On Some Construction Methods for Bivariate Copulas

Radko Mesiar, Jozef Komorník, and Magda Komorníková

Abstract. We propose a rather general construction method for bivariate copulas, generalizing some construction methods known from the literature. In some special cases, the constraints ensuring the output of the proposed method to be a copula are given. Our approach opens several new problems in copula theory.

Keywords: Farlie–Gumbel–Morgenstern copulas, Mayor–Torrens copulas, construction method for copula.

1 Introduction

We suppose readers to be familiar with the basics of copula theory. In the opposite case, we recommend the lecture notes [12]. Recently, several construction methods for bivariate copulas have been proposed. Recall, for example, conic copulas [10], univariate conditioning method proposed in [15], UCS (univariate conditioning stable) copulas [8], a method proposed by Rodríguez–Lallena and Úbeda–Flores [17] and its generalization in [11], another method introduced by Aguilló et al. in [1], quadratic construction introduced in [13], several construction methods based on diagonal or horizontal (vertical) sections discussed in [5, 3, 7], etc.

Radko Mesiar · Magda Komorníková
Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11
813 68 Bratislava, Slovakia
e-mail: {mesiar,magda}@math.sk

Jozef Komorník
Faculty of Management Comenius University, Odbojárov 10, P.O. BOX 95
820 05 Bratislava, Slovakia
e-mail: Jozef.Komornik@fm.uniba.sk

H. Bustince et al. (eds.), *Aggregation Functions in Theory and in Practise*, Advances in Intelligent Systems and Computing 228,
DOI: 10.1007/978-3-642-39165-1_7,

We recall two of the above mentioned methods. Recall that a function $N:[0,1]\to[0,1]$ which is a decreasing involution is called a *strong negation*. We denote its unique fixed point as e, $N(e)=e$. Due to [1, Theorem 23], the next result holds.

Proposition 1. *Let $N:[0,1]\to[0,1]$ be a strong negation such that it is 1–Lipschitz on the interval $[e,1]$. Then the function $C_N:[0,1]^2\to[0,1]$ given by*

$$C_N(x,y)=\max\left(0,x\wedge y-N(x\vee y)\right) \tag{1}$$

is a copula.

Inspired by the form of the Farlie–Gumbel–Morgenstern copulas (FGM–copulas)

$$C_\lambda^{FGM}(x,y)=x\cdot y+\lambda x\cdot y\cdot(1-x)\cdot(1-y), \tag{2}$$

where $\lambda\in[-1,1]$, Kim et al. have studied in [11] the constraints for λ so that the function $C:[0,1]^2\to[0,1]$ given by

$$C(x,y)=C^*(x,y)+\lambda f(x)\cdot g(y) \tag{3}$$

is a copula, where C^* is an a priori given copula, $f,g:[0,1]\to[0,1]$ are Lipschitz continuous functions satisfying $f(0)=g(0)=f(1)=g(1)=0$.

Note that a special case of (3) when $C^*=\Pi$ under consideration was the product copula, was studied in [17].

The aim of this paper is to find a formula generalizing all above introduced formulae (5), (2), (3) and to study some of its new instances.

The paper is organized as follows. In the next section, we introduce our general formula and discuss a special case of (5) yielding Mayor–Torrens copulas [14] and open a related problem based on our generalized formula. Section 3 is focused on the product copula based constructions exploiting our generalized formula. Finally, in concluding remarks we sketch some problems for further investigations.

2 A General Formula for Constructing Bivariate Copulas

Observe that denoting by M the strongest bivariate copula, $M=\min$, formula (5) can be rewritten as

$$C_N(x,y)=\max\left(0,M(x,y)-M\left(N(x),N(y)\right)\right). \tag{4}$$

Similarly, formulae (2) and (3) can be written as

$$C(x,y)=\max\left(0,C^*(x,y)+\lambda\Pi(f(x),g(y))\right). \tag{5}$$

Denote by $\mathbf{p}$ the pentuple (C_1,C_2,λ,f,g), where $C_1,C_2:[0,1]^2\to[0,1]$ are bivariate copulas, λ is a real constant and $f,g:[0,1]\to[0,1]$ are real functions. It is evident that the formula

$$C_{\mathbf{p}}(x,y) = \max\left(0, C_1(x,y) + \lambda C_2(f(x), g(y))\right) \tag{6}$$

is a well defined real function, $C_{\mathbf{p}} : [0,1]^2 \to \Re$. Clearly, formulae (5) and (4) correspond to $\mathbf{p} = (M, M, -1, N, N)$, while formula (2) is linked to $\mathbf{p} = (\Pi, \Pi, \lambda, f, f)$ with $\lambda \in [-1,1]$ and $f(x) = x(1-x)$. Similarly, formulae (3) and (5) are related to $\mathbf{p} = (C^*, \Pi, \lambda, f, g)$. Recall a trivial result $C_{\mathbf{p}} = C_1$ whenever $\lambda = 0$.

