2n + 2m$ such that* $\mathrm{SAT}(\Phi)$ *is the projection of* $\mathrm{SUBSETSUM}(A, b)$.

*Proof.* Suppose formula $\Phi$ is defined in terms of variables $x_1, x_2, ..., x_n$ and clauses $C_1, C_2, ..., C_m$. We use a standard reduction from 3SAT to subset sum (e.g., [6], Section 34.5.5). We define $A(\Phi)$ and $b$ as follows. Every integer in $A(\Phi)$ as well as $b$ is an $(n+m)$-digit number (in base 10). The first $n$ bits correspond to the variables and the last $m$ bits correspond to each of the clauses. $b_j = 1$, if $1 \leqslant j \leqslant n$ and $b_j = 4$, if $n + 1 \leqslant j \leqslant n + m$.

Next we construct $2n$ integers $v_i, v_i'$ for $i \in \{1, \ldots, n\}$, and $2m$ integers $s_i, s_i'$ for $i \in \{1, \ldots, m\}$ as follows: $v_{ij} = 1$, if $j = i$ or $x_i \in C_{j-n}$ and 0, otherwise, $v_{ij}' = 1$, if $j = i$ or $\overline{x}_i \in C_{j-n}$ and 0, otherwise. $s_{ij} = 1$, if $j = n + i$ and 0 otherwise, $s_{ij}' = 2$, if $j = n + i$ and 0 otherwise.

We define the set $A(\Phi) = \{v_1, \ldots, v_n, v_1', \ldots, v_n', s_1, \ldots, s_m, s_1', \ldots, s_m'\}$.

Consider the subset-sum instance with $A(\Phi), b$ as constructed above for any 3SAT instance $\Phi$. Let $S$ be any subset of $A(\Phi)$. If the elements of $S$ sum exactly to $b$ then it is clear that for each $i \in \{1, \ldots, n\}$ exactly one of $v_i, v_i'$ belong to $S$. Furthermore, setting $x_i = 1$ if $v_i \in S$ or $x_i = 0$ if $v_i' \in S$ satisfies every clause. Thus the characteristic vector of $S$ restricted to $\{v_1, \ldots, v_n\}$ is a satisfying assignment for the corresponding SAT formula.

Also, if $\Phi$ is satisfiable then the instance of subset sum thus created has a solution corresponding to each satisfying assignment: Pick $v_i$ if $x_i = 1$ or $v_i'$ if $x_i = 0$ in an assignment. Since the assignment is satisfying, every clause is satisfied and so the sum of digits corresponding to each clause is at least 1. Therefore, for a clause $C_j$ either $s_j$ or $s_j'$ or both can be picked to ensure that the sum of the corresponding digits is exactly 4. Note that there is unique way to do this.

This shows that every vertex of the subset sum polytope SUBSETSUM$(A(\Phi), b)$ projects to a vertex of SAT$(\Phi)$ and every vertex of SAT$(\Phi)$ can be lifted to a vertex of SUBSETSUM$(A(\Phi), b)$,. The projection is defined by dropping every coordinate except those corresponding to the numbers $v_i$ in the reduction described above. The lifting is defined by the procedure in the proceeding paragraph. Hence, SAT$(\Phi)$ is a projection of SUBSETSUM$(A(\Phi), b)$. □

Combining the preceding two theorems we obtain the following.

**Corollary 1.** *For every natural number* $n \geqslant 1$, *there exists an instance* $A, b$ *of the subset-sum problem with* $O(n)$ *integers in* $A$ *such that* $\mathrm{xc}(\mathrm{SUBSETSUM}(A, b)) \geqslant 2^{\Omega(\sqrt{n})}$.

