On the Construction of Semiquadratic Copulas

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Abstract. We introduce several classes of semiquadratic copulas (i.e. copulas that are quadratic in at least one coordinate of any point of the unit square) of which the diagonal section or the opposite diagonal section are given functions. These copulas are constructed by quadratic interpolation on segments connecting the diagonal (resp. opposite diagonal) of the unit square to the boundaries of the unit square. We provide for each class the necessary and sufficient conditions on a diagonal (resp. opposite diagonal) function and two auxiliary real functions f and g to obtain a copula which has this diagonal (resp. opposite diagonal) function as diagonal (resp. opposite diagonal) section.

1 Introduction

Bivariate copulas (briefly copulas) [10] are binary operations on the unit interval having 0 as absorbing element and 1 as neutral element and satisfying the condition of 2-increasingness, i.e. a copula is a function $C : [0,1]^2 \to [0,1]$ satisfying the following conditions:

1. for all $x \in [0,1]$, it holds that

$$C(x,0) = C(0,x) = 0, \quad C(x,1) = C(1,x) = x;$$

2. for all $x, x', y, y' \in [0,1]$ such that $x \leq x'$ and $y \leq y'$, it holds that

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$$V_C([x,x'] \times [y,y']) := C(x,y) + C(x',y') - C(x,y') - C(x',y) \geq 0\,.$$

$V_C([x,x'] \times [y,y'])$ is called the C-volume of the rectangle $[x,x'] \times [y,y']$. The copulas M and W, defined by $M(x,y) = \min(x,y)$ and $W(x,y) = \max(x+y-1,0)$, are called the Fréchet–Hoeffding upper and lower bounds: for any copula C it holds that $W \leq C \leq M$. A third important copula is the non-singular product copula Π defined by $\Pi(x,y) = xy$.

The diagonal section of a copula C is the function $\delta_C : [0,1] \to [0,1]$ defined by $\delta_C(x) = C(x,x)$. A diagonal function δ is a $[0,1] \to [0,1]$ function that satisfies the following conditions:

(D1) $\delta(0) = 0$, $\delta(1) = 1$;
(D2) for all $x \in [0,1]$, it holds that $\delta(x) \leq x$;
(D3) δ is increasing;
(D4) δ is 2-Lipschitz continuous, i.e. for all $x, x' \in [0,1]$, it holds that

$$|\delta(x') - \delta(x)| \leq 2|x' - x|\,.$$

Note that (D4) implies that δ is absolutely continuous, and hence differentiable almost everywhere. The diagonal section δ_C of a copula C is a diagonal function. Conversely, for any diagonal function δ there exists at least one copula C with diagonal section $\delta_C = \delta$. For example, the copula K_δ, defined by $K_\delta(x,y) = \min(x,y,(\delta(x)+\delta(y))/2)$, is the greatest symmetric copula with a given diagonal section δ [11] (see also [5, 7]). Moreover, the Bertino copula B_δ defined by

$$B_\delta(x,y) = \min(x,y) - \min\{t - \delta(t) \mid t \in [\min(x,y), \max(x,y)]\}\,,$$

is the smallest copula with a given diagonal section δ. Note that B_δ is symmetric.

Similarly, the opposite diagonal section of a copula C is the function $\omega_C : [0,1] \to [0,1]$ defined by $\omega_C(x) = C(x, 1-x)$. An opposite diagonal function [2] is a function $\omega : [0,1] \to [0,1]$ that satisfies the following conditions:

(OD1) for all $x \in [0,1]$, it holds that $\omega(x) \leq \min(x, 1-x)$;
(OD2) ω is 1-Lipschitz continuous, i.e. for all $x, x' \in [0,1]$, it holds that

$$\left|\omega(x') - \omega(x)\right| \leq \left|x' - x\right|\,.$$

Note that (OD2) implies that ω is absolutely continuous, and hence differentiable almost everywhere. The opposite diagonal section ω_C of a copula C is an opposite diagonal function. Conversely, for any opposite diagonal function ω, there exists at least one copula C with opposite diagonal section $\omega_C = \omega$. For instance, the copula F_ω defined by

$$F_\omega(x,y) = \max(x+y-1,0) + \min\{\omega(t) \mid t \in [\min(x,1-y), \max(x,1-y)]\}\,,$$

is the greatest copula with opposite diagonal section ω [2, 9].

Diagonal functions and opposite diagonal functions have been used recently to construct several classes of copulas [1, 2, 3, 4, 5, 6, 7, 12].