Example 1. Consider $\mathbf{p} = (C^H, \Pi, \lambda, f, f)$, where C^H is the Hamacher product, i.e., a copula given by

$$C^H(x,y) = \frac{x \cdot y}{x + y - x \cdot y}$$

whenever $x \cdot y \neq 0$, and $f(x) = x^2 \cdot (1-x)$. After some processing with software *MATHEMATICA* it can be shown that $C_{\mathbf{p}}$ is a copula if and only if $\lambda \in [-2,1]$. Moreover, then $C_{\mathbf{p}}$ is an absolutely continuous copula given by

$$C_{\mathbf{p}}(x,y) = \frac{x \cdot y}{x + y - x \cdot y} + \lambda x^2 \cdot y^2 \cdot (1-x) \cdot (1-y).$$

On the other hand, putting $\mathbf{r} = (C^H, \Pi, \lambda, g, g)$, where $g(x) = x \cdot (1-x)$, compare (2), $C_{\mathbf{r}}$ is a copula only if $\lambda = 0$, i.e., when $C_{\mathbf{r}} = C^H$.

Consider an arbitrary copula $C : [0,1]^2 \to [0,1]$ and its diagonal section $\delta : [0,1] \to [0,1]$ given by $\delta(x) = C(x,x)$. Recall that δ is non–decreasing, 2–Lipschitz, $\delta(0) = 0$, $\delta(1) = 1$ and $\delta(x) \leq x$ for all $x \in [0,1]$. Then the function $f : [0,1] \to [0,1]$ given by $f(x) = x - \delta(x)$ is 1–Lipschitz, $f(0) = f(1) = 0$. For $\mathbf{p} = (M, M, -1, f, f)$, it holds

$$C_{\mathbf{p}}(x,y) = \max\left(0, M(x,y) - M(x - \delta(x), y - \delta(y))\right). \tag{7}$$

Applying formula (5) considering $N = f$, one gets

$$C_f(x,y) = \max\left(0, x \wedge y - f(x \vee y)\right) = \max\left(0, \delta(x \vee y) - |x - y|\right) = C_\delta^{MT}(x,y),$$

where C_δ^{MT} is a Mayor–Torrens copula [14] derived from the diagonal section δ.

On the other hand, $C_{\mathbf{p}}$ given by formula (7) for diagonal sections of 3 basic copulas W, Π, M yields copulas $W, C_{\mathbf{q}}, M$, where $\mathbf{q} = (M, M, -1, f_\Pi, f_\Pi)$, $f_\Pi(x) = x \cdot (1-x)$. Observe that the copula $C_{\mathbf{q}} : [0,1]^2 \to [0,1]$ is described in Fig. 1.

On the other side, consider the ordinal sum copula $C = \left(\left\langle 0, \frac{1}{2}, W \right\rangle, \left\langle \frac{1}{2}, 1, W \right\rangle\right)$, i.e., $f : [0,1] \to [0,1]$ given by

$$f(x) = \begin{cases} \min\left(x, \frac{1}{2} - x\right) & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \min\left(x - \frac{1}{2}, 1 - x\right) & \text{else .} \end{cases}$$

Then for $\mathbf{r} = (M, M, -1, f, f)$ the resulting function $C_{\mathbf{r}} : [0,1]^2 \to [0,1]$ satisfies

$$C_{\mathbf{r}}(x, \frac{3}{4}) = 2x - 1 \quad \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right],$$

violating the 1–Lipschitz property of $C_{\mathbf{r}}$. Thus $C_{\mathbf{r}}$ is not a copula.

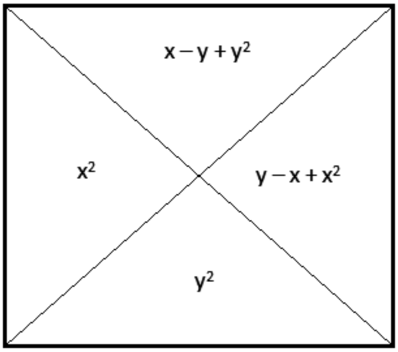

Fig. 1 Formulae for the copula $C_{\mathbf{q}}$

We open a problem of characterizing all diagonal sections δ of bivariate semicopulas such that the formula (7) yields a copula. Note also that if a function $C_{\mathbf{p}}$ given by (7) is a copula, then $C_{\mathbf{q}}$ with $\mathbf{q} = (M, M, \lambda, f, f)$ is a copula for any $\lambda \in [-1, 0]$.

