Implications Satisfying the Law of Importation with a Given T-norm

S. Massanet and J. Torrens

Abstract. The main goal of this paper is to characterize all fuzzy implications with continuous natural negation that satisfy the law of importation with a given continuous t-norm T. Particular cases when the fixed t-norm T is the minimum, the product and the Łukasiewicz t-norm are deduced from the general result and the corresponding characterizations are presented separately.

Keywords: Implication function, Law of importation, t-norm, t-conorm.

1 Introduction

One of the most important connectives used in fuzzy control and approximate reasoning are fuzzy implications. This is because they are the generalization of binary implications in classical logic to the framework of fuzzy logic and consequently they are used to perform fuzzy conditionals [15, 18, 24]. In addition of modelling fuzzy conditionals, they are also used to perform backward and forward inferences in any fuzzy rules based system through the inference rules of modus ponens and modus tollens [17, 24, 33].

Moreover, fuzzy implications have proved to be useful in many other fields like fuzzy relational equations [24], fuzzy DI-subsethood measures and image processing [11, 1], fuzzy morphological operators [19], computing with words [24], data mining [41] and rough sets [32], among others. Thus, it is not surprising that fuzzy implications have attracted the efforts of many researchers not only from the point of view of their applications, but also from the purely theoretical perspective. See for instance the surveys [6] and [24] and the book [5], entirely devoted to fuzzy implications.

S. Massanet · J. Torrens
University of the Balearic Islands, Palma de Mallorca 07122, Spain
e-mail: {s.massanet,jts224}@uib.es

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From this theoretical point of view, there are several lines of research that have been specially developed. Among them we can highlight the following ones:

1. The study of the different classes of fuzzy implications and their axiomatic characterization (see [5] and the references therein, but also the recent works [1, 8]).
2. The relationship among these classes and the intersections between them (see again [5] and the references therein, as well as the recent works [9] and [26]).
3. The study of new construction methods of fuzzy implications (see [5, 27, 28, 31, 30, 36]).
4. The analysis of additional properties of fuzzy implications.

In the last line, there are a lot of properties that have been studied in detail by many authors. In almost all the cases the interest of each property comes from its specific applications and its theoretical study usually reduces to the solution of a functional equation. Some of the most studied properties are:

a) *The modus ponens*, because it becomes crucial in the inference process through the compositional rule of inference (CRI). Some works on this property are [21, 38, 39, 40].
b) *The distributivity properties* over conjunctions and disjunctions. In this case, these distributivities allow to avoid the combinatorial rule explosion in fuzzy systems ([13]). They have been extensively studied again by many authors, see [2, 3, 7, 10, 34, 35, 37].
c) *The law of importation*. This property is extremely related to the exchange principle (see [25]) and it has proved to be useful in simplifying the process of applying the CRI in many cases, see [16] and [5]. It can be written as

$$I(T(x,y),z) = I(x,I(y,z)) \quad \text{for all} \quad x,y,z \in [0,1],$$

where T is a t-norm (or a more general conjunction) and I is a fuzzy implication. The law of importation has been studied in [5, 16, 22, 23, 25]. Moreover, in this last article the law of importation has also been used in new characterizations of some classes of implications like (S,N)-implications and R-implications. Finally, it is a crucial property to characterize Yager's implications (see [29]).

Although all these works devoted to the law of importation, there are still some open problems involving this property. In particular, given any t-norm T, it is an open problem to find all fuzzy implications I such that they satisfy the law of importation with respect to this fixed t-norm T. That is, find all fuzzy implications such that

$$I(T(x,y),z) = I(x,I(y,z)) \quad \text{for all} \quad x,y,z \in [0,1], \qquad \textbf{(LI)}$$

being T any fixed (continuous) t-norm.

In this paper we want to deal with this problem and we will give some partial solutions (in the sense that we will find all solutions involving fuzzy implications with an additional property). Specifically, we will characterize all fuzzy implications with continuous natural negation that satisfy the law of importation with any given continuous t-norm T. Particular cases when the fixed t-norm T is the minimum, the product and the Łukasiewicz t-norm are deduced from the general result and the corresponding characterizations are presented separately.