Copulas with a given diagonal section are important tools for modelling upper (λ_U) and lower (λ_L) tail dependence [1, 3], which can be expressed as

$$\lambda_U = 2 - \delta_C'(1^-) \quad \text{and} \quad \lambda_L = \delta_C'(0^+)\,.$$

On the other hand, copulas with a given opposite diagonal section are important tools for modelling upper-lower (λ_{UL}) and lower-upper (λ_{LU}) tail dependence [2], which can be expressed as

$$\lambda_{UL} = 1 + \omega_C'(1^-) \quad \text{and} \quad \lambda_{LU} = 1 - \omega_C'(0^+)\,.$$

The above tail dependences are used in the literature to model the dependence between extreme events [13].

This paper is organized as follows. In the next section we introduce lower, upper, horizontal and vertical semiquadratic functions with a given diagonal section and characterize the corresponding classes of copulas. In Section 3 we introduce in a similar way lower-upper, upper-lower, horizontal and vertical semiquadratic functions with a given opposite diagonal section and characterize the corresponding classes of copulas. Finally, some conclusions are given.

2 Semiquadratic Copulas with a Given Diagonal Section

2.1 Lower and Upper Semiquadratic Copulas with a Given Diagonal Section

Lower (resp. upper) semiquadratic copulas with a given diagonal section are constructed by quadratic interpolation on segments connecting the diagonal of the unit square to the left (resp. right) and lower (resp. upper) boundary of the unit square. The quadratic interpolation scheme for theses classes is depicted in Fig. 1.

For any two functions $f, g :]0,1] \to \mathbb{R}$ that are absolutely continuous and satisfy

$$\lim_{\substack{x\to 0\\ 0\le y\le x}} y(x-y)f(x) = 0 \quad \text{and} \quad \lim_{\substack{y\to 0\\ 0\le x\le y}} x(y-x)g(y) = 0\,, \tag{1}$$

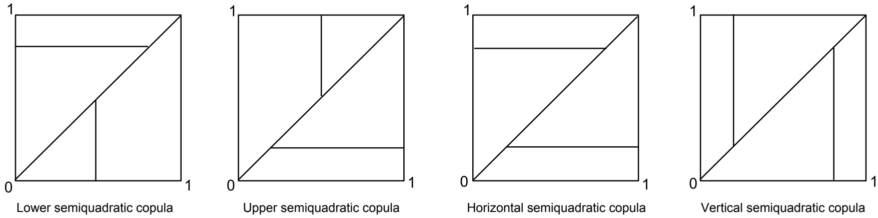

Fig. 1 Semiquadratic copulas with a given diagonal section

and any diagonal function δ, the function $C^{f,g}_{l,\delta} : [0,1]^2 \to \mathbb{R}$ defined by

$$C^{f,g}_{l,\delta}(x,y) = \begin{cases} \frac{x}{y}\delta(y) - x(y-x)g(y) & , \text{if } 0 < x \leq y, \\ \frac{y}{x}\delta(x) - y(x-y)f(x) & , \text{if } 0 < y \leq x, \end{cases} \tag{2}$$

with $C^{f,g}_{l,\delta}(t,0) = C^{f,g}_{l,\delta}(0,t) = 0$ for all $t \in [0,1]$, is well defined. Note that the limit conditions on f and g ensure that $C^{f,g}_{l,\delta}$ is continuous. The function $C^{f,g}_{l,\delta}$ is called *a lower semiquadratic function with diagonal section* δ since it satisfies $C^{f,g}_{l,\delta}(t,t) = \delta(t)$ for all $t \in [0,1]$, and since it is quadratic in x on $0 \leq x \leq y \leq 1$ and quadratic in y on $0 \leq y \leq x \leq 1$. Obviously, symmetric functions are obtained when $f = g$. Note that for $f = g = 0$, the definition of a lower semilinear function [5] is retrieved.

We now investigate the conditions to be fulfilled by the functions f, g and δ such that the lower semiquadratic function $C^{f,g}_{l,\delta}$ is a copula. Note that f and g, being absolutely continuous, are differentiable almost everywhere.