***3d-Matching.*** Consider a hypergraph $G = ([n], E)$, where $E$ contains triples for some $i, j, k \in [n]$ where $i, j, k$ are distinct. A subset $E' \subseteq E$ is said to be a 3-dimensional matching if all the triples in $E'$ are disjoint. The 3$d$-matching polytope 3DM$(G)$ is defined as the convex hull of the characteristic vectors of every 3$d$-matching of $G$. That is, $\mathrm{3DM}(G) := \mathrm{conv}(\{\chi(E') \mid E' \subseteq E \text{ is a } 3d\text{-matching}\})$

The 3d-matching problem asks: given a hypergraph $G$, does there exist a 3d-matching that covers all vertices? This problem is known to be $NP$-complete and was one of Karp's 21 problems proved to be $NP$-complete [9, 11]. Note that this problem can be solved by linear optimization over the polytope 3DM$(G)$ and therefore it is to be expected that 3DM$(G)$ would not have a polynomial size extended formulation.

Now we show that the 3d-matching polytope has superpolynomial extension complexity in the worst case. We prove this using a standard reduction from 3SAT to 3d-Matching used in the NP-completeness proof for the later problem (See [9]). The form of this reduction, which is very widely used, employs a gadget for each variable along with a gadget for each clause. We omit the exact details for the reduction here because we are only interested in the correctness of the reduction and the variable gadget (See Figure 1).

In the reduction, any 3SAT formula $\Phi$ is converted to an instance of a 3d-matching by creating a set of hyperedges for every variable (See Figure 1) along with some other hyperedges that does not concern us for our result. The crucial property that we require is the following: any satisfiable assignment of $\Phi$ defines some (possibly more than one) 3d-matching. Furthermore, in any maximal matching either only the light hyperedges or only the dark hyperedges are picked,

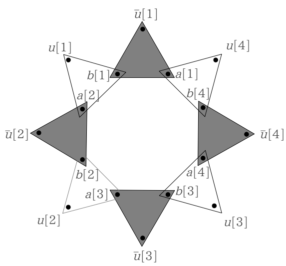


**Fig. 1.** Gadget for a variable

corresponding to setting the corresponding variable to, say, true or false respectively. Using these facts we can prove the following:

**Theorem 4.** *Let $\Phi$ be an instance of 3SAT and let $H$ be the hypergraph obtained by the reduction above. Then* $\mathrm{SAT}(\Phi)$ *is the projection of a face of* $3\mathrm{DM}(H)$.

*Proof.* Let the number of hyperedges in the gadget corresponding to a variable $x$ be $2k(x)$. Then, the number of hyperedges picked among these hyperedges in any matching in $H$ is at most $k(x)$. Therefore, if $y_1, \ldots, y_{2k(x)}$ denote the variables corresponding to these hyperedges in the polytope $3\mathrm{DM}(H)$ then $\sum_{i=1}^{2k(x)} y_i \leqslant k(x)$ is a valid inequality for $3\mathrm{DM}(H)$. Consider the face $F$ of $3\mathrm{DM}(H)$ obtained by adding the equality $\sum_{i=1}^{2k(x)} y_i = k(x)$ corresponding to each variable $x$ appearing in $\Phi$.

Any vertex of $3\mathrm{DM}(H)$ lying in $F$ selects either all light hyperedges or all dark hyperedges. Therefore, projecting out all variables except one variable $y_i$ corresponding to any fixed (arbitrarily chosen) light hyperedge for each variable in $\Phi$ gives a valid satisfying assignment for $\Phi$ and thus a vertex of $\mathrm{SAT}(\Phi)$. Alternatively, any vertex of $\mathrm{SAT}(\Phi)$ can be extended to a vertex of $3\mathrm{DM}(H)$ lying in $F$ easily.

Therefore, $\mathrm{SAT}(\Phi)$ is the projection of $F$. □

The number of vertices in $H$ is $O(nm)$ where $n$ is the number of variables and $m$ the number of clauses in $\Phi$. Considering only the 3SAT formulae with high extension complexity, we have $m = O(n)$. Therefore, considering only the hypergraphs arising from such 3SAT formulae and using propositions 1 and 2, we have that

**Corollary 2.** *For every natural number $n \geqslant 1$, there exists a hypergraph $H$ with $O(n)$ vertices such that* $\mathrm{xc}(3\mathrm{DM}(H)) \geqslant 2^{\Omega(n^{1/4})}$.

