Conditional Preference Nets and Possibilistic Logic

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Abstract. CP-nets (Conditional preference networks) are a well-known compact graphical representation of preferences in Artificial Intelligence, that can be viewed as a qualitative counterpart to Bayesian nets. In case of binary attributes it captures specific partial orderings over Boolean interpretations where strict preference statements are defined between interpretations which differ by a single flip of an attribute value. It respects preferential independence encoded by the ceteris paribus property. The popularity of this approach has motivated some comparison with other preference representation setting such as possibilistic logic. In this paper, we focus our discussion on the possibilistic representation of CP-nets, and the question whether it is possible to capture the CP-net partial order over interpretations by means of a possibilistic knowledge base and a suitable semantics. We show that several results in the literature on the alleged faithful representation of CP-nets by possibilistic bases are questionable. To this aim we discuss some canonical examples of CP-net topologies where the considered possibilistic approach fails to exactly capture the partial order induced by CP-nets, thus shedding light on the difficulties encountered when trying to reconcile the two frameworks.

1 Introduction

The representation and the handling of preferences has been extensively studied in artificial intelligence (AI), operations research, and data bases; see [1] for an introductory survey. "CP-nets" [2] have been especially popular in AI as a framework for expressing conditional preferences, based on a graphical representation. CP-nets express that in a given context, a partially described situation is strictly preferred to another partially described situation, every other variable having the same value in both situations; this is the *ceteris paribus* condition.

However the systematic application of the ceteris paribus principle introduces restrictions in the expression of preferences. This has motivated the comparison between CP-nets and possibilistic logic [3] since the latter provides another flexible setting for representing preferences [4, 5]. In possibilistic logic, classical propositions state goals, and weights are priority levels that express how imperative are these goals. A merit of a logic-based representation of preferences is also the capability of reasoning about preferences and in particular to deal with their possible inconsistency. A series of publications [6–10] have dealt with the question of representing CP-nets by means of a possibilistic logic base. Since CP-nets may leave some interpretations non comparable, a possibilistic logic

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representation of them should use partially ordered symbolic weights [11] that leave room for incomparability. It has been also noticed that CP-nets implicitly privilege the preference constraints associated with father nodes with respect to the ones associated to children nodes in the graphical representation.

However, the possibilistic logic representation of CP-nets advocated in [8–10] is not always completely faithful and may remain locally approximate. The aim of this paper is to fully investigate this state of facts, also highlighting when the existing approach does provide an exact representation for CP-nets.

The paper is organized as follows. First, a short background on possibilistic logic, on CP-nets and its encoding with possibilistic logic formulas having symbolic weights is provided in Sections 2 and 3. Then in Section 4 we discuss the different partial orders that can be used for comparing the vectors of symbolic weights which reflect the violation of preferences and are associated with each interpretation. Used as such, each of the considered orders are successful for retrieving the CP-net ordering on specific graphical structures and fail on others, as shown in Section 5. Section 6 identifies on which particular structures the existing possibilistic representation is exact, and shows more generally how lower and upper representations can be obtained. Section 7 briefly discusses the related work and exhibits a final example that points out the difficulty of capturing the CP-net ordering exactly in a logical way.

2 Possibilistic Logic

We consider a propositional language where formulas are denoted by p_1, ..., p_n, and Ω is its set of interpretations. Let $B^N = \{(p_j, \alpha_j) \mid j = 1, \ldots, m\}$ be a possibilistic logic base where p_j is a propositional logic formula and $\alpha_j \in \mathcal{L} \subseteq [0, 1]$ is a priority level [3]. The logical conjunctions and disjunctions are denoted $\wedge$ and $\vee$. Each formula (p_j, α_j) means that $N(p_j) \geq \alpha_j$, where N is a necessity measure, i.e., a set function satisfying the property $N(p \wedge q) = \min(N(p), N(q))$. A necessity measure is associated to a possibility distribution π (a mapping $\Omega \to [0, 1]$ here expressing preference) as follows:

$$N(p) = \min_{\omega \notin M(p)}(1 - \pi(\omega)) = 1 - \Pi(\neg p),$$

where Π is the possibility measure associated to N and $M(p)$ is the set of models induced by the underlying propositional language for which p is true.