Proposition 1. *Let δ be a diagonal function and let f and g be two absolutely continuous functions that satisfy conditions (1). Then the lower semiquadratic function $C^{f,g}_{l,\delta}$ defined by (2) is a copula with diagonal section δ if and only if*

(i) $f(1) = g(1) = 0$,

(ii) $\max\left(f(t) + t\,|f'(t)|\,, g(t) + t\,|g'(t)|\right) \leq \left(\frac{\delta(t)}{t}\right)'$,

(iii) $f(t) + g(t) \geq t\left(\frac{\delta(t)}{t^2}\right)'$,

for all $t \in]0,1]$ where the derivatives exist.

Example 1. Let δ_Π be the diagonal section of the product copula Π, i.e. $\delta_\Pi(t) = t^2$ for all $t \in [0,1]$. Let f and g be defined by $f(t) = g(t) = 1 - t$ for all $t \in]0,1]$. One easily verifies that the conditions of Proposition 1 are satisfied and hence, $C^{f,g}_{l,\delta_\Pi}$ is a lower semiquadratic copula with diagonal section δ_Π.

Upper semiquadratic copulas with a given diagonal section can be obtained easily from lower semiquadratic copulas with an appropriate given diagonal section.

Proposition 2. *Let δ be a diagonal function and $\hat{\delta}$ be the diagonal function defined by $\hat{\delta}(x) = 2x - 1 + \delta(1-x)$. Let $\hat{f}$ and $\hat{g}$ be two absolutely continuous functions that satisfy conditions (1) and f and g be two functions defined by $f(x) = \hat{f}(1-x)$ and $g(x) = \hat{g}(1-x)$. The function $C^{f,g}_{u,\delta} : [0,1]^2 \to [0,1]$ defined by*

$$C^{f,g}_{u,\delta}(x,y) = x + y - 1 + C^{\hat{f},\hat{g}}_{l,\hat{\delta}}(1-x, 1-y)\,, \tag{3}$$

is a copula with diagonal section δ if and only if

(i) $f(0) = g(0) = 0$,

(ii) $\max\left(f(t) + (1-t)\,|f'(t)|\,, g(t) + (1-t)\,|g'(t)|\right) \leq \left(\frac{t-\delta(t)}{1-t}\right)'$,

(iii) $f(t) + g(t) \geq (1-t)\left(\frac{2t-1-\delta(t)}{(1-t)^2}\right)'$,

for all $t \in [0,1[$ where the derivatives exist.

The function $C^{f,g}_{u,\delta}$ defined by (3) is called *an upper semiquadratic function with diagonal section δ.*

2.2 Horizontal and Vertical Semiquadratic Copulas with a Given Diagonal Section

Horizontal (resp. vertical) semiquadratic copulas with a given diagonal section are constructed by quadratic interpolation on segments connecting the diagonal of the unit square to the left (resp. lower) and right (resp. upper) boundary of the unit square. The quadratic interpolation scheme for theses classes is depicted in Fig. 1.

For any two functions $f : [0,1[\to \mathbb{R}$ and $g :]0,1] \to \mathbb{R}$ that are absolutely continuous and satisfy

$$\lim_{\substack{y\to 0\\ 0\leq x\leq y}} x(y-x)g(y) = 0 \quad \text{and} \quad \lim_{\substack{y\to 1\\ 0\leq y\leq x}} (1-x)(y-x)f(y) = 0\,, \tag{4}$$

and any diagonal function δ, the function $C^{f,g}_{h,\delta} : [0,1]^2 \to \mathbb{R}$ defined by

$$C^{f,g}_{h,\delta}(x,y) = \begin{cases} \dfrac{x}{y}\delta(y) - x(y-x)g(y) & \text{,if } 0 < x \leq y\,,\\[2ex] y - \dfrac{1-x}{1-y}(y-\delta(y)) - (1-x)(x-y)f(y) & \text{,if } y \leq x < 1\,, \end{cases} \tag{5}$$

with $C^{f,g}_{h,\delta}(0,t) = 0$ and $C^{f,g}_{h,\delta}(1,t) = t$ for all $t \in [0,1]$, is well defined. Note that the limit conditions on f and g ensure that $C^{f,g}_{h,\delta}$ is continuous. The function $C^{f,g}_{h,\delta}$ is called *a horizontal semiquadratic function with diagonal section δ* since it satisfies $C^{f,g}_{h,\delta}(t,t) = \delta(t)$ for all $t \in [0,1]$, and since it is quadratic in x on $[0,1]^2$. Note that for $f = g = 0$, the definition of a horizontal semilinear function [1] is retrieved.

We now investigate the conditions to be fulfilled by the functions f, g and δ such that the horizontal semiquadratic function $C^{f,g}_{h,\delta}$ is a copula.

