Norm Aggregations and OWA Operators

José M. Merigó and Ronald R. Yager

Abstract. The ordered weighted average (OWA) is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum. This paper studies the use of the OWA operator with norms. Several extensions and generalizations are suggested including the use of the induced OWA operator and the OWA weighted average. This approach represents a general frameworkof the aggregation operators when dealing with distance and similarity measures. Some key particular cases are studied including the addition OWA and the subtraction OWA operator

1 Introduction

The ordered weighted average (OWA) [17] is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum. It has been used in a wide range of applications [1, 28] and has been extended and generalized in a wide range of directions. For example, Fodor et al. [2] presented a generalization by using quasi-arithmetic means. Yager and Filev [27] introduced the induced OWA (IOWA) operator providing a more general reordering process. Merigó and Gil-Lafuente [9] extended this approach by using generalized and quasi-arithmetic means. Other authors have studied the use of distance measures with the OWA operator [3, 10, 15, 29]. In this direction, it is worth noting a recent development by Yager [28] regarding the use of OWA operators with norms that generalizes distance and similarity measures under the same framework.

José M. Merigó
Manchester Business School, University of Manchester, Booth Street West, M15 6PB, Manchester, UK
e-mail: jose.merigolindahl@mbs.ac.uk

Ronald R. Yager
Machine Intelligence Institute, Iona College 10801, New Rochelle, NY, USA
e-mail: yager@panix.com

H. Bustince et al. (eds.), *Aggregation Functions in Theory and in Practise*, Advances in Intelligent Systems and Computing 228,
DOI: 10.1007/978-3-642-39165-1_17,

A further interesting approach is those aggregation operators that integrate the OWA operator with the weighted average. Several approaches have been proposed in this direction by many authors including the weighted OWA (WOWA) [12], the hybrid average [19], the importance OWA [21] and the immediate weights [11, 6]. Recently, Merigó [6] has suggested the OWA weighted average (OWAWA) as a generalization that unifies both concepts in the same formulation and considering the degree of importance they have in the specific aggregation taken into account.

The aim of this paper is to develop further extensions regarding the use of OWA operators with norms. It is presented the use of the IOWA operator with norms forming the IOWA norm (IOWAN) aggregation. Thus, it is possible to represent a wide range of norm aggregations from the minimum to the maximum and under complex reordering processes. Next, it is introduced the use of the OWAWA operator obtaining the OWAWA norm (OWAWAN) that provides a unified framework between the usual weighted average norm and the OWAN operator. Several families and particular cases are studied including the addition OWAWA (A-OWAWA), the subtraction OWAWA (S-OWAWA) and many other cases. These operators seem to be of great importance because they may provide a new methodology for dealing with arithmetic operations. The use of the variance [20, 25] as a particular type of norm is also considered.

This paper is organized as follows. Section 2 reviews some basic preliminaries regarding the OWA and the OWAWA operator and norm aggregations. Section 3 introduces the use of the OWAWA operator with norms and Section 4 studies a wide range of particular cases. Section 5 ends the paper summarizing the main conclusions of the paper.

2 Preliminaries

This section briefly reviews the OWA operator, the OWAWA operator and norm aggregations.

2.1 The OWA Operator

The OWA operator is an aggregation operator that considers a wide range of averaging operators that move between the minimum and the maximum. It permits to aggregate the information considering the degree of optimism or pessimism that a decision maker wants to use in the aggregation. It has been used in a wide range of applications including soft computing, decision making and statistics [1, 28]. It can be defined as follows.

Definition 1. An OWA operator of dimension n is a mapping $OWA : R^n \rightarrow R$ that has an associated weighting W and $\sum_{j=1}^{n} w_j = 1$, such that:

$$OWA(a_1,...,a_n) = \sum_{j=1}^{n} w_j b_j \quad (1)$$

where b_j is the jth largest of the a_i .

Several properties could be studied including different families of OWA operators and measures for characterizing the weighting vector [17, 18]. Note that in most of the OWA literature, the arguments are reordered according to a weighting vector. However, it is also possible to reorder the weighting vector according to the initial positions of the arguments a_i [22] and this is of great importance in order to integrate the weighted average and the OWA in the same formulation.