The base B^N is associated to the possibility distribution

$$\pi_B^N(\omega) = \min_{j=1,\ldots,m} \pi_{(p_j,\alpha_j)}(\omega)$$

on the set of interpretations, where $\pi_{(p_j,\alpha_j)}(\omega) = 1$ if $\omega \in M(p_j)$, and $\pi_{(p_j,\alpha_j)}$ $(\omega) = 1 - \alpha_j$ if $\omega \notin M(p_j)$. An interpretation ω is all the more possible as it does not violate any formula p_j having a higher priority level α_j. So, if $\omega \notin M(p_j)$, $\pi_B^N(\omega) \leq 1 - \alpha_j$, and if $\omega \in \bigcap_{j \in J} M(\neg p_j)$, $\pi_B^N(\omega) \leq \min_{j \in J}(1 - \alpha_j)$. It is a description "from above" of π_B^N, which is the least specific possibility distribution in agreement with the knowledge base B^N. A possibilistic base B^N can be transformed in a base where the formulas p_i are clauses (without altering the distribution π_B^N). We can still see B^N as a conjunction of weighted clauses, i.e., as an extension of the conjunctive normal form.

3 CP-Nets and Their Encoding in Possibilistic Logic

A CP-net [2] is graphical in nature, and exploits conditional preferential independence in structuring the preferences provided by a user. The model is reminiscent of a Bayes net; however, the nature of the relation between nodes within a network is generally quite weak, compared with the probabilistic relations in Bayes nets. The aim in using the graph is to capture statements of qualitative conditional preferential independence.

Definition 1. *A CP-net $\mathcal{N}$ over the set of Boolean variables $V = \{X_1, \cdots, X_n\}$ is a directed graph over the nodes $X_1, \cdots, X_n$, and there is a directed edge from X_i to X_j if the preference over the value X_j is conditioned on the value of X_i. Each node $X_i \in V$ is associated with a conditional preference table $CPT(X_i)$ that associates a strict preference ($x_i > \neg x_i$ or $\neg x_i > x_i$) with each possible instantiation u_i of the parents of X_i (if any).*

A complete (preference) ordering of interpretations satisfies a CP-net $\mathcal{N}$ iff it satisfies each conditional preference expressed in $\mathcal{N}$. In this case, the ordering is said to be *consistent* with $\mathcal{N}$. We denote by $Pa(X)$ the set of direct parent variables of X, and by $Ch(X)$ the set of direct successors (children) of X. The set of interpretations of a group of variables $S \subseteq V$ is denoted by $Ast(S)$, with $\Omega = Ast(V)$. Given a CP-net $\mathcal{N}$, for each node $X_i, i = 1, \ldots, n$, each entry in a conditional preference table CPT_i is of the form $\phi = u : \star x_i > \star\neg x_i$, where $u \in Ast(Pa(X_i)), \star$ is blank if the preference is $x_i > \neg x_i$ and is $\neg$ otherwise. This is encoded by a constraint of the form $N(\neg u \vee \star x_i) \geq \alpha_i > 0$, in possibility theory, where N is a necessity measure [3]. The weight α_i stands for the priority of the formula $\neg u \vee \star x_i$. Although valued on [0, 1] this priority is not instantiated, that is, α_i is a variable attached to node i. It expresses that having $\neg\star x_i$ is somewhat not satisfactory in context u, as the possibility of $\neg \star x_i \wedge u$ is upper bounded by $1 - \alpha_i$. Clearly, satisfying $\neg \star x_i \wedge u$ is all the more impossible as α_i is large.

