Deciding the Winner of an Arbitrary Finite Poset Game Is PSPACE-Complete

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Abstract. A poset game is a two-player game played over a partially ordered set (poset) in which the players alternate choosing an element of the poset, removing it and all elements greater than it. The first player unable to select an element of the poset loses. Polynomial time algorithms exist for certain restricted classes of poset games, such as the game of Nim. However, until recently the complexity of arbitrary finite poset games was only known to exist somewhere between NC^1 and PSPACE. We resolve this discrepancy by showing that deciding the winner of an arbitrary finite poset game is PSPACE-complete. To this end, we give an explicit reduction from Node Kayles, a PSPACE-complete game in which players vie to chose an independent set in a graph.

1 Introduction

A partially ordered set, or poset, is a set of elements with a binary relation (denoted ≤) indicating the ordering of elements that is reflexive, transitive, and antisymmetric. A poset game is an impartial two-player game played over some poset. Each turn, a player selects an element of the poset, removing it and all elements greater than it. A player loses when faced with the empty set. Equivalently, the last player able to select an element wins. We will assume that the number of elements in the poset is finite, which ensures that the game will eventually end in such a manner.

Poset games have been studied in various forms since a complete analysis of the game of Nim was given in 1901 by C. Bouton [2]. Other poset games with explicit polynomial time strategies include Von Neumann's Hackendot [17] and impartial Hackenbush on trees [1]. The above games have no induced subposet of cardinality four that form an 'N'. In fact, it is shown in [4] that all N-free poset games can be solved in polynomial time.

However there are several other well-studied poset games played over specific structures with unknown complexity [8]. Perhaps the most popular is the game of Chomp, which was introduced by Gale in 1974 and is played on the cross product of two Nim stacks [10]. Work by Byrnes [3] shows that certain Chomp

positions exhibit periodic behavior, but a quick general solution still does not exist. In Subset Takeaway [11], introduced by Gale in 1982, the players take turns removing a set and all its supersets from a collection of sets. In Shuh's Game of Divisors [15], the players alternate removing a divisor of n and its multiples. In fact, both Chomp and Subset Takeaway are special cases of the Game of Divisors, with n the product of at most two primes and n square-free, respectively.

In this paper, we discuss the complexity of deciding the winner of an arbitrary finite poset game, which has remained a longstanding question in the attempt to classify the tractability of combinatorial games [8,9]. Let PG be the language consisting of poset games with a winning strategy for the first player. Let $\langle P, \leq \rangle \in PG$ and $P = \{p_1, p_2, \ldots, p_n\}$. We will assume that $\langle P, \leq \rangle$ is represented as input in some explicit manner, such as a 0-1 matrix A where $a_{ij} = 1$ iff $p_i \leq p_j$. Although not all matrices of this type describe a legitimate poset, it is easy to check the validity of a matrix representation of a poset in polynomial time.

In [13], Kalinich shows that PG is at least as hard as NC^1 under AC^0 reductions by creating a correspondence with boolean circuits. Weighted poset games, which are a generalization of poset games, were shown to be PSPACE-complete in [12]. That result, which uses a completely different technique than the one described in this paper, along with another proof in [16], clearly show that PG is in PSPACE. We show that PG is indeed PSPACE-complete.

In [14], Schaefer shows that the two-player game Node Kayles is PSPACE-complete. In Node Kayles, the players take turns removing a vertex and all neighbors of that vertex from a graph. The first player unable to move loses.

In Section 2 we will give two constructions that serve as the basis for a reduction from Node Kayles to PG. We will then give a variety of lemmas demonstrating the desirable properties of these constructions in Section 3. In Section 4 we will combine these lemmas to show that PG is PSPACE-complete.

2   Constructions

Below we will give two constructions, ψ and φ. When applied in succession, they reduce an instance of Node Kayles into an instance of PG such that the winning player is preserved. Let G be the class of finite simple graphs and P be the class of finite posets. For $g \in G$ we will write $g = (V, E)$ where V is the set of vertices and E is the set of edges. We will use K_n to denote the complete graph on n vertices.

2.1 ψ-Construction

Define $\psi : G \to G$ such that

- $|E|$ is odd $\implies \psi(g) = g \uplus K_2 \uplus K_2$
- $|E|$ is even $\implies \psi(g) = g \uplus K_2 \uplus K_4$
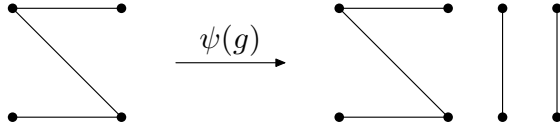
Fig. 1. Example of ψ-construction when $|E|$ is odd
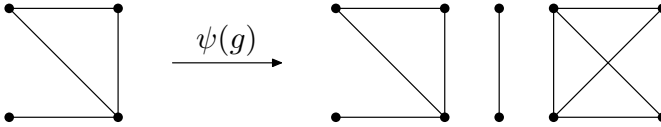

Fig. 2. Example of ψ-construction when $|E|$ is even

This construction serves two purposes. First, the edge cardinality of the resulting graph is always odd. Second, for every vertex, there is an edge that is not incident to it. It is also important to note that the winning player of the Node Kayles game does not change (see Lemma 1).