***Stable Set for Cubic Planar Graphs.*** Now we show that $\mathrm{STAB}(G)$ can have superpolynomial extension complexity even when $G$ is a cubic planar graph. Our starting point is the following result proved by Fiorini et. al. [8].

**Theorem 5 ([8]).** *For every natural number $n \geqslant 1$ there exists a graph $G$ such that $G$ has $O(n)$ vertices and $O(n)$ edges, and* $\mathrm{xc}(\mathrm{STAB}(G)) \geqslant 2^{\Omega(\sqrt{n})}$.

We start with this graph and convert it into a cubic planar graph $G'$ with $O(n^2)$ vertices and extension complexity at least $2^{\Omega(\sqrt{n})}$.

***Making a Graph Planar.*** For making any graph $G$ planar without reducing the extension complexity of the associated stable set polytope, we use the same gadget used by Garey, Johnson and Stockmeyer [10] in the proof of NP-completeness of finding maximum stable set in planar graph. Start with any planar drawing of $G$ and replace every crossing with the gadget $H$ with 22 vertices shown in Figure 2 to obtain a graph $G'$. The following theorem shows that $\mathrm{STAB}(G)$ is the projection of a face of $\mathrm{STAB}(G')$.

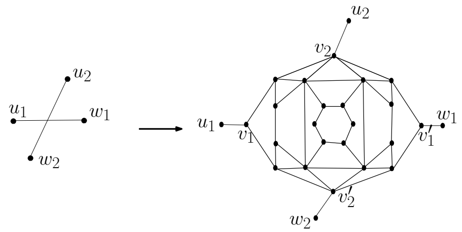


**Fig. 2.** Gadget to remove a crossing

**Table 1.** Values of $s_{ij}$

| $i\backslash j$ | 2 | 1 | 0 |
|---|---|---|---|
| 2 | 9 | 8 | 7 |
| 1 | 9 | 9 | 8 |
| 0 | 8 | 8 | 7 |

Using the face $F := \mathrm{STAB}(G') \bigcap_{i=1}^{k} \{x \mid \sum_{j \in V_{H_i}} x_j = 9\}$, where $k$ denotes the number of gadgets itroduced in $G$ and a proof similar to that of Theorem 1, we have the following:

**Theorem 6.** *Let $G$ be a graph and let $G'$ be obtained from a planar embedding of $G$ by replacing every edge intersection with a gadget shown in Figure 2. Then,* $\mathrm{STAB}(G)$ *is the projection of a face of* $\mathrm{STAB}(G')$.

Since for any graph $G$ with $O(n)$ edges, the number of gadgets introduced $k \leqslant O(n^2)$, we have that the graph $G'$ in the above theorem has at most $O(n^2)$ vertices and edges. Therefore we have a planar graph $G'$ with at most $O(n^2)$ vertices and $O(n^2)$ edges. This together with Theorem 5, Theorem 6 and propositions 1 and 2 yields the following corollary.

**Corollary 3.** *For every $n$ there exists a planar graph $G$ with $O(n^2)$ vertices and $O(n^2)$ edges such that* $\mathrm{xc}(\mathrm{STAB}(G)) \geqslant 2^{\Omega(\sqrt{n})}$.

***Making a Graph Cubic.*** Suppose we have a graph $G$ and we transform it into another graph $G'$ by performing one of the following operations:

**ReduceDegree:** Replace a vertex $v$ of $G$ of degree $\delta \geqslant 4$ with a cycle $C_v = (v_1, v_1', \ldots, v_\delta, v_\delta')$ of length $2\delta$ and connect the neighbours of $v$ to alternating vertices $(v_1, v_2, \ldots, v_\delta)$ of the cycle.
**RemoveBridge:** Replace any degree two vertex $v$ in $G$ by a four cycle $v_1, v_2, v_3, v_4$. Let $u$ and $w$ be the neighbours of $v$ in $G$. Then, add the edges $(u, v_1)$ and $(v_3, w)$. Also add the edge $(v_2, v_4)$ in the graph.
**RemoveTerminal:** Replace any vertex with degree either two or three with a triangle. In case of degree one, attach any one vertex of the triangle to the erstwhile neighbour.