The encoding of a CP-net in possibilistic logic is performed as follows:

- According to the above conventions, each entry of the form $u : \star x_i > \star\neg x_i$ in the conditional preference table CPT_i of each node $X_i, i = 1, \ldots, n$ is encoded by the possibilistic logic clause $(\neg u \vee \star x_i, \alpha_i)$, where $\alpha_i > 0$ is a symbolic weight.
- Since the same weight is attached to each clause built from CPT_i, the set of weighted clauses induced from CPT_i is thus equivalent to the weighted conjunction $\phi_i = (\bigwedge_{u \in Ast(Pa(X_i))} (\neg u \vee \star x_i), \alpha_i)$, one per variable, or to the pair of weighted clauses (ϕ_i^+, ϕ_i^-) of the form:
$$(\neg(\vee_{u \in A_i^+} u) \vee x_i, \alpha_i), (\neg(\vee_{u \in A_i^-} u) \vee \neg x_i, \alpha_i),$$
where $\{A_i^+, A_i^-\}$ is a partition of $Ast(Pa(X_i))$, such that $x_i > \neg x_i$ on A_i^+ and $\neg x_i > x_i$ on A_i^-.
- Additional constraints over weights are added. The weight α_i attached to each node X_i, is supposed to be strictly smaller than the weight of each of its parents α_i^* (thus leading to constraints of the form $\max(\{\alpha_i\}) < \alpha_i^*$).

A partially ordered possibilistic base $(\Sigma, \succeq_\Sigma)$ is built from a CP-net in this way, where $\succeq_\Sigma$ stands for the order relation over weights. Let us denote by $\mathcal{F}_\omega \subseteq \Sigma$, the set of formulas falsified by the interpretation $\omega \in \Omega$. For each interpretation ω, we associate a vector $\boldsymbol{\omega}(\Sigma)$ obtained as follows. For each weighted formula $\phi_i^+ \wedge \phi_i^-$ in the possibilistic base Σ satisfied by ω, we put 1 in the i^{th} component of the vector, and $1-\alpha_i$ otherwise, in agreement with possibilistic logic semantics [3]. By construction, $L = \{1, 1-\alpha_i, i = 1\ldots, n\}$, with $1 > 1-\alpha_i, \forall i$. Vector $\boldsymbol{\omega}(\Sigma)$ has a specific format. Namely its component v_i (one per CP-net node) lies in $\{1, 1-\alpha_i\}$ for $i = 1, \ldots, n$. We consider different possible partial orders for comparing such vectors in the next section.

Example 1: [2]. Fig. 1(a) illustrates a CP-net about preferences for evening dress. It involves variables J, P, and S, standing for the jacket, pants, and shirt:

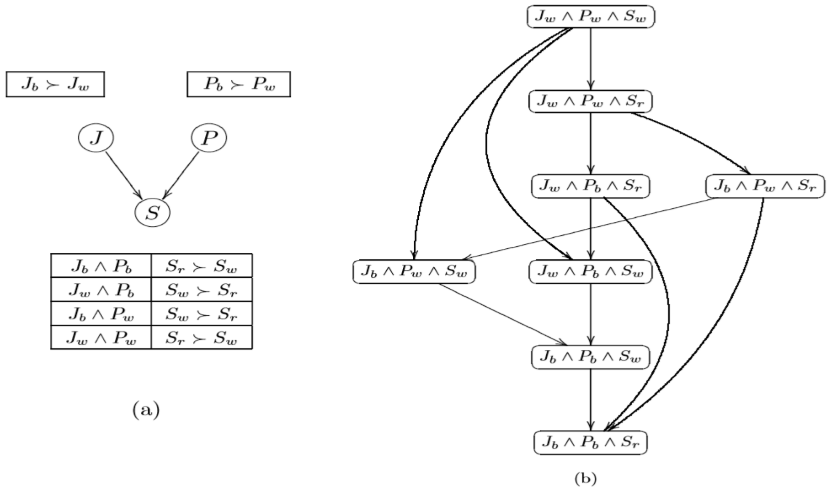

Fig. 1. CP-net and partial order induced by it

- preferred color is black (b) rather than white (w) for J and P: $P_b > P_w$, which yields formula $\phi_P = (P_b, \alpha)$, and $J_b > J_w$, which yields formula $\phi_J = (J_b, \beta)$.
- the preference between the red and white shirts is conditioned on the combination of jacket and pants: if they have the same color, then a white shirt will make my outfit too colorless, thus a red shirt is preferred: $P_b \wedge J_b : S_r > S_w$; $P_w \wedge J_w : S_r > S_w$, which yields formula $\phi_S^- = (\neg(J = P) \vee S_r, \gamma)$.
- Otherwise, if the jacket and the pants are of different colors, then a red shirt will probably make the outfit too flashy, thus a white shirt is preferred. $P_b \wedge J_w : S_w > S_r$; $P_w \wedge J_b : S_w > S_r$, which yields formula $\phi_S^+ = ((J = P) \vee S_w, \gamma)$. Moreover, we assume $\alpha > \gamma$ and $\beta > \gamma$ since P and S are father nodes of J.