The OWA operator can be extended by using induced aggregation operators [9, 24] forming the induced OWA (IOWA) operator [27]. Therefore, it is possible to consider a more general reordering process that deals with complex situations.

2.2 The OWAWA Operator

The ordered weighted averaging - weighted average (OWAWA) [6] is a model that unifies the OWA operator and the weighted average in the same formulation considering the degree of importance that each concept has in the analysis. Therefore, both concepts can be seen as a particular case of a more general one. One of its key advantages is that it can be reduced to the usual weighted average or to the OWA. Therefore, any study that uses the OWA or the weighted average can be revised and extended with the OWAWA operator provides a more complete analysis of the information considered. It can be defined as follows.

Definition 2. An OWAWA operator of dimension n is a mapping $OWAWA : R^n \to R$ that has an associated weighting W of dimension n such that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$OWAWA(a_1,...,a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \quad (2)$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) w_j$ with $\beta \in [0,1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth largest of the a_i.

It provides a parameterized family of aggregation operators from the minimum to the maximum. The difference against the OWA is that it also considers subjective information. Thus, it is possible to consider a partial boundary condition that considers the minimum and maximum adjusted with the weighted average [6]. Note that if $\beta = 1$, we get the OWA operator and if $\beta = 0$, the WA. The OWAWA operator accomplishes similar properties than the usual OWA aggregation operators including the symmetry, the use of mixture operators and so on [6].

2.3 Norm Aggregations

Norm aggregations provide a more general representation of the aggregation when dealing with distance measures because they allow us to include more complex operations in the analysis. A norm associates with some vector or tuple $X = (x_1, x_2, ..., x_n)$ a unique non-negative scalar. A norm is a function $f : R^n \to [0, \infty)$ that has the following properties [1, 28]:

1. $f(x_1, x_2, ..., x_n) = 0$ if and only if all $x_i = 0$.
2. $f(aX) = |a| f(X)$.
3. $f(X) + f(Y) \geq f(X+Y)$, that is, the triangle inequality.

When dealing with averaging functions, norms can be used following a similar methodology as it is used with distance measures [7]. Thus, with the weighted average it can be formulated the following expression:

$$f(a_1, a_2, ..., a_n) = G(|a_1|, |a_2|, ..., |a_n|) = \sum_{i=1}^{n} w_i |a_i|, \tag{3}$$

Recently, Yager [25] has suggested the use of norms in the OWA operator by using:

$$f(a_1, a_2, ..., a_n) = G(|a_1|, |a_2|, ..., |a_n|) = \sum_{j=1}^{n} w_j N_j, \tag{4}$$

where N_j is the jth largest of the $|a_i|$.

Note that norms can be used in order to get a distance or a metric function assuming that if f is a norm then $d(X,Y) = f(a = |X-Y|)$.

3 Norms with OWAWA Operators

Norms are useful in a wide range of situations because they include many aggregation operators and distance measures as particular cases. Among others, it is worth noting the usual average, the Hamming distance [4] and the variance.This paper presents several extensions and generalizations by using a wide range of averaging aggregation operators with norms. First, let us consider the use of the induced OWA (IOWA) operator with norms forming the IOWA norm (IOWAN) operator. Its main advantage is that it considers complex reordering processes in the aggregation of norms providing a parameterized family of norms from the minimum to the maximum one. Note that this is of great interest because when dealing with norms, not always the highest or the lowest one is the preferred one. It can be defined as follows.

Definition 3. An IOWAN operator of dimension n is a mapping $IOWAN : R^n \times R^n \to R$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$IOWAN\left(\langle u_1, |a_1|\rangle, \langle u_2, |a_2|\rangle, ..., \langle u_n, |a_n|\rangle\right) = \sum_{j=1}^{n} w_j N_j, \tag{5}$$

where N_j is the a_i value of the IOWAN pair $\langle u_i, a_i \rangle$ having the jth largest u_i, u_i is the order-inducing variable and $|a_i|$ is the argument variable represented in the form of individual norms.