2 Preliminaries

We will suppose the reader to be familiar with the theory of t-norms. For more details in this particular topic, we refer the reader to [20]. To make this work self-contained, we recall here some of the concepts and results used in the rest of the paper. First of all, the definition of fuzzy negation is given.

Definition 1. (Definition 1.1 in [14]) A decreasing function $N : [0,1] \to [0,1]$ is called a *fuzzy negation*, if $N(0) = 1, N(1) = 0$. A fuzzy negation N is called

(i) *strict*, if it is strictly decreasing and continuous,
(ii) *strong*, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

Next lemma plays an important role in the results presented in this paper. Essentially, given a fuzzy negation, it defines a new fuzzy negation which in some sense can perform the role of the inverse of the original negation.

Lemma 1. *(Lemma 1.4.10 in [5]) If N is a continuous fuzzy negation, then the function* $\mathfrak{R}_N : [0,1] \to [0,1]$ *defined by*

$$\mathfrak{R}_N(x) = \begin{cases} N^{(-1)}(x) & \text{if } x \in (0,1], \\ 1 & \text{if } x = 0, \end{cases}$$

where $N^{(-1)}$ *stands for the pseudo-inverse of N given by*

$$N^{(-1)}(x) = \sup\{z \in [0,1] \mid N(z) > x\}$$

for all $x \in [0,1]$*, is a strictly decreasing fuzzy negation. Moreover,*

$$\mathfrak{R}_N^{(-1)} = N,$$

$$N \circ \mathfrak{R}_N = id_{[0,1]},$$

$$\mathfrak{R}_N \circ N|_{Ran(\mathfrak{R}_N)} = id|_{Ran(\mathfrak{R}_N)},$$

where $Ran(\mathfrak{R}_N)$ *stands for the range of function* $\mathfrak{R}_N$.

Next, we introduce the concept of automorphism and conjugate function.

Definition 2. A function $\varphi : [0,1] \to [0,1]$ is an automorphism if it is continuous and strictly increasing and satisfies the boundary conditions $\varphi(0) = 0$ and $\varphi(1) = 1$, i.e., if it is an increasing bijection from $[0,1]$ to $[0,1]$.

Definition 3. Let $\varphi : [0,1] \to [0,1]$ be an automorphism. Two functions $f, g : [0,1]^n \to [0,1]$ are φ-conjugate if $g = f_\varphi$, where

$$f_\varphi(x_1,\ldots,x_n) = \varphi^{-1}(f(\varphi(x_1),\ldots,\varphi(x_n))), \quad x_1,\ldots,x_n \in [0,1].$$

Note that given an automorphism $\varphi : [0,1] \to [0,1]$, the φ-conjugate of a t-norm T, that is T_φ, and the φ-conjugate of an implication I (see Definition 4), that is I_φ, are again a t-norm and an implication, respectively.

Now, we recall the definition of fuzzy implications.

Definition 4. (Definition 1.15 in [14]) A binary operator $I : [0,1]^2 \to [0,1]$ is said to be a *fuzzy implication* if it satisfies:

(I1) $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$.
(I2) $I(x,y) \leq I(x,z)$ when $y \leq z$, for all $x \in [0,1]$.
(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not derived from the definition. Fuzzy implications can satisfy additional properties coming from tautologies in crisp logic. In this paper, we are going to deal with the law of importation, already presented in the introduction.

The natural negation of a fuzzy implication will be also useful in our study.

Definition 5. (Definition 1.4.15 in [5]) Let I be a fuzzy implication. The function N_I defined by $N_I(x) = I(x,0)$ for all $x \in [0,1]$, is called the *natural negation* of I.

Remark 1
(i) If I is a fuzzy implication, N_I is always a fuzzy negation.
(ii) Given a binary function $F : [0,1]^2 \to [0,1]$, we will denote by $N_F(x) = F(x,0)$ for all $x \in [0,1]$ its 0-horizontal section. In general, N_F is not a fuzzy negation. In fact, it is trivial to check that N_F is a fuzzy negation if, and only if, $F(x,0)$ is a decreasing function satisfying $F(0,0) = 1$ and $F(1,0) = 0$.