3 Product–Based Construction of Copulas

Inspired by (5), consider a pentuple $\mathbf{p} = (C, C, -1, N, N)$ where N is a strong negation, i.e., consider a function $C_{\mathbf{p}} : [0,1]^2 \to [0,1]$ given by

$$C_{\mathbf{p}}(x,y) = \max\left(0, C(x,y) - C(N(x), N(y))\right). \tag{8}$$

Evidently, $C_{\mathbf{p}}$ is non–decreasing in both coordinates and satisfies the boundary conditions for copulas, i.e., $C_{\mathbf{p}}$ is a semicopula [2, 4]. For arbitrary Frank copula [9] and the standard negation $N_s : [0,1] \to [0,1]$ given by $N_s(x) = 1 - x$, we see that $C_{\mathbf{p}} = W$ is a copula.

For the 3 basic copulas, the case $C = M$ was discussed in [1], see Proposition 1. For the case $C = W$, observe that $C_{\mathbf{p}}$ is a copula if and only if $N \circ N_s \leq N_s \circ N$ and then $C_{\mathbf{p}} = W$ (this is, e.g., the case of a convex strong negation N). We focus now on the third basic copula $C = \Pi$, i.e., we will consider $\mathbf{p} = (\Pi, \Pi, -1, N, N)$, i.e., $C_{\mathbf{p}} : [0,1]^2 \to [0,1]$ given by

$$C_{\mathbf{p}}(x,y) = \max\left(0, x \cdot y - N(x) \cdot N(y)\right). \tag{9}$$

Proposition 2. *Let $N : [0,1] \to [0,1]$ be a differentiable convex strong negation. Then the function $C_{\mathbf{p}}$ given by (9) is a negative quadrant dependent copula.*

Proof. Observe first that under requirements of this proposition, $C_{\mathbf{p}}(x,y) = 0$ if and only if $y \leq N(x)$. Moreover, if $x \cdot y = N(x) \cdot N(y)$, then $N'(x) \cdot N'(y) = 1$. As for as $C_{\mathbf{p}}$ is a semicopula, it is enough to show its 2–increasingness on its positive area. Consider $0 < x_1 < x_2 \leq 1$, $0 < y_1 < y_2 \leq 1$ such that $x_1 \cdot y_1 \geq N(x_1) \cdot N(y_1)$. Then the volume $V_{C_{\mathbf{p}}}$ of the rectangle $[x_1, x_2] \times [y_1, y_2]$ is non–negative if and only if

$$(x_2 - x_1) \cdot (y_2 - y_1) \geq (N(x_1) - N(x_2)) \cdot (N(y_1) - N(y_2)) =$$

$$= (x_1 - x_2) \cdot N'(x_0) \cdot (y_1 - y_2) \cdot N'(y_0),$$

where x_0 is some point from $]x_1, x_2[$ and y_0 is some point from $]y_1, y_2[$.

Equivalently, it should hold $N'(x_0) \cdot N'(y_0) \leq 1$. Due to the fact that $x_0 \cdot y_0 > x_1 \cdot y_1$ and $N(x_0) < N(x_1)$, $N(y_0) < N(y_1)$, it holds $x_0 \cdot y_0 > N(x_0) \cdot N(y_0)$.

Consequently, $x_0 \cdot y_0 > x_0 \cdot y = N(x_0) \cdot N(y)$, where $y = N(x_0) < y_0$. Due to the convexity and monotonicity of N, it holds

$$N'(y) < N'(y_0) < 0,\ N'(x_0) < 0,$$

and hence

$$N'(x_0) \cdot N'(y_0) < N'(x_0) \cdot N'(y) = 1.$$

Thus $C_{\mathbf{p}}$ is a copula. Obviously, $C_{\mathbf{p}} \leq \Pi$, i.e., $C_{\mathbf{p}}$ is a NQD copula. □

Remark 1. As a by–product of the proof of Proposition 2, we see that for a convex differentiable strong negation N, the copula $C_{\mathbf{p}}$ given by (9) has its zero–area bounded by the graph of the function N. The same zero area is also obtained by some other kinds of constructing copulas by means of N. For example, this is the case of conic copulas based on N [3], or UCS copulas introduced by Durante and Jaworski in [8], $C_N(x,y) = x \cdot N\left(\min\left(1, \frac{N(y)}{x}\right)\right)$.