Note that the IOWAN operator is reduced to the usual IOWA operator when $|a_i| = a_i$. It can also be seen as a distance measure when $d(X,Y) = f(a = |X - Y|)$ becoming the induced OWA distance (IOWAD) operator [7]. Furthermore, it is also possible to formulate it by using generalized means forming the induced generalized OWA (IGOWA) operator ($|a| = a^{\lambda}$) [9] and the induced Minkowski OWA distance (IMOWAD) ($d(X,Y) = f(a = |X - Y|^{\lambda})$) [8].

Next, let us look into a more general framework by using the OWAWA operator with norms forming the OWAWA norm (OWAWAN) operator. Thus, it is possible to integrate norms with weighted averages (Eq. 3) and OWA operators in the same formulation and considering the degree of importance that each concept has in the aggregation. Thus, all the particular types of norms used previously can also be included in this framework including the use of distance and similarity measures. It can be defined as follows.

Definition 4. An OWAWAN operator is a mapping $OWAWAN : R^n \to R$ of dimension n, if it has an associated weighting vector W, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$ and a weighting vector V that affects the WA, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, such that:

$$OWAWAN\left(|a_1|, |a_2|, ..., |a_n|\right) = \beta \sum_{j=1}^{n} w_j N_j + (1-\beta) \sum_{i=1}^{n} v_i |a_i| \tag{6}$$

where N_j is the jth smallest of the $|a_i|$, each argument $|a_i|$ is the argument variable represented in the form of individual norms and $\beta \in [0,1]$.

Note that the OWAWAN operator can be formulated integrating both equations into a single one as it is done with the OWAWA operator [6] as follows:

$$OWAWAN\left(|a_1|, |a_2|, ..., |a_n|\right) = \sum_{j=1}^{n} \hat{v}_j N_j, \tag{7}$$

where N_j is the jth smallest of the $|a_i|$, each argument $|a_i|$ is the argument variable represented in the form of individual norms and has an associated weight v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$ and v_j is the weight (WA) v_i ordered according to N_j, that is, according to the jth smallest of the $|a_i|$.

Observe that this is possible only when dealing with arithmetic averaging functions. If it is used a generalized or quasi-arithmetic mean with $\lambda \neq 1$ or $g(a) \neq a$, it is not possible to integrate it in this way.

As we can see, if $\beta = 1$, the OWAWAN operator becomes the OWAN operator and if $\beta = 0$, the weighted averaging norm (WAN). The OWAWAN operator accomplishes similar properties than the norm aggregation operators [28].

Note that Eq. (6) has been presented adapting the ordering of the weighted average to the OWA operator. However, it is also possible to formulate the OWAWA operator adapting the ordering of the OWA operator to the weighted average as:

$$OWAWAN\left(|a_1|, |a_2|, \ldots, |a_n|\right) = \beta \sum_{i=1}^{n} w_i\,|a_i| + (1-\beta) \sum_{i=1}^{n} v_i\,|a_i|, \tag{8}$$

where each argument a_i has an associated weight w_i that represents the weight w_j ordered according to the ordering of the arguments a_i and $\beta \in [0,1]$.

A further interesting issue appears when the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$ or $V = \sum_{i=1}^{n} v_i \neq 1$. In these situations and without considering the concept of heavy aggregations [23], the OWAWAN operator can be formulated in the following way:

$$OWAWAN\left(|a_1|, |a_2|, \ldots, |a_n|\right) = \frac{\beta}{W} \sum_{j=1}^{n} w_j N_j + \frac{(1-\beta)}{V} \sum_{i=1}^{n} v_i |a_i|, \tag{9}$$

Similarly to the IOWAN, the OWAWAN operator can also be seen as a distance metric by using $d(X,Y) = f(a = |X-Y|)$. Thus, it becomes the OWAWA distance (OWAWAD) operator [11] that can be formulated as follows:

$$OWAWAD\left(|x_1, y_1|, |x_2, y_2|, \ldots, |x_n, y_n|\right) = \beta \sum_{j=1}^{n} w_j D_j + (1-\beta) \sum_{i=1}^{n} v_i |x_i - y_i|, \tag{10}$$

where D_j is the jth smallest of the $|x_i - y_i|$, each argument $|x_i - y_i|$ is the argument variable represented in the form of individual distances and $\beta \in [0,1]$.