3 On the Satisfaction of (LI) with a Given T-norm T

In this section, we want to characterize all fuzzy implications with a continuous natural negation which satisfy the Law of Importation (LI) with a fixed t-norm T. Until now, all the previous studies on the law of importation have focused on the satisfaction of (LI) by concrete classes of fuzzy implications. Thus, some results involving this property and (S,N)-implications are presented in [16] and [25]; R-implications in [16]; QL-implications in [5] and [22] and Yager's implications in [4]. On the other hand, fixed a concrete t-norm T, it is still an open problem to know which fuzzy implications satisfy (LI) with this T.

First of all, it is worth to study if fixed a concrete t-norm T, any fuzzy negation can be the natural negation of a fuzzy implication satisfying (LI) with T. In fact, there exists some dependence between the t-norm T and the natural negation of the fuzzy implication I. Thus, not all fuzzy negations can be natural negations of a fuzzy implication satisfying (LI) with a concrete t-norm. To characterize which fuzzy negations are compatible with a t-norm T in this sense, the following property will be considered:

$$\text{if } N(y) = N(y') \text{ for some } y, y' \in [0,1], \text{ then } N(T(x,y)) = N(T(x,y')) \quad \forall x \in [0,1]. \tag{1}$$

Proposition 1. *Let $I : [0,1]^2 \to [0,1]$ be a binary function such that N_I is a fuzzy negation. If I satisfies (LI) with a t-norm T, then N_I and T satisfy Property (1).*

The following example illustrates the previous result.

Example 1. Let N be the continuous (non-strict) fuzzy negation given by

$$N_1(x) = \begin{cases} -2x+1 & \text{if } 0 \le x < 0.25, \\ 0.5 & \text{if } 0.25 \le x \le 0.75, \\ 2-2x & \text{otherwise,} \end{cases}$$

and $T = T_P$, the product t-norm. Consider now a fuzzy implication I with $N_I = N_1$. Then it can not satisfy (LI) with T_P since in this case Property (1) does not hold: $N_1(0.25) = 0.5 = N_1(0.75)$ but

$$N_1(T_P(0.1,0.25)) = N_1(0.025) = 0.95 \neq 0.85 = N_1(0.075) = N_1(T_P(0.1,0.75)).$$

This implies that on the one hand,

$$I(0.1, I(0.25,0)) = I(0.1, N_1(0.25)) = I(0.1, N_1(0.75)) = I(0.1, I(0.75,0)),$$

but on the other hand,

$$I(T_P(0.1,0.25),0) = N_1(T_P(0.1,0.25)) \neq N_1(T_P(0.1,0.75)) = I(T_P(0.1,0.75),0),$$

and (LI) does not hold.

Next result gives the expression of any binary function with N_I a continuous fuzzy negation satisfying (LI) with a t-norm T. The binary function only depends on the t-norm T and its natural negation.

Proposition 2. *Let $I : [0,1]^2 \to [0,1]$ be a binary function with N_I a continuous fuzzy negation satisfying (LI) with a t-norm T. Then*

$$I(x,y) = N_I(T(x, \mathfrak{R}_{N_I}(y))).$$

From now on, we will denote these implications generated from a t-norm T and a fuzzy negation N by $I_{N,T}(x,y) = N(T(x, \mathfrak{R}_N(y)))$.

Remark 2. Instead of $\mathfrak{R}_{N_I}$, we can consider any function N_1 such that $N_1^{(-1)} = N_I$ and $N_I \circ N_1 = \mathrm{id}_{[0,1]}$. This is a straightforward consequence of the satisfaction of Property (1) in this case. Since $N_I(\mathfrak{R}_{N_I}(y)) = N_I(N_1(y))$, then using the aforementioned property, $N_I(T(x,\mathfrak{R}_{N_I}(y))) = N_I(T(x,N_1(y)))$ and therefore, $I_{N_I,T}$ can be computed using either $\mathfrak{R}_{N_I}$ or N_1.

This class of implications is contained into the class of (S,N)-implications generated from a continuous negation N. This fact is coherent with the characterization of (S,N)-implications, where (LI) is involved, given in Theorem 22 in [25].