4 Partial Order Relations over Vectors

In this section we will present a number of partial order relations with the purpose to use them to generate a particular ordering over interpretations.

In Section 3, we have shown how to encode a CP-net in a possibilistic logic format. Since we can associate a vector to each interpretation with respect to formulas in the possibilistic base, comparing two interpretations amounts to comparing their associated vectors. We first give definitions of some order relations over vectors, and then discuss how to capture CP-net orderings when we interpret possibilistic logic bases based on these vector comparison techniques. Let $\boldsymbol{v} = (v_1, ..., v_k), \boldsymbol{v'} = (v'_1, ..., v'_k) \in L^k$ be two vectors, where L is a scale partially ordered by $>$:

Definition 2 (Pareto). $\boldsymbol{v} \succ_{Pareto} \boldsymbol{v'}$ *if and only if* $\forall i, v_i \geq v'_i$ *and* $\exists j, v_j > v'_j$.

Definition 3 (symmetric Pareto). $\boldsymbol{v} \succ_{SP} \boldsymbol{v'}$ *if and only if there exists a permutation* σ *the components of* $\boldsymbol{v'}$*, yielding vector* $\boldsymbol{v'}^{\sigma}$*, such that* $\boldsymbol{v} \succ_{Pareto} \boldsymbol{v'}^{\sigma}$.

The discrimin order, denoted by $\succ_{discrimin}$ is defined for totally ordered scales in the following way: identical vector components are discarded, and the minimum of the remaining components for each vector are compared. Since here the minimum does not always correspond to a single value, but to subsets of L^k, we propose the following procedure for comparing the vectors:

Definition 4 (discrimin). *Let* $\mathfrak{D}(\boldsymbol{v}, \boldsymbol{v'}) = \{j | v_j \neq v'_j\}$ *be the set of component indices where the two vectors* $\boldsymbol{v}$ *and* $\boldsymbol{v'}$ *differ. Then* $v \succ_{discrimin} v'$ *iff*
$\min(\{v_i | i \in \mathfrak{D}(\boldsymbol{v}, \boldsymbol{v'})\} \cup \{v'_i | i \in \mathfrak{D}(\boldsymbol{v}, \boldsymbol{v'})\}) \subseteq \{v'_i | i \in \mathfrak{D}(\boldsymbol{v}, \boldsymbol{v'})\} \setminus \{v_i | i \in \mathfrak{D}(\boldsymbol{v}, \boldsymbol{v'})\}$.
where min *here returns the subset of the smallest incomparable values (wrt* $>$*).*

In the standard case of a totally ordered scale, the leximin order is defined by first reordering the vectors in an increasing way and then applying the discrimin order to the reordered vectors. Since we deal with a partial order, the reordering of vectors is no longer unique, and we have to generalize the definition:

Definition 5 (leximin). *First, delete all pairs* (v_i, v'_j) *such that* $v_i = v'_j$ *in* $\boldsymbol{v}$ *and* $\boldsymbol{v'}$ *(each deleted component can be used only one time in the deletion process). Thus, we get two non overlapping sets* $r(\boldsymbol{v})$ *and* $r(\boldsymbol{v'})$ *of remaining components, namely* $r(\boldsymbol{v}) \cap r(\boldsymbol{v'}) = \emptyset$*. Then,* $v \succ_{lex} v'$ *iff* $\min(r(\boldsymbol{v}) \cup r(\boldsymbol{v'})) \subseteq r(\boldsymbol{v'})$.