Note that the main advantage of the OWAWAD is that it integrates the weighted Hamming distance (WHD) and the OWA distance (OWAD) [10, 15] in the same formulation considering the degree of importance that each concept has in the formulation. As we can see, if $\beta = 1$, the OWAWAD operator becomes the OWAD operator and if $\beta = 0$, the WHD.

Furthermore, it is also possible to use generalized and quasi-arithmetic means in the analysis. Thus, the OWAWAN operator becomes the GOWAWA and the Quasi-OWAWA operator. With generalized means this is obtained when $|a| = a^{\lambda}$ and with quasi-arithmetic means if $|a| = g(a)$.

An interesting issue when analysing these aggregation operators is to characterize the weighting vector. This can be done following the OWA literature where it is considered the degree of orness (attitudinal character) [6, 17], the entropy of dispersion [6, 17] and the divergence of the weights [6, 23]. When dealing with OWAWA operators, the degree of orness can be formulated from two different perspectives. A first perspective assumes that the weighted average can also be studied with this measure being the objective to determine the tendency of the aggregation to the minimum or to the maximum:

$$\alpha(\hat{V}) = \beta \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right) + (1-\beta) \sum_{j=1}^{n} v_j \left(\frac{n-j}{n-1}\right). \tag{11}$$

It is straightforward to calculate the andness measure by using the dual. That is, $Andness(\hat{V}) = 1 - \alpha(\hat{V})$. The other perspective is focussed on the attitudinal character. In this case, the weighted average is seen as a neutral aggregation because it only considers the subjective opinion. Thus, it is reasonable to assume that the orness measure for this part of the equation should be 0.5 obtaining the following expression:

$$\alpha(\hat{V}) = \beta \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right) + (1-\beta) \times 0.5. \tag{12}$$

In this case it is also trivial to form the andness measure or the degree of pessimism.

The entropy of dispersion measures the amount of information being used in the aggregation. If we extend this approach to the OWAWAN operator, it is obtained the following formulation:

$$H(\hat{V}) = -\left(\beta \sum_{j=1}^{n} w_j \ln(w_j) + (1-\beta) \sum_{i=1}^{n} v_i \ln(v_i)\right). \tag{13}$$

As we can see, if $\beta = 1$, we obtain the Yager entropy of dispersion for the OWAN operator and if $\beta = 0$, we get the classical Shannon entropy [13].

The divergence [6, 23] measures the divergence of the weights against the attitudinal character. It is useful in various situations, especially when the attitudinal character and the entropy of dispersion are not enough to correctly analyse the weighting vector of an aggregation. If we extend the divergence to the OWAWAN operator, we get the following divergence:

$$Div(\hat{V}) = \beta \left(\sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2\right) + (1-\beta) \left(\sum_{j=1}^{n} v_j \left(\frac{n-j}{n-1} - \alpha(V)\right)^2\right). \tag{14}$$

Note that if $\beta = 1$, we get the OWAN divergence and if $\beta = 0$, the WAN divergence. Moreover, it is also possible to consider a variation of Eq. (14) by using Eq. (12). In this case, the divergence of the weighted average is 0 and it is only considered the divergence of the OWAN operator.

Finally, let us consider the use of norms under other frameworks that unifies the OWA operator and the weighted average in the same formulation. Among others, let us consider the use of hybrid averages [19], WOWA operators [15] and immediate weights [11, 6]. By using the hybrid average it is formed the hybrid averaging norm (HAN) that can be formulated as follows.

$$HAN(|a_1|, |a_2|, ..., |a_n|) = \sum_{j=1}^{n} w_j N_j, \tag{15}$$

where N_j is the jth smallest of the $|\hat{a}_i|$ $(\hat{a}_i = n\omega_i |a_i|, i = 1,2,...,n)$, $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weighting vector of the $|a_i|$, with $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. With the WOWA operator it is obtained the WOWA norm (WOWAN) operator. It is formulated in the following way.