Theorem 1. *Let N be a continuous negation and T a t-norm satisfying Property (1). Then $I_{N,T}$ is an (S,N)-implication generated from $S(x,y) = N(T(\mathfrak{R}_N(x),\mathfrak{R}_N(y)))$ and N.*

Moreover, this class of implications satisfies (LI) with the same t-norm T from which they are generated.

Proposition 3. *Let N be a continuous fuzzy negation and T a t-norm satisfying Property (1). Then $I_{N,T}$ satisfies (LI) with T.*

Now, we are in condition to fully characterize the binary functions I with N_I a continuous fuzzy negation satisfying (LI) with a t-norm T.

Theorem 2. *Let $I : [0,1]^2 \to [0,1]$ be a binary function with N_I a continuous fuzzy negation and T a t-norm. Then*

$$I \text{ satisfies (LI) with } T \Leftrightarrow N_I \text{ and } T \text{ satisfy Property (1) and } I = I_{N_I,T}.$$

Note that it remains to know when N_I and T satisfy Property (1). From now on, we will try given a concrete continuous t-norm T, to determine which fuzzy negations satisfy the property with T.

4 Characterization of Fuzzy Implications Satisfying (LI) with a Continuous T-norm

In the previous section, Example 1 shows that Property (1) does not hold for any t-norm and fuzzy negation. Consequently, given a fixed t-norm T, in order to characterize all fuzzy implications with a continuous natural negation satisfying (LI) with T, we need to know which fuzzy negations are compatible with the t-norm T. In this section, we will answer this question for some continuous t-norms presenting for each one, which fuzzy negations can be considered and which fuzzy implications satisfying (LI) with T are generated in that case.

First of all, some negations satisfy Property (1) with any t-norm T (not necessarily continuous).

Proposition 4. *Let N be a continuous fuzzy negation. If there exists $x_0 \in [0,1)$ such that $N(x_0) = 1$ and N is strictly decreasing in $(x_0,1)$ then Property (1) holds for any t-norm T.*

Remark 3. Note that the previous result includes strict fuzzy negations which are compatible with any t-norm T.

4.1 Minimum T-norm

The first t-norm we are going to study is the minimum t-norm $T_M(x,y) = \min\{x,y\}$. This t-norm performs well with any continuous negation not restricting the choice of the fuzzy negation.

Proposition 5. *If $T = T_M$, then Property (1) holds for any continuous negation N.*

At this point, we can characterize all fuzzy implications with continuous natural negation satisfying (LI) with T_M.

Theorem 3. *Let $I : [0,1]^2 \to [0,1]$ be a binary function with N_I a continuous fuzzy negation. Then the following statements are equivalent:*

(i) I satisfies (LI) with T_M.
(ii) I is given by $I(x,y) = \max\{N_I(x), y\}$.

Remark 4. The fuzzy implications satisfying (LI) with T_M are, in fact, the so-called generalized Kleene-Dienes implications. In particular, if $N_I(x) = N_C(x) = 1-x$, we retrieve the Kleene-Dienes implication $I_{KD}(x,y) = \max\{1-x, y\}$. On the other hand, note that there are other implications satisfying (LI) with T_M than those given in Theorem 3. Of course, they must have non-continuous natural negation like the Gödel implication given by

$$I_{GD}(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ y & \text{if } x > y, \end{cases}$$

which satisfies (LI) with T_M using Theorem 7.3.5 in [5].

4.2 Continuous Archimedean T-norms

In contrast with the minimum t-norm, not all continuous fuzzy negations are compatible with Archimedean t-norms.

Proposition 6. *If T is an Archimedean t-norm, Property (1) holds if, and only if, N is a continuous fuzzy negation being strictly decreasing for all $x \in (0,1)$ such that $N(x) < 1$.*

Remark 5. Note that if we consider Archimedean t-norms, compatible fuzzy negations are strict ones and those given by the expression:

$$N(x) = \begin{cases} 1 & \text{if } x \in [0, x_0], \\ N'\left(\frac{x-x_0}{1-x_0}\right) & \text{otherwise}, \end{cases} \tag{2}$$

where $x_0 \in (0,1)$ and N' is a strict negation. Thus, if the fuzzy negation is a continuous (non-strict) negation, it can only have a unique constant region and it must value 1 there.