2.2 φ-Construction

Let $\varphi : G \to P$ be a function from simple graphs to posets, where $\varphi(g) = A \cup B \cup C$ is a three-level poset with disjoint levels A, B, and C from lowest to highest. That is, for any $a \in A$, $b \in B$, and $c \in C$, $b \nleq a$, $c \nleq b$, and $c \nleq a$. Furthermore, any two elements on the same level are incomparable.

Fix $g = (V, E)$. The elements of the poset $\varphi(g)$ are as follows:

- The elements of C are the edges of g. That is, $C = E$.
- The elements of B are the vertices of g. That is, $B = V$.
- The elements of A are copies of the edges of g. To represent this, let $\gamma : C \to A$ be a 1-1 correspondence between the elements of C and the elements of A.

For each edge $e = (v_1, v_2)$ and $b \in B$, the $\leq$ relationship of the poset $\varphi(g)$ is as follows:

- $b \leq e$ iff $b = v_1$ or $b = v_2$. That is, e lies directly above its endpoints in B.
- $\gamma(e) \leq b$ iff $b \neq v_1$ and $b \neq v_2$. That is, $\gamma(e)$ is less than all the elements in B except the endpoints of e.

3   Lemmas

Lemma 1. *Player 1 wins the Node Kayles game on g iff Player 1 wins the Node Kayles game on $\psi(g)$.*
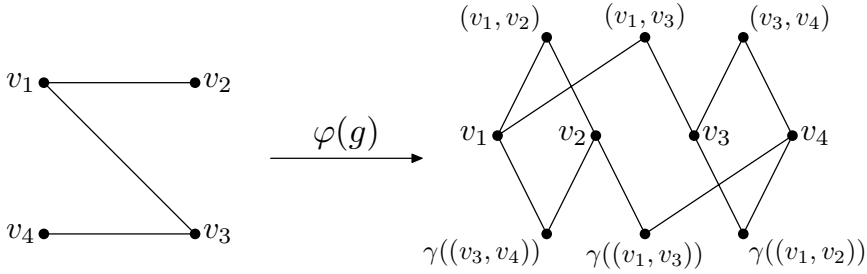
Fig. 3. Example φ-construction. Note that the left picture is an undirected graph representing a Node Kayles game, and the right picture is a Hasse Diagram representing the resultant poset game.

Proof. Suppose that the Node Kayles game played on g is a win for Player 1, who we will assume by convention is the first to play. We will show that this gives Player 1 an explicit winning strategy on $\psi(g)$. Player 1 first chooses the winning move in g. If Player 2 chooses a vertex in g, Player 1 can always respond with another move in g because Player 1 has the winning strategy on g. If Player 2 chooses a vertex in one of the complete graphs, Player 1 can respond with a vertex in the other complete graph, removing both complete graphs from consideration for the remainder of the game. Because Player 1 can respond to any move of Player 2, Player 1 will eventually win. Of course, this argument holds if Player 2 has the winning strategy in g, and similarly shows that a player has a winning strategy on g if he has a winning strategy on $\psi(g)$.

In terms of Sprague-Grundy theory, the disjoint union of the two complete graphs has Grundy number zero. Adding a game of Grundy number zero to an existing game does not change the winner of the original game [1]. In particular, the Grundy number of g is equal to the Grundy number of $\psi(g)$. □

Let $g = (V, E)$ be a finite simple graph and $e = (v_1, v_2)$ be an arbitrary edge in $\psi(g)$. For the following lemmas, assume that two players are playing the poset game on $\varphi(\psi(g))$. Also assume, for simplicity, that the players are Alice and Bob.

Lemma 2. *Assume no moves in A or C have yet been chosen. If both v_1 and v_2 have been chosen, then $\gamma(e)$ is a winning move.*

Proof. Because the ψ-construction always leaves a graph with an odd number of edges, choosing $\gamma(e)$ leaves an even number of incomparable points in A. □

Lemma 3. *Assume no moves in A or C have yet been chosen. If exactly one of v_1 and v_2 has been chosen, then $\gamma(e)$ is a losing move.*

Proof. First notice that e has already been removed from the poset because both $v_1 \le e$ and $v_2 \le e$. Because $\gamma(e) \not\le v_1$ and $\gamma(e) \not\le v_2$, choosing $\gamma(e)$ leaves a single point (either v_1 or v_2) in B. Thus, the next player can win by choosing the lone element in B, leaving an even number of incomparable points in A. □

Lemma 4. *Assume no moves in A or C have yet been chosen. If neither v_1 nor v_2 has been chosen, then both e and $\gamma(e)$ are losing moves.*

Proof. Assume that either player, say Alice, chooses $\gamma(e)$, which results in an even number of incomparable points in A, v_1 and v_2 in B, and e in C. Bob can then respond by choosing e. If Alice responds with v_1, then Bob can respond with v_2 (and vice versa), resulting in an even number of points in A, which is a win for Bob.