Definition 5. Let P and W be two weighting vectors of dimension n, with $P = (p_1, p_2, ..., p_n)$ and $W = (w_1, w2, ..., w_n)$, such that $p_i \in [0,1]$ and $\sum_{i=1}^{n} p_i = 1$, and $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. A mapping $WOWAN : R^n \times R^n \to R$ is a WOWAN operator of dimension n if:

$$WOWAN(|a_1|, |a_2|, ..., |a_n|) = \sum_{i=1}^{n} \omega_i |a_{\sigma(i)}|, \tag{16}$$

where $\{\sigma(1), ..., \sigma(n)\}$ is a permutation of $\{1, ..., n\}$ such that $a_{\sigma(i-1)} \geq a_{\sigma(i)}$ for all $i = 2, ..., n$, and the weight ω_i is defined as:

$$\omega_i = w* \left(\sum_{j \leq i} p_{\sigma(j)} \right) - w* \left(\sum_{j < i} p_{\sigma(j)} \right), \tag{17}$$

with $w*$ a monotone increasing function that interpolates the points $(i/n, \sum_{j \leq i} w_j)$ together with the point $(0,0)$. $w*$ is required to be a straight line when the points can be interpolated in this way.

The use of immediate weights forms the immediate weighted averaging norm (IWAN) and it is constructed by using the following expression:

$$IWAN(|a_1|, |a_2|, ..., |a_n|) = \sum_{j=1}^{n} \hat{v}_j |N_j|, \tag{18}$$

where N_j is the jth smallest of the $|a_i|$, each $|a_i|$ has associated a weight v_i, v_j is the associated weight of N_j, and $\hat{v}_j = (w_j v_j / \sum_{j=1}^{n} w_j v_j)$.

Note that this expression can only be used when dealing with arithmetic averaging operators as in the OWAWAN operator because with generalized aggregation operators, the formulation may have some incorrect deviations.

Furthermore, observe that similar extensions and generalizations could also be studied with induced and generalized aggregation operators [9, 30] and by using distance measures as it has been explained before.

4 Families of OWAWAN Operators

The OWAWAN operator includes a wide range of particular cases. First, it is possible to study several families by analyzing the weighting vector [6, 18]. Thus, it is possible to form the following cases:

- Simple averaging norm: $w_j = 1/n$ and $v_i = 1/n$, for all i, j.
- Arithmetic WAN (AWAN): $w_j = 1/n$, for all j.
- Arithmetic OWAN (AOWAN): $v_i = 1/n$, for all i.
- Min-WAN: $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$.
- Max-WAN: $w_n = 1$ and $w_j = 0$, for all $j \neq n$.
- Step OWAWAN: $w_k = 1$ and $w_j = 0$, for all $j \neq k$.
- Hurwicz WAN: $w_1 = 1 - \alpha$, $w_n = \alpha$ and $w_j = 0$ for all $j \neq 1, n$.
- Window OWAWAN: $w_j = 1/m$ for $k \leq j \leq k+m-1$ and $w_j = 0$ for all $j > k+m, j < k$.
- Median odd OWAWAN: If n is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others.
- Median even OWAWAN: If n is even we assign $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_j = 0$ for all others.
- Olympic OWAWAN: $w_1 = w_n = 0$, and for others $w_j = 1/(n-2)$.

Some other interesting cases are found by analyzing a different expression in the norm. Among others, it is worth noting the following ones:

- If $OWAWAN(X,Y) = f(X+Y)$, we obtain the addition OWAWA (A-OWAWA) operator that can be formulated as:

$$A - OWAWA\left(|x_1 + y_1|, \ldots, |x_n + y_n|\right) = \beta \sum_{j=1}^{n} w_j A_j + (1-\beta) \sum_{i=1}^{n} v_i |x_i + y_i|, \quad (19)$$

  where A_j is the jth smallest of the $|x_i + y_i|$ and $\beta \in [0,1]$.
- If $OWAWAN(X,Y) = f(X-Y)$, the subtraction OWAWA (S-OWAWA) operator and it is expressed as follows:

$$S - OWAWA\left(|x_1 - y_1|, \ldots, |x_n - y_n|\right) = \beta \sum_{j=1}^{n} w_j S_j + (1-\beta) \sum_{i=1}^{n} v_i (x_i - y_i), \quad (20)$$

  where S_j is the jth smallest of the $(x_i - y_i)$ and $\beta \in [0,1]$.
- If $OWAWAN(X,Y) = f(X \times Y)$, we get the multiplication OWAWA (M-OWAWA) operator that is defined in the following way:

$$M - OWAWA\left(|x_1 \times y_1|, \ldots, |x_n \times y_n|\right) = \beta \sum_{j=1}^{n} w_j M_j + (1-\beta) \sum_{i=1}^{n} v_i |x_i \times y_i|, \quad (21)$$

  where M_j is the jth smallest of the $|x_i \times y_i|$ and $\beta \in [0,1]$.
- If $OWAWAN(X,Y) = f(X \div Y)$, we obtain the division OWAWA (D-OWAWA) operator:

$$D - OWAWA\left(|x_1 \div y_1|, \ldots, |x_n \div y_n|\right) = \beta \sum_{j=1}^{n} w_j D_j + (1-\beta) \sum_{i=1}^{n} v_i |x_i \div y_i|, \quad (22)$$

  where D_j is the jth smallest of the $|x_i \div y_i|$ and $\beta \in [0,1]$.

- If $OWAWAN(X,Y) = f((X-Y)^2)$, it is formed the variance OWAWA (Var-OWAWA) operator:

$$Var-OWAWA\left((x_1-y_1)^2,\ldots,(x_n-y_n)^2\right) = \beta\sum_{j=1}^{n} w_j A_j + (1-\beta)\sum_{i=1}^{n} v_i(x_i-y_i)^2, \tag{23}$$

where A_j is the jth smallest of the $(x_i-y_i)^2$ and $\beta \in [0,1]$.

Note that all these cases can be reduced to the OWA and the weighted average version forming the addition OWA, subtraction OWA, multiplication OWA, addition WA, and so on. Moreover, observe that many other cases could be studied including the OWAWAD operator explained in Section 3 and OWAWA operators with other similarity measures or operations.

5 Conclusions

This paper has suggested the use of OWAWA operators with norms. The main advantage of this approach is that it considers a unified framework between the OWA and the weighted average when aggregating information with norms. Thus, it is possible to consider subjective opinions and the attitudinal character of the decision maker in the same formulation. Several fundamental properties have been studied. It has been shown that the OWAWAD operator is also a particular case of this approach.

Some other extensions have also been considered including the use of induced aggregation operators (IOWAN operator) and the use of other approaches for unifying the OWA with the weighted average that have formed the HAN operator, the WOWA operator and the IWAN operator. Several key families of OWAWAN operators have also been studied including the addition OWAWA, the subtraction OWAWA, the multiplication OWAWA, the division OWAWA and the variance OWAWA operator. These aggregation operators have shown the potential for developing a new framework for arithmetic operations.

Acknowledgements. Support from the European Commission (PIEF-GA-2011-300062)is gratefully acknowledged.

References

[1] Beliakov, G., Pradera, A., Calvo, T.: Aggregation functions: A guide for practitioners. STUDFUZZ, vol. 221. Springer, Heidelberg (2007)
[2] Fodor, J., Marichal, J.L., Roubens, M.: Characterization of the ordered weighted averaging operators. IEEE Trans. Fuzzy Syst. 3, 236–240 (1995)
[3] Grabisch, M., Marichal, J.L., Mesiar, R., Pap, E.: Aggregation functions: Means. Inform. Sci. 181, 1–22 (2011)
[4] Hamming, R.W.: Error-detecting and error-correcting codes. Bell Syst. Tech. J. 29, 147–160 (1950)