**Theorem 7.** *Let $G$ be any graph and let $G'$ be obtained by performing any number of operation ReduceDegree, RemoveBridge, or RemoveTerminal described above on $G$. Then* $\mathrm{STAB}(G)$ *is the projection of a face of* $\mathrm{STAB}(G')$.

*Proof.* Omitted. □

If $G$ has $n$ vertices and $m$ edges then first applying operation ReduceDegree until every vertex has degree at most 3, and then applying operation RemoveBridge and RemoveTerminal repeatedly until no vertex of degree 0, 1 or 2 is left, produces a graph that has $O(n + m)$ vertices and $O(n + m)$ edges. Furthermore, any application of the three operations do not make a planar graph non-planar. Combining this fact with Theorem 7, Corollary 3 and propositions 1 and 2, we have

**Corollary 4.** *For every natural number $n \geqslant 1$ there exists a cubic planar graph $G$ with $O(n)$ vertices and edges such that* $\mathrm{xc}(\mathrm{STAB}(G)) \geqslant 2^{\Omega(n^{1/4})}$.

## 4 Extended Formulations for $\mathrm{CUT}^{\square}(G)$ and Its Relatives

We use the results described in the previous section to obtain bounds on the extension complexity of the cut polytope of graphs. We begin by reviewing the result in [8] for $\mathrm{CUT}_n^{\square}$ using a direct argument that avoids introducing correlation polytopes. For any integer $n \geq 2$ consider the integers $b_1 = ... = b_{n-1} = 1$ and $b_n = 3 - n$. Let $b = (b_1, b_2, ..., b_n)$ be the corresponding $n$-vector. Inequality (1) for this $b$-vector is easily seen to be of negative type and can be written as

$$\sum_{1 \leq i < j \leq n-1} x_{ij} \leq 1 + (n-3) \sum_{i=1}^{n-1} x_{in}. \tag{3}$$

**Lemma 2.** *Let $S$ be any cut in $K_n$ not containing vertex $n$ and let $\delta(S)$ be its corresponding cut vector. Then the slack of $\delta(S)$ with respect to (3) is $(|S|-1)^2$.*

Let us label a cut $S$ by a binary $n$-vector $a$ where $a_i = 1$ if and only if $i \in S$. Under the conditions of the lemma we observe that the slack $(|S| - 1)^2 = (a^T b - 1)^2$ since we have $a_n = 0$ and $b_1 = ... = b_{n-1} = 1$. Now consider consider any subset $T$ of $\{1, 2, ..., n-1\}$ and set $b_i = 1$ for $i \in T$, $b_n = 3 - |T|$ and $b_i = 0$ otherwise. We form a $2^{n-1}$ by $2^{n-1}$ matrix $M$ as follows. Let the rows and columns be indexed by subsets $T$ and $S$ of $\{1, 2, ..., n-1\}$, labelled by the $n$-vectors $a$ and $b$ as just described. A straight forward application of Lemma 2 shows that $M = M^*(n-1)$. Hence using the fact that the non-negative rank of a matrix is at least as large as that of any of its submatrices, we have that every extended formulation of $\mathrm{CUT}_n^{\square}$ has size $2^{\Omega(n)}$.

Recall the hypermetric polytope, defined in Section 2.1, is the intersection of all hypermetric inequalities. As remarked, nonnegative type inequalities are weaker than hypermetric inequalities and so valid for this polytope. In addition all cut vertices satisfy all hypermetric inequalities. Therefore $M = M^*(n-1)$ is also a submatrix of a slack matrix for the hypermetric polytope on $n$ points. So this polytope also has extension complexity at least $2^{\Omega(n)}$.

Finally let us consider the polytope, which we denote $P_n$, defined by the inequalities used to define rows of the slack matrix $M$ above. We will show that membership testing for $P_n$ is co-NP-complete.