We expect that Proposition 2 is also valid for convex strong negations N which are not differentiable.

Example 2. For $c \in]0,1[$, define a function $N_c : [0,1] \to [0,1]$ by

$$N_c(x) = \begin{cases} 1 - \frac{1-c}{c}x & \text{if } x \in [0,c] \\ \frac{c(1-x)}{1-c} & \text{else .} \end{cases}$$

Then N_c is a strong negation which is convex if and only if $c \in]0, \frac{1}{2}]$.

Applying formula (9), we see that $C_{\mathbf{p}} : [0,1]^2 \to [0,1]$ is given by

$$C_{\mathbf{p}}(x,y) = \begin{cases} \frac{c^2x + c^2y - c^2 + (1-2c)x \cdot y}{(1-c)^2} & \text{if } (x,y) \in [c,1]^2, \\ \frac{(1-c)x + (y-1)c}{1-c} & \text{if } x \in [0,c] \text{ and } y \geq \frac{c-(1-c)x}{c}, \\ \frac{(1-c)y + (x-1)c}{1-c} & \text{if } y \in [0,c] \text{ and } x \geq \frac{c-(1-c)y}{c}, \\ 0 & \text{else .} \end{cases}$$

Observe that for each $c \in]0,1[$, $C_{\mathbf{p}}$ is a semicopula which is Lipschitz with constant $\max\left(1, \left(\frac{c}{1-c}\right)^2\right)$, i.e., for $c > \frac{1}{2}$, $C_{\mathbf{p}}$ is not a copula. On the other hand, for each $c \in]0, \frac{1}{2}]$, $C_{\mathbf{p}}$ is a copula.

Open problems:

i) For each convex strong negation N, putting $\mathbf{p} = (\Pi, \Pi, \lambda, N, N)$, the function $C_\mathbf{p}$ is a copula for $\lambda \in \{-1, 0\}$. Is this claim valid for each $\lambda \in [-1, 0]$? Are there some other constant λ so that $C_\mathbf{p}$ is a copula?

ii) For two convex strong negations N_1, N_2, and some $\lambda \in \Re$, does $\mathbf{p} = (\Pi, \Pi, \lambda, N_1, N_2)$ generate a copula $C_\mathbf{p}$ applying (9)?

Example 3. Consider the standard negation N_s. Applying formula (9) to $\mathbf{p} = (\Pi, \Pi, \lambda, N_s, N_s)$, it holds

$$C_\mathbf{p}(x,y) = \max(0, x \cdot y + \lambda(1-x) \cdot (1-y)) = \max(0, (1+\lambda)x \cdot y - \lambda(x+y-1)),$$

which is a copula (Sugeno–Weber t-norm, see [12]) for each $\lambda \in [-1, 0]$. For $\lambda > 0$, $C_\mathbf{p}$ is not monotone and thus not a copula (even it is not an aggregation function). For $\lambda < -1$, $C_\mathbf{p}$ is Lipschitz with constant $-\lambda$, and thus not a copula.

As another interesting fact consider the pentuple $\mathbf{p} = (\Pi, M, \lambda, f, f)$ with $f(x) = x(1-x)$. After a short processing it is not difficult to check that then $C_\mathbf{p}$ given by (6) is a copula if and only if $\lambda = 0$ and then $C_\mathbf{p} = \Pi$.

This observation opens another problem, namely whether it can be shown that for any differentiable functions f, g such that $f(0) = f(1) = g(0) = g(1) = 0$ and $f'(0), f'(1), g'(0), g'(1)$ are different from 0, compare [11], $C_\mathbf{p}$ for $\mathbf{p} = (\Pi, M, \lambda, f, g)$ is a copula only if $\lambda = 0$ (and then $C_\mathbf{p} = \Pi$).