[5] Karayiannis, N.: Soft learning vector quantization and clustering algorithms based on ordered weighted aggregation operators. IEEE Trans. Neural Networks 11, 1093–1105 (2000)
[6] Merigó, J.M.: A unified model between the weighted average and the induced OWA operator. Expert Syst. Applic. 38, 11560–11572 (2011)
[7] Merigó, J.M., Casanovas, M.: Decision making with distance measures and induced aggregation operators. Computers & Indust. Engin. 60, 66–76 (2011)
[8] Merigó, J.M., Casanovas, M.: A new Minkowski distance based on induced aggregation operators. Int. J. Computational Intelligence Syst. 4, 123–133 (2011)
[9] Merigó, J.M., Gil-Lafuente, A.M.: The induced generalized OWA operator. Inform. Sci. 179, 729–741 (2009)
[10] Merigó, J.M., Gil-Lafuente, A.M.: New decision-making techniques and their application in the selection of financial products. Inform. Sci. 180, 2085–2094 (2010)
[11] Merigó, J.M., Gil-Lafuente, A.M.: Decision making techniques in business and economics based on the OWA operator. SORT - Stat. Oper. Res. Trans. 36, 81–101 (2012)
[12] Merigó, J.M., Xu, Y.J., Zeng, S.Z.: Group decision making with distance measures and probabilistic information. Knowledge-Based Syst. 40, 81–87 (2013)
[13] Shannon, C.E.: A mathematical theory of communication. Bell Syst. Tech. J. 27, 379–423 (1948)
[14] Torra, V.: The weighted OWA operator. Int. J. Intelligent Syst. 12, 153–166 (1997)
[15] Xu, Z.S., Chen, J.: Orderedweighted distance measure. J. Syst. Sci. Syst. Engin. 17, 432–445 (2008)
[16] Xu, Z.S., Da, Q.L.: An overview of operators for aggregating information. Int. J. Intelligent Syst. 18, 953–969 (2003)
[17] Yager, R.R.: On ordered weighted averaging aggregation operators in multi-criteria decision making. IEEE Trans. Syst. Man Cybern. B 18, 183–190 (1988)
[18] Yager, R.R.: Families of OWA operators. Fuzzy Sets Syst. 59, 125–148 (1993)
[19] Yager, R.R.: On the inclusion of variance in decision making under uncertainty. Int. J. Uncert. Fuzz. Knowledge-Based Syst. 4, 401–419 (1996)
[20] Yager, R.R.: Including importances in OWA aggregation using fuzzy systems modelling. IEEE Trans. Fuzzy Syst. 6, 286–294 (1998)
[21] Yager, R.R.: New modes of OWA information fusion. Int. J. Intelligent Syst. 13, 661–681 (1998)
[22] Yager, R.R.: Heavy OWA operators. Fuzzy Optim. Decision Making 1, 379–397 (2002)
[23] Yager, R.R.: Induced aggregation operators. Fuzzy Sets Syst. 137, 59–69 (2003)
[24] Yager, R.R.: Generalizing variance to allow the inclusion of decision attitude in decision making under uncertainty. Int. J. Approximate Reasoning 42, 137–158 (2006)
[25] Yager, R.R.: Norms induced from OWA operators. IEEE Trans. Fuzzy Syst. 18, 57–66 (2010)
[26] Yager, R.R., Engemann, K.J., Filev, D.P.: On the concept of immediate probabilities. Int. J. Intell. Syst. 10, 373–397 (1995)
[27] Yager, R.R., Filev, D.P.: Induced ordered weighted averaging operators. IEEE Trans. Syst. Man Cybern. B 29, 141–150 (1999)
[28] Yager, R.R., Kacprzyk, J., Beliakov, G.: Recent developments on the ordered weighted averaging operators: Theory and practice. Springer, Heidelberg (2011)
[29] Zeng, S.Z., Su, W., Le, A.: Fuzzy generalized ordered weighted averaging distance operator and its application to decision making. Int. J. Fuzzy Syst. 14, 402–412 (2012)
[30] Zhou, L.G., Chen, H.Y., Liu, J.P.: Generalized power aggregation operators and their applications in group decision making. Computers & Indust. Engin. 62, 989–999 (2012)