Proposition 3. *Let δ be a diagonal function and let f and g be two absolutely continuous functions that satisfy conditions (4). Then the horizontal semiquadratic function $C^{f,g}_{h,\delta}$ defined by (5) is a copula with diagonal section δ if and only if*

(i) $f(0) = g(1) = 0$,
(ii) $f(t) + (1-t)\,|f'(t)| \leq \left(\frac{t-\delta(t)}{1-t}\right)'$,
(iii) $g(t) + t\,|g'(t)| \leq \left(\frac{\delta(t)}{t}\right)'$,
(iv) $tf(t) + (1-t)g(t) \geq \frac{t^2-\delta(t)}{t(1-t)}$,

for all $t \in]0,1[$ *where the derivatives exist.*

Example 2. Let δ_Π be the diagonal section of the product copula. Let f be defined by $f(t) = t$ for all $t \in [0,1[$ and g be defined by $g(t) = 1-t$ for all $t \in]0,1]$. One easily verifies that the conditions of Proposition 3 are satisfied and hence, $C^{f,g}_{h,\delta_\Pi}$ is a horizontal semiquadratic copula with diagonal section δ_Π.

Vertical semiquadratic copulas with a given diagonal section are trivially obtained as the converses of horizontal semiquadratic copulas with a given diagonal section.

Proposition 4. *Let* δ *be a diagonal function and let* f *and* g *be two absolutely continuous functions that satisfy conditions (4). The function* $C^{f,g}_{v,\delta} : [0,1]^2 \to [0,1]$*, defined by*

$$C^{f,g}_{v,\delta}(x,y) = C^{f,g}_{h,\delta}(y,x)\,, \tag{6}$$

is a copula with diagonal section δ *if and only if* $C^{f,g}_{h,\delta}$ *is a copula, i.e. under the conditions of Proposition 3.*

The function $C^{f,g}_{v,\delta}$ defined by (6) is called *a vertical semiquadratic function with diagonal section* δ.

3 Semiquadratic Copulas with a Given Opposite Diagonal Section

3.1 Lower-Upper and Upper-Lower Semiquadratic Copulas with a Given Opposite Diagonal Section

Lower-upper (resp. upper-lower) semiquadratic copulas with a given opposite diagonal section are constructed by quadratic interpolation on segments connecting the opposite diagonal of the unit square to the left (resp. right) and upper (resp. lower) boundary of the unit square. The quadratic interpolation scheme for these classes is depicted in Fig. 2.

For any two functions $f :]0,1] \to \mathbb{R}$ and $g : [0,1[\to \mathbb{R}$ that are absolutely continuous and satisfy

$$\lim_{\substack{y\to 0\\ 0\leq 1-x\leq y}} (1-y)(x+y-1)f(x) = 0 \quad \text{and} \quad \lim_{\substack{y\to 1\\ y\leq 1-x\leq 1}} x(1-x-y)g(y) = 0\,, \tag{7}$$

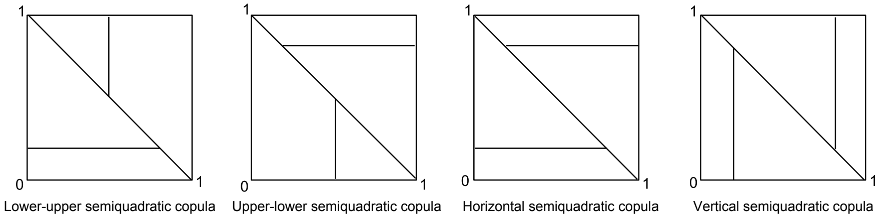

Fig. 2 Semiquadratic copulas with a given opposite diagonal section

and any opposite diagonal function ω, the function $C^{f,g}_{lu,\omega} : [0,1]^2 \to \mathbb{R}$ defined by $C^{f,g}_{lu,\omega}(x,y) =$

$$\begin{cases} \dfrac{x}{1-y}\omega(1-y) - x(1-x-y)g(y) & ,\text{if } y \leq 1-x < 1\,, \\ x+y-1+\dfrac{1-y}{x}\omega(x) - (1-y)(x+y-1)f(x) & ,\text{if } 0 < 1-y \leq x\,, \end{cases} \tag{8}$$

with $C^{f,g}_{lu,\omega}(0,t) = 0$ and $C^{f,g}_{lu,\omega}(1,t) = t$ for all $t \in [0,1]$, is well defined. Note that the limit conditions on f and g ensure that $C^{f,g}_{lu,\omega}$ is continuous. The function $C^{f,g}_{lu,\omega}$ is called *a lower-upper semiquadratic function with opposite diagonal section* ω since it satisfies $C^{f,g}_{lu,\omega}(t,1-t) = \omega(t)$ for all $t \in [0,1]$, and since it is quadratic in x on $0 \leq x+y \leq 1$ and quadratic in y on $1 \leq x+y \leq 2$. Note that for $f = g = 0$, the definition of a lower-upper semilinear function with a given opposite diagonal section [8] is retrieved.