Recall that continuous Archimedean t-norms are divided in two subsets: nilpotent t-norms and strict t-norms. So, from now on, we will study these two cases separately.

4.2.1 Nilpotent T-norms

Nilpotent t-norms are φ-conjugated with the Łukasiewicz t-norm $T_{LK}(x,y) = \max\{x+y-1,0\}$, i.e., $T = (T_{LK})_\varphi$ for some automorphism φ. The following result characterizes completely fuzzy implications satisfying (LI) with these t-norms.

Theorem 4. *Let $I : [0,1]^2 \to [0,1]$ be a binary function with N_I a continuous fuzzy negation and $\varphi : [0,1] \to [0,1]$ an automorphism. Then the following statements are equivalent:*

(i) I satisfies (LI) with $(T_{LK})_\varphi$.
(ii) One of the following two cases hold:

(a) If N_I is strict, then I is given by

$$I(x,y) = \begin{cases} 1 & \text{if } y > (N_I \circ (N_C)_\varphi)(x), \\ f^{-1}(f(N_I(x)) + f(y) - 1) & \text{if } y \le (N_I \circ (N_C)_\varphi)(x). \end{cases}$$

where $f = \varphi \circ N_I^{-1}$, and $N_C(x) = 1 - x$ denotes the classical negation.
(b) If N_I is given by Equation (2) with $x_0 \in (0,1)$ and N' a strict negation, then I is given by $I(x,y) =$

$$= \begin{cases} 1 & \text{if } y > N'\left(\frac{\varphi^{-1}(1-\varphi(x)+\varphi(x_0))-x_0}{1-x_0}\right), \\ N'\left(\dfrac{\varphi^{-1}(\varphi(x)+\varphi(x_0+(1-x_0)N'^{-1}(y))-1)-x_0}{1-x_0}\right) & \text{if } y \le N'\left(\frac{\varphi^{-1}(1-\varphi(x)+\varphi(x_0))-x_0}{1-x_0}\right). \end{cases}$$

Taking $\varphi(x) = x$, a particular result for the Łukasiewicz t-norm can be deduced.

Corollary 1. *Let $I : [0,1]^2 \to [0,1]$ be a binary function with N_I a continuous fuzzy negation. Then the following statements are equivalent:*

(i) I satisfies (LI) with T_{LK}.
(ii) One of the two following cases hold:

(a) If N_I is strict, then I is given by

$$I(x,y) = \begin{cases} 1 & \text{if } y > (N_I \circ N_C)(x), \\ N_I(x + N_I^{-1}(y) - 1) & \text{if } y \le (N_I \circ N_C)(x). \end{cases}$$

(b) If N_I is given by Equation (2) with $x_0 \in (0,1)$ and N' a strict negation, then

$$I(x,y) = \begin{cases} 1 & \text{if } y > N'\left(\frac{1-x}{1-x_0}\right), \\ N'\left(N'^{-1}(y) - \dfrac{1-x}{1-x_0}\right) & \text{if } y \leq N'\left(\frac{1-x}{1-x_0}\right). \end{cases}$$

Remark 6. Taking $N_I = N_C$, the Łukasiewicz implication $I_{LK}(x,y) = \min\{1, 1-x+y\}$ is obtained from case (a) in the previous corollary. Again note that there are other implications satisfying (LI) with T_{LK} than those given in Corollary 1. Of course, they must have non-continuous natural negation like the Weber implication, which satisfies (LI) with T_{LK} by Example 7.3.3-(ii) in [5] and is given by

$$I_{WB}(x,y) = \begin{cases} 1 & \text{if } x < 1, \\ y & \text{if } x = 1. \end{cases}$$

4.2.2 Strict T-norms

Strict t-norms are those t-norms T which are φ-conjugated with the product t-norm $T_P(x,y) = xy$, i.e., $T = (T_P)_\varphi$ for some automorphism φ. Example 1 shows that strict t-norms do not satisfy Property (1) with every fuzzy negation, as it is stated in Proposition 6.

The following result allows us to characterize fuzzy implications satisfying (LI) with a fixed strict t-norm.