**Theorem 8.** *Let $P_n$ be the polytope defined as above, and let $x \in \mathbb{R}^{n(n-1)/2}$. Then it is co-NP-complete to decide if $x \in P_n$.*

*Proof.* Clearly if $x \notin P_n$ then this can be witnessed by a violated inequality of type (3), so the problem is in co-NP.

To see the hardness we do a reduction from the clique problem: given graph $G = (V, E)$ on $n$ vertices and integer $k$, does $G$ have a clique of size at least $k$? Since a graph has a clique of size $k$ if and only if its suspension has a clique of size $k+1$ we can assume *wlog* that $G$ is a suspension with vertex $v_n$ connected to every other vertex.

Form a vector $x$ as follows: $x_{ij} = 1/k$, if $j = n$, $x_{ij} = 2/k$, if $j \neq n$ and $ij \in E$ and $x_{ij} = -n^2$otherwise

Fix an integer $t$, $2 \leq t \leq n$ and consider a $b$-vector with $b_n = 3 - t$, and with $t-1$ other values of $b_i = 1$. Without loss of generality we may assume these are lablelled $1, 2, ..., t-1$. Let $T$ be the induced subgraph of $G$ on these vertices. The corresponding non-negative type inequality is:

$$\sum_{1 \leq i < j \leq t-1} x_{ij} \leq 1 + (t-3) \sum_{i=1}^{t-1} x_{in}. \tag{4}$$

Suppose $T$ is a complete subgraph. Then the left hand side minus the right hand side of (4) is $\frac{2(t-1)(t-2)}{2k} - (1 + \frac{(t-3)(t-1)}{k}) = \frac{t-k-1}{k}$. This will be positive if and only if $t \geq k+1$, in which case $x$ violates (4). On the other hand if $T$ is not a complete subgraph then the left hand side of (4) is always negative and so the inequality is satisfied. Therefore $x$ satisfies all inequalities defining rows of $M$ if an only if $G$ has no clique of size at least $k$. □

***Cut polytope for minors of a graph.*** A graph $H$ is a *minor* of a graph $G$ if $H$ can be obtained from $G$ by contracting some edges, deleting some edges and isolated vertices, and relabeling. In the introduction we noted that if an $n$ vertex graph $G$ has no $K_5$-minor then $\mathrm{CUT}^{\square}(G)$ has $O(n^3)$ extension complexity. The following Lemma shows that the extension complexity of a graph $G$ can be bounded from below in terms of its largest clique minor.

**Lemma 3.** *Let $G$ be a graph and let $H$ be obtained by deleting an edge of $G$, or deleting a vertex of $G$, or contracting an edge of $G$, Then,* $\mathrm{xc}(\mathrm{CUT}^{\square}(G)) \geqslant \mathrm{xc}(\mathrm{CUT}^{\square}(H))$.

*Proof.* Omitted. □

Therefore, we get the following theorem that can be proved by induction over a sequence of minor-producing steps.

**Theorem 9.** *Let $G$ be a graph and $H$ be a minor of $G$. Then,* $\mathrm{xc}(\mathrm{CUT}^{\square}(G)) \geqslant \mathrm{xc}(\mathrm{CUT}^{\square}(H))$.

Using the above theorem together with the result of [8] that the extension complexity of $\mathrm{CUT}^{\square}(K_n)$ is at least $2^{\Omega(n)}$ we get the following result.

**Corollary 5.** *The extension complexity of* $\mathrm{CUT}^{\square}(G)$ *for a graph $G$ with a $K_n$ minor is at least $2^{\Omega(n)}$.*

Using Theorem 9 and the fact that $K_{n+1}$ is a minor of $K_{1,n,n}$ we can immediately prove that the Bell inequality polytopes mentioned in the introduction have exponential complexity.