We now investigate the conditions to be fulfilled by the functions f, g and ω such that the lower-upper semiquadratic function $C^{f,g}_{lu,\omega}$ is a copula.

Proposition 5. *Let ω be an opposite diagonal function and let f and g be two absolutely continuous functions that satisfy conditions (7). Then the lower-upper semiquadratic function $C^{f,g}_{lu,\omega}$ defined by (8) is a copula with opposite diagonal section ω if and only if*

(i) $f(1) = g(0) = 0$,
(ii) $f(t) - t\,|f'(t)| \geq \left(\frac{\omega(t)}{t}\right)'$,
(iii) $g(t) - (1-t)\,|g'(t)| \geq -\left(\frac{\omega(1-t)}{1-t}\right)'$,
(iv) $f(t) + g(1-t) \leq t\left(\frac{\omega(t)-t}{t^2}\right)'$,

for all $t \in]0,1[$ where the derivatives exist.

Example 3. Let ω_Π be the opposite diagonal section of the product copula Π, i.e. $\omega_\Pi(t) = t(1-t)$ for all $t \in [0,1]$. Let f be defined by $f(t) = 1-t$ for all $t \in]0,1]$ and g be defined by $g(t) = -t$ for all $t \in [0,1[$. One easily verifies that the conditions of

Proposition 5 are satisfied and hence, $C^{f,g}_{lu,\omega_\Pi}$ is a lower-upper semiquadratic copula with opposite diagonal section ω_Π.

Upper-lower semiquadratic copulas with a given opposite diagonal section can be obtained easily from lower-upper semiquadratic copulas with an appropriate given opposite diagonal section.

Proposition 6. *Let ω be an opposite diagonal function and $\hat{\omega}$ be the opposite diagonal function defined by $\hat{\omega}(x) = \omega(1-x)$. Let f and g be two absolutely continuous functions that satisfy conditions (7). The function $C^{f,g}_{ul,\omega} : [0,1]^2 \to [0,1]$, defined by*

$$C^{f,g}_{ul,\omega}(x,y) = C^{f,g}_{lu,\hat{\omega}}(y,x)\,, \tag{9}$$

is a copula with opposite diagonal section ω if and only if

(i) $f(1) = g(0) = 0\,,$
(ii) $f(t) - t\,|f'(t)| \geq \left(\frac{\omega(1-t)}{t}\right)'\,,$
(iii) $g(t) - (1-t)\,|g'(t)| \geq -\left(\frac{\omega(t)}{1-t}\right)'\,,$
(iv) $f(1-t) + g(t) \leq (1-t)\left(\frac{1-t-\omega(t)}{(1-t)^2}\right)'\,,$

for all $t \in]0,1[$ where the derivatives exist.

The function $C^{f,g}_{ul,\omega}$ defined by (9) is called *an upper-lower semiquadratic function with opposite diagonal section ω.*

3.2 Horizontal and Vertical Semiquadratic Copulas with a Given Opposite Diagonal Section

Horizontal (resp. vertical) semiquadratic copulas with a given opposite diagonal section are constructed by quadratic interpolation on segments connecting the opposite diagonal of the unit square to the left (resp. lower) and right (resp. upper) boundary of the unit square. The quadratic interpolation scheme for these classes is depicted in Fig. 2.