In the following, we shall apply these relations to the particular vectors associated to the possibilistic encoding of CP-nets, as explained in Section 3, where the possible values of a vector component i are either 1 or $1 - \alpha_i$ (the α_i being distinct variables), and $L = \{1, 1 - \alpha_i, i = 1, \ldots, n\}$ such that $1 > 1 - \alpha_i$.

Proposition 1. Leximin *and* discrimin *orders coincide on these particular vectors.*

Proof. Indeed, since the value of a vector component is either '1' or '$1 - \alpha_i$', and since each possibilistic formula attached to a node in the CP-net is associated with a different weight α_i, we are sure that a given '$1 - \alpha_i$' is present only in one component position. With these hypotheses, the difference between *leximin* and *discrimin* procedures is that *leximin* deletes some components with value '1' because it is the only component value that can be in different ranks. But we know that ‘1’ is the greatest component value, so this cannot affect the result of the final application of min operator in each case. Thus, *leximin* and *discrimin* orders coincide on these particular vectors.

These relations have been previously used for capturing the CP-nets ordering: symmetric Pareto (SP), discrimin in [8, 9], or leximin in [10] or min order in [6, 7]. In the next section, we provide a comparative discussion of these proposals and we point out when each ordering fails to exactly retrieve the CP-net ordering.

5 CP-Nets vs. Possibilistic Logic: Counterexamples

It has been claimed that CP-net orderings can be captured by using the encoding explained in Section 3 and applying the *symmetric Pareto* order [8, 9] recalled in Section 4, or the *leximin* order [10], to vectors $\boldsymbol{\omega}(\Sigma)$. This is in fact true only for special families of CP-nets, as shown in the example below. But the possibilistic encoding of CP-nets together with the use of one of the previously cited orders do not always lead to an exact representation of CP-nets in the general case, as we shall see on further examples.

Considering Ex. 1 again, Table 2 gives the satisfaction levels for the possibilistic clauses encoding the 3 elementary preferences, and the 8 possible interpretations (choices), where α, β, γ are the weights of nodes J, P, S respectively.

Table 1. Possible alternative choices in Example 1

Ω	ϕ_P	ϕ_J	ϕ_S
$P_b J_b S_r$	1	1	1
$P_b J_b S_w$	1	1	1- γ
$P_b J_w S_w$	1	1-β	1
$P_w J_b S_w$	1-α	1	1
$P_b J_w S_r$	1	1-β	1-γ
$P_w J_b S_r$	1-α	1	1-γ
$P_w J_w S_r$	1-α	1-β	1-γ
$P_w J_w S_w$	1-α	1-β	1

We introduce the following constraints, $\alpha > \gamma$ and $\beta > \gamma$ between the symbolic weights, which give priority to the constraint associated to father nodes J, P over the ones corresponding to the child node S. Then, the application of symmetric Pareto order or leximin order, allows us to rank-order interpretations. It can be checked that the ordering of interpretations obtained by these two orders applied to vectors $\boldsymbol{\omega}(\Sigma)$ coincide with the ordering $\succ_{\mathcal{N}}$ induced by the CP-net $\mathcal{N}$, as indicated in Fig. 1(b) (for short, $P_b J_b S_r$ is denoted *bbr*, etc.):

- $bbr \succ_{\mathcal{N}} bbw \succ_{\mathcal{N}} bww \succ_{\mathcal{N}} bwr \succ_{\mathcal{N}} bwr \succ_{\mathcal{N}} wwr \succ_{\mathcal{N}} www$.
- $bbr \succ_{\mathcal{N}} bbw \succ_{\mathcal{N}} wbw \succ_{\mathcal{N}} wbr \succ_{\mathcal{N}} wwr \succ_{\mathcal{N}} www$.