**Corollary 6.** *The extension complexity of* $\mathrm{CUT}^{\square}(K_{1,n,n})$ *is at least* $2^{\Omega(n)}$.

***Cut Polytope for $K_6$ minor-free graphs.*** Let $G = (V, E)$ be any graph with $V = \{1, \ldots, n\}$. Consider the suspension $G'$ of $G$ obtained by adding an extra vertex labelled 0 with edges to all vertices $V$.

**Theorem 10.** *Let $G = (V, E)$ be a graph and let $G'$ be a suspension over $G$. Then* $\mathrm{STAB}(G)$ *is the projection of a face of* $\mathrm{CUT}^{\square}(G')$.

*Proof.* The polytope $\mathrm{STAB}(G)$ is defined over variables $x_i$ corresponding to each of the vertex $i \in V$ whereas the polytope $\mathrm{CUT}^{\square}(G')$ is defined over the variables $x_{ij}$ for $i, j \in \{0, \ldots, n\}$.

Any cut vertex $C$ of $\mathrm{CUT}^{\square}(G')$ defines sets $S, \overline{S}$ such that $x_{ij} = 1$ if and only if $i \in S, j \in \overline{S}$. We may assume that $0 \in \overline{S}$ by interchanging $S$ and $\overline{S}$ if necessary. For every edge $e = (k, l)$ in $G$ consider an inequality $h_e := \{x_{0k} + x_{0l} - x_{kl} \geqslant 0\}$. It is clear that $h_e$ is a valid inequality for $\mathrm{CUT}^{\square}(G')$ for all edges $e$ in $G$. Furthermore, $h_e$ is tight for a cut vector in $G'$ if and only if either $k, l$ do not lie in the same cut set or $k, l$ both lie in the cut set containing 0. Therefore consider the face $F := \mathrm{CUT}^{\square}(G') \bigcap_{(i,j)\in E}\{x_{0i} + x_{0j} - x_{ij} = 0\}$.

Each vertex in $F$ can be projected to a valid stable set in $G$ by projecting onto the variables $x_{01}, x_{02}, \ldots, x_{0n}$. Furthermore, every stable set $S$ in $G$ can be extended to a cut vector for $G'$ by taking the cut vector corresponding to $S, \overline{S} \cup \{0\}$. Therefore, $\mathrm{STAB}(G)$ is the projection of a face of $\mathrm{CUT}^{\square}(G')$. □

Using this theorem it is easy to show the existence of graphs with a linear number of edges that do not have $K_6$ as a minor and yet have a high extension complexity. In fact we get a slightly sharper result.

**Theorem 11.** *For every $n \geqslant 2$ there exists a graph $G$ which is a suspension of a planar graph and for which* $\mathrm{xc}(\mathrm{CUT}^{\square}(G)) \geqslant 2^{\Omega(n^{1/4})}$.

*Proof.* Consider a planar graph $G = (V, E)$ with $n$ vertices for which $\mathrm{xc}(\mathrm{STAB}(G)) \geqslant 2^{\Omega(n^{1/4})}$. Corollary 3 guarantees the existence of such a graph for every $n$. Then the suspension over $G$ has $n+1$ vertices and a linear number of edges. The theorem then follows by applying Theorem 10 together with Propositions 1 and 2. □

The above theorem provides a sharp contrast for the complexity of the cut polytope for graphs in terms of their minors. As noted in the introduction, for any $K_5$ minor-free graph $G$ with $n$ vertices $\mathrm{CUT}^{\square}(G)$ has an extension of size $O(n^3)$ whereas the above result shows that there are $K_6$ minor free graphs whose cut polytope has superpolynomial extension complexity.

## 5 Concluding Remarks

We have a given a simple polyhedral procedure for proving lower bounds on the extension complexity of a polytope. Using this procedure and some standard NP-completeness reductions we were able to prove lower bounds on the extension complexity of various well known combinatorial polytopes. For the cut polytope in particular, we are able to draw a sharp line, in terms of minors, for when this complexity becomes super polynomial.

Nevertheless the procedure is not completely 'automatic' in the sense that any NP-completeness reduction of a certain type, say using gadgets, automatically gives a result on the extension complexity of related polytopes. This would seem to be a very promising line of future research.

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