4 Concluding Remarks

We have proposed a rather general formula (6) transforming a given copula into a real function, which in several special cases leads to new parametric families of copulas. We have discussed some of such families, but also some negative cases leading to trivial solutions only. Our proposal opens several problems for a deeper study. For example, problems of fitting copulas with special properties, such as symmetric copulas which are NQD but with Spearman's rho close to 0 (then copulas discussed in Example 1 can be of use). For several special types of $\mathbf{p}$ with fixed C_1, C_2, f, g, the problem of characterizing all constants λ such that $C_\mathbf{p}$ is a copula generalizes the problem opened by Kim et al. in [11]. For example, consider $\mathbf{p} = (M, M, \lambda, f, f)$ with $f : [0,1] \to [0,1]$ non–increasing and $f(0) = 0$. Obviously, $C_\mathbf{p}$ is then a semicopula if and only if $\lambda \leq 0$, independently of non–zero function f. As another particular problem, we can consider pentuples $\mathbf{p}_1, \mathbf{p}_2$ applied consecutively. Indeed, for $\mathbf{p}_1 = (C_1, C_2, \lambda, f, g)$ such that $C_{\mathbf{p}_1}$ is a copula, and $\mathbf{p}_2 = (C_{\mathbf{p}_1}, C_3, \tau, h, q)$ one can define $C_{\mathbf{p}_1,\mathbf{p}_2} = (C_{\mathbf{p}_1})_{\mathbf{p}_2}$, which in the case $\lambda, \tau \leq 0$ can be written as

$$C_{\mathbf{p}_1,\mathbf{p}_2}(x,y) = \max(0, C_1(x,y) + \lambda C_2(f(x), g(y)) + \tau C_3(h(x), q(y))).$$

Acknowledgements. The research summarized in this paper was supported by the Grants VEGA 1/0143/11 and APVV–0496–10.

References

[1] Aguiló, I., Suňer, J., Torrens, J.: A construction method of semicopulas from fuzzy negations. Fuzzy Sets and Systems (to appear)
[2] Bassano, B., Spizzichino, F.: Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. J. Multivariate Anal. 93(2), 313–339 (2005)
[3] De Baets, B., De Meyer, H., Kalická, J., Mesiar, R.: Flipping and cyclic shifting of binary aggregation functions. Fuzzy Sets and Systems 160(6), 752–765 (2009)
[4] Durante, F., Sempi, C.: Semicopulae. Kybernetika 41(3), 315–328 (2005)
[5] Durante, F., Kolesárová, A., Mesiar, R., Sempi, C.: Copulas with given diagonal sections: novel constructions and applications. International Journal of Uncertainty, Fuzziness and Knowlege-Based Systems 15(4), 397–410 (2007)
[6] Durante, F., Kolesárová, A., Mesiar, R., Sempi, C.: Copulas with given values on a horizontal and a vertical section. Kybernetika 43(2), 209–220 (2007)
[7] Durante, F., Kolesárová, A., Mesiar, R., Sempi, C.: Semilinear copulas. Fuzzy Sets and Systems 159(1), 63–76 (2008)
[8] Durante, F., Jaworski, P.: Invariant dependence structure under univariate truncation. Statistics 46, 263–267 (2012)
[9] Frank, M.J.: On the simultaneous associativity of F(x, y) and x+y-F(x, y). Aequationes Math. 19(2-3), 194–226 (1979)
[10] Jwaid, T., De Baets, B., Kalická, J., Mesiar, R.: Conic aggregation functions. Fuzzy Sets and Systems 167(1), 3–20 (2011)
[11] Kim, J.-M., Sungur, E.A., Choi, T., Heo, T.-Y.: Generalized Bivariate Copulas and Their Properties. Model Assisted Statistics and Applications-International Journal 6, 127–136 (2011)
[12] Klement, E.P., Mesiar, R., Pap, E.: Triangular norms. Trends in Logic-Studia Logica Library, vol. 8. Kluwer Academic Publishers, Dordrecht (2000)
[13] Kolesárová, A., Mesiar, R., Kalická, J.: On a new construction of 1–Lipschitz aggregation functions, quasi–copulas and copulas. Fuzzy Sets and Systems (2013), doi:10.1016/j.fss.2013.01.005
[14] Mayor, G., Torrens, J.: On a class of binary operations: Non–strict Archimedean aggregation functions. In: Proceedings of ISMVL 1988, Palma de Mallorca, Spain, pp. 54–59 (1988)
[15] Mesiar, R., Jágr, V., Juráňová, M., Komorníková, M.: Univariate conditioning of copulas. Kybernetika 44(6), 807–816 (2008)
[16] Nelsen, R.B.: An introduction to copulas, 2nd edn. Springer Series in Statistics. Springer, New York (2006)
[17] Rodríguez–Lallena, J.A., Úbeda–Flores, M.: A new class of bivariate copulas. Statistics & Probability Letters 66, 315–325 (2004)