If, however, Alice responds with a point $a \in A$, there are three cases: $a \le v_1$ and $a \le v_2$, $a \le v_1$ and $a \not\le v_2$, or $a \le v_2$ and $a \not\le v_1$. Note that, by construction, there is no point a such that $a \not\le v_1$ and $a \not\le v_2$. That is, the only point that is not less than both v_1 and v_2 is $\gamma(e)$, which has already been taken by assumption. So first assume that $a \le v_1$ and $a \le v_2$. This would leave an odd number of elements in A, resulting in a win for Bob. Consider then that $a \le v_1$ and $a \not\le v_2$ or $a \le v_2$ and $a \not\le v_1$. Without loss of generality we can assume $a \le v_1$ and $a \not\le v_2$. Because $\psi(g)$ has at least two distinct components, each having at least one edge, there exists an edge e_2 that is not incident to either v_1 or v_2. By construction, $\gamma(e_2) \le v_2$. Thus, Bob can choose $\gamma(e_2)$, leaving only an even number of elements in A, resulting in a win for Bob.

If Alice had initially chosen e instead of $\gamma(e)$, then Bob could have responded with $\gamma(e)$, which leads to the same game as played as above, which was a win for Bob.   □

4   Main Theorem

Theorem 1. *PG is PSPACE-complete.*

Proof. It is straightforward to check and demonstrated explicitly in [16] that PG is in PSPACE. We will next give a reduction from Node Kayles to PG to show that the latter is also PSPACE-hard. First note that $\varphi(\psi(g))$ is computable in polynomial time.

We will argue inductively that Player 1 has a winning strategy for the poset game played on $\varphi(\psi(g))$ iff Player 1 has a winning strategy for the Nodes Kales game played on g. The idea behind the construction is that both players are forced to play elements in B until two elements v_1 and v_2 representing adjacent vertices in $\psi(g)$ have been chosen. At this point the following player can win by choosing the element $\gamma((v_1, v_2))$ in A.

Assume that the poset game played on $\varphi(\psi(g))$ has been played in the prescribed manner so far. That is, no elements from A or C have yet been chosen. Lemma 2 ensures that choosing a vertex neighboring a vertex that has already been chosen is a losing move. Lemma 3 and Lemma 4 ensure that choosing any point in A or C before two neighboring vertices have been chosen is a losing move. Thus, a player has a winning strategy on $\varphi(\psi(g))$ iff that player has a winning strategy on $\psi(g)$, since there is an obvious correspondence between the moves in $\varphi(\psi(g))$ and the moves in $\psi(g)$. Lemma 1 ensures that a player has a winning strategy on $\psi(g)$ iff he has a winning strategy on g.   □
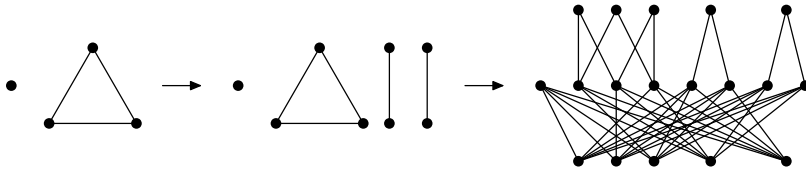
Fig. 4. Example of full reduction from g to $\psi(g)$ to $\varphi(\psi(g))$

5   Future Work

Using the above theorem, it follows easily that deciding the winner of a finite poset game with any height $k \geq 3$ is PSPACE-complete. In contrast, determining the winner of single-level poset games is trivially obtained by considering the parity of the poset elements. There are also polynomial time algorithms for some two-level poset games. In [7], Fraenkel and Aviezri give a polynomial time algorithm for finding the Grundy number of poset games played over a restricted class of two-level posets whose upper elements act like edges of a hypergraph. In [5], Fenner, Gurjar, Korwar, and Thierauf give a natural generalization of that algorithm and explore other possible avenues for finding the winner in polynomial time. However, neither of these results yield a general algorithm, and the complexity of two-level poset games remains an open problem.

This work has also spawned a new PSPACE-complete game on sets invented by Fenner and Fortnow [6]. Given a collection of finite sets $S_1, \ldots, S_k$, each player takes turns picking a non-empty set S_i, removing the elements of S_i from all the sets S_j. The player who empties all the sets wins. To reduce a poset game into an instance of set-game, simply take the sets as the upper cones of the poset. That is, each set consists of an element and all elements greater than it. However, if the cardinality of the sets is bounded, the complexity is still open.

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