For any two functions $f :]0,1] \to \mathbb{R}$ and $g : [0,1[\to \mathbb{R}$ that are absolutely continuous and satisfy

$$\lim_{\substack{y\to 0\\ 0\leq 1-x\leq y}} (1-x)(x+y-1)f(y) = 0 \quad \text{and} \quad \lim_{\substack{y\to 1\\ y\leq 1-x\leq 1}} x(1-x-y)g(y) = 0\,, \tag{10}$$

and any opposite diagonal function ω, the function $C^{f,g}_{h,\omega} : [0,1]^2 \to \mathbb{R}$ defined by $C^{f,g}_{h,\omega}(x,y) =$

$$\begin{cases} \dfrac{x}{1-y}\omega(1-y) - x(1-x-y)g(y) & \text{,if } y \leq 1-x < 1\,, \\ x+y-1+\dfrac{1-x}{y}\omega(1-y) - (1-x)(x+y-1)f(y) & \text{,if } 0 < y \leq 1-x\,, \end{cases} \tag{11}$$

with $C_{h,\omega}^{f,g}(0,t) = 0$ and $C_{h,\omega}^{f,g}(1,t) = t$ for all $t \in [0,1]$, is well defined. Note that the limit conditions on f and g ensure that $C_{h,\omega}^{f,g}$ is continuous. The function $C_{h,\omega}^{f,g}$ is called *a horizontal semiquadratic function with opposite diagonal section* ω since it satisfies $C_{h,\omega}^{f,g}(t,1-t) = \omega(t)$ for all $t \in [0,1]$, and since it is quadratic in x on $[0,1]^2$. Note that for $f = g = 0$, the definition of a horizontal semilinear function with a given opposite diagonal section [8] is retrieved.

We now investigate the conditions to be fulfilled by the functions f, g and ω such that the horizontal semiquadratic function $C_{h,\omega}^{f,g}$ is a copula.

Proposition 7. *Let* ω *be an opposite diagonal function and let* f *and* g *be two absolutely continuous functions that satisfy conditions (10). Then the horizontal semiquadratic function* $C_{h,\omega}^{f,g}$ *defined by (11) is a copula with opposite diagonal section* ω *if and only if*

(i) $f(1) = g(0) = 0$,
(ii) $f(t) - t\,|f'(t)| \geq \left(\frac{\omega(1-t)}{t}\right)'$,
(iii) $g(t) - (1-t)\,|g'(t)| \geq -\left(\frac{\omega(1-t)}{1-t}\right)'$,
(iv) $(1-t)f(1-t) + tg(1-t) \leq 1 - \frac{\omega(t)}{t(1-t)}$,

for all $t \in]0,1[$ *where the derivatives exist.*

Example 4. Let ω_Π be the opposite diagonal section of the product copula Π. Let f be defined by $f(t) = 1-t$ for all $t \in]0,1]$ and g be defined by $g(t) = -t$ for all $t \in [0,1[$. One easily verifies that the conditions of Proposition 7 are satisfied and hence, $C_{h,\omega_\Pi}^{f,g}$ is a horizontal semiquadratic copula with opposite diagonal section ω_Π.

Vertical semiquadratic copulas with a given opposite diagonal section are trivially obtained as the converses of horizontal semiquadratic copulas with an appropriate given opposite diagonal section.

Proposition 8. *Let* ω *be an opposite diagonal function and* $\hat{\omega}$ *be the opposite diagonal function defined by* $\hat{\omega}(x) = \omega(1-x)$. *Let* f *and* g *be two absolutely continuous functions that satisfy conditions (10). The function* $C_{v,\omega}^{f,g} : [0,1]^2 \to [0,1]$, *defined by*

$$C_{v,\omega}^{f,g}(x,y) = C_{h,\hat{\omega}}^{f,g}(y,x)\,, \tag{12}$$

is a copula with opposite diagonal section ω *if and only if*

(i) $f(1) = g(0) = 0$,
(ii) $f(t) + t\,|f'(t)| \geq \left(\frac{\omega(t)}{t}\right)'$,
(iii) $g(t) + (1-t)\,|g'(t)| \geq -\left(\frac{\omega(t)}{1-t}\right)'$,
(iv) $tf(1-t) + (1-t)g(1-t) \leq 1 - \frac{\omega(t)}{t(1-t)}$,

for all $t \in]0,1[$ *where the derivatives exist.*

The function $C^{f,g}_{v,\omega}$ defined by (12) is called *a vertical semiquadratic function with opposite diagonal section* ω.

4 Conclusions

We have introduced the classes of lower, upper, horizontal and vertical semiquadratic functions with a given diagonal section as well as the classes of lower-upper, upper-lower, horizontal and vertical semiquadratic functions with a given opposite diagonal section. Moreover, we have identified the necessary and sufficient conditions on a diagonal (resp. opposite diagonal) function and two auxiliary real functions f and g to obtain a copula that has this diagonal (resp. opposite diagonal) function as diagonal (resp. opposite diagonal) section.

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