Theorem 5. *Let $I : [0,1]^2 \to [0,1]$ be a binary function with N_I a continuous fuzzy negation and $\varphi : [0,1] \to [0,1]$ an automorphism. Then the following statements are equivalent:*

(i) I satisfies (LI) with $(T_P)_\varphi$.
(ii) One of the two following cases hold:

(a) If N_I is strict, then I is given by

$$I(x,y) = g^{-1}(g(N_I(x)) \cdot g(y))$$

where $g = \varphi \circ N_I^{-1}$.
(b) If N_I is given by Equation (2) with $x_0 \in (0,1)$ and N' a strict negation, then

$$I(x,y) = \begin{cases} 1 & \text{if } y \geq N'\left(\dfrac{\varphi^{-1}\left(\frac{\varphi(x_0)}{\varphi(x)}\right) - x_0}{1-x_0}\right), \\ N'\left(\dfrac{\varphi^{-1}(\varphi(x)\cdot\varphi(x_0 + (1-x_0)\cdot N'^{-1}(y))) - x_0}{1-x_0}\right) & \text{if } y < N'\left(\dfrac{\varphi^{-1}\left(\frac{\varphi(x_0)}{\varphi(x)}\right) - x_0}{1-x_0}\right). \end{cases}$$

Remark 7. The fuzzy implications obtained in case (a) of the previous result are in fact φ-conjugates of Yager's f-generated implications with $f(0) < \infty$ such that $f = g \circ \varphi^{-1}$, since

$$(I_f)_\varphi(x,y) = \varphi^{-1}(f^{-1}(\varphi(x) \cdot f(\varphi(y)))) = g^{-1}(g(N_I(x)) \cdot g(y)).$$

Recall that φ-conjugates of Yager's f-generated implications with $f(0) < \infty$ are characterized as the only binary operations satisfying (LI) with $(T_P)_\varphi$ and with N_I a strict fuzzy negation (see Theorem 8 in [29]).

Taking $\varphi(x) = x$, a particular result for the product t-norm can be deduced.

Corollary 2. *Let $I : [0,1]^2 \to [0,1]$ be a binary function with N_I a continuous fuzzy negation. Then the following statements are equivalent:*

(i) I satisfies (LI) with T_P.
(ii) One of the two following cases hold:

(a) If N_I is strict, then I is given by

$$I(x,y) = N_I(x \cdot N_I^{-1}(y)).$$

(b) If N_I is given by Equation (2) with $x_0 \in (0,1)$ and N' a strict negation, then

$$I(x,y) = \begin{cases} 1 & \text{if } y \geq N'\left(\frac{x_0(1-x)}{x(1-x_0)}\right), \\ N'\left(\dfrac{x\cdot(x_0+(1-x_0)\cdot N'^{-1}(y))-x_0}{1-x_0}\right) & \text{if } y < N'\left(\frac{x_0(1-x)}{x(1-x_0)}\right). \end{cases}$$

Remark 8. The fuzzy implications obtained in case (a) of the previous result are in fact Yager's f-generated implications with $f(0) < \infty$ such that $f = N_I^{-1}$. Recall that Yager's f-generated implications with $f(0) < \infty$ are characterized as the only binary operations satisfying (LI) with T_P and with N_I a strict fuzzy negation (see Theorem 6 in [29]).

Remark 9. Note that there are other implications satisfying (LI) with T_P than those given in Corollary 2. Of course, they must have non-continuous natural negation like the Yager implication given by

$$I_{YG}(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ and } y = 0, \\ y^x & \text{if } x > 0 \text{ or } y > 0, \end{cases}$$

which satisfies (LI) with T_P using Theorem 7.3.4 in [5].

5 Conclusions and Future Work

In this paper, we have characterized all fuzzy implications satisfying (LI) with a t-norm T when the natural negation of the implication is continuous. Moreover, we have determined in particular the expression of these implications when some continuous t-norms are considered: the minimum t-norm and the Archimedean continuous ones. The fuzzy implications obtained in these cases are always (S,N)-implications but often, they belong also to other well-known classes as Yager's f-generated implications with $f(0) < \infty$ and their conjugate implications.

As a future work, we want to study the case when an ordinal sum t-norm is considered in order to cover all continuous t-norms. In addition, some non-continuous t-norms as the drastic t-norm and the nilpotent minimum worth to be studied.

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