In order to provide a clear discussion about the possibilistic logic representation, we first establish that a preference between interpretation vectors differing by a single variable flip only depends on the instantiations of the corresponding variable and its children:

Proposition 2. *Let X_i be a node in a CP-net $\mathcal{N}$ and $Y_i = V \setminus \{\{X_i\} \cup Pa(X_i)\}$. Let $(\Sigma, \succeq_\Sigma)$ be the partially ordered possibilistic base associated with $\mathcal{N}$ using the procedure of Section 3. If the CP-net contains the statement $u : x_i > \neg x_i$ (resp: $u : \neg x_i > x_i$), the preference only depends on the instantiations of variable x_i and its children nodes.*

Proof: Let $\omega^+ = u_i x_i y_i$ and $\omega^- = u_i \neg x_i y_i$, $u_i \in A_i^+$. Since they share the same assignment of variables in $Pa(X_i)$, both models satisfy either ϕ_j^+ or $\phi_j^-, \forall X_j \in Pa(X)$. We denote by $\mathcal{F}^{Pa}$ the set of formulas $\phi_j^+, \phi_j^-, X_j \in Pa(X_i)$ falsified by ω^+, ω^- (they are the same); and by $\mathcal{F}^Y$ the set of formulas $\phi_j^+, \phi_j^-, X_j \in Y_i \setminus Ch(X_i)$, (i.e. X_j is a neither a direct descendant of X_i nor one of its parents) and falsified by ω^+, ω^-; and by $\mathcal{F}_{\omega^+}^{Ch}$ the set of formulas $\phi_j^+, \phi_j^-, X_j \in Ch(X_i)$ falsified by ω^+ and $\mathcal{F}_{\omega^-}^{Ch}$ the set of formulas falsified by ω^-. Then, $\mathcal{F}_{\omega^+} = \mathcal{F}^{Pa} \cup \mathcal{F}^Y \cup \mathcal{F}_{\omega^+}^{Ch}$ and $\mathcal{F}_{\omega^-} = \mathcal{F}^{Pa} \cup \{\phi_i^+\} \cup \mathcal{F}^Y \cup \mathcal{F}_{\omega^-}^{Ch}$. So we have $\mathcal{F}_\omega \setminus \mathcal{F}_{\omega'} = \mathcal{F}_{\omega^+}^{Ch}$ and $\mathcal{F}_{\omega'} \setminus \mathcal{F}_\omega = \{\phi_i^+\} \cup \mathcal{F}_{\omega^-}^{Ch}$. Following the construction of $(\Sigma, \succeq_\Sigma)$ we have that ϕ_i^+ is strictly preferred to all formulas in $\mathcal{F}_{\omega^+}^{Ch} \cup \mathcal{F}_{\omega^-}^{Ch}$. Then $\forall \phi \in \mathcal{F}_\omega \setminus \mathcal{F}_{\omega'}, \phi_i^+ \succ_\Sigma \phi$.

Let X_k be a child of X_i. Note that by construction, $\omega^+ \models \phi_k^+$ and $\omega^- \models \phi_k^-$. Besides, $\omega^+ \models \neg\phi_k^-$ if and only if $\omega^+ \models u_k$, and $\omega^- \models \neg\phi_k^+$ if and only if $\omega^- \models u_k$. Hence there are three cases for the child X_k:

- either $\omega^+ \models u_k$ and $\omega^- \models \neg u_k$ (then $\phi_k^- \in \mathcal{F}_{\omega^+}^{Ch}$, but $\phi_k^+ \notin \mathcal{F}_{\omega^+}^{Ch}$);
- or $\omega^+ \models \neg u_k$ and $\omega^- \models u_k$ (then $\phi_k^+ \in \mathcal{F}_{\omega^-}^{Ch}$, but $\phi_k^- \notin \mathcal{F}_{\omega^-}^{Ch}$);
- or $\omega^+ \models \neg u_k$ and $\omega^- \models \neg u_k$, and $\mathcal{F}_{\omega^-}^{Ch} \cup \mathcal{F}_{\omega^+}^{Ch}$ does not contain any formula pertaining to variable X_k.

Now, it becomes clear that $\boldsymbol{\omega}^+(\Sigma)$ and $\boldsymbol{\omega}^-(\Sigma)$ only differ on components pertaining to children nodes of X_i and to X_i itself. □

Due to the specific structure of CP-nets, and since we have shown that a preference is only related to a variable node and their children nodes (Proposition 2), we have to consider the three following elementary cases:

- *Case a*: Two father nodes and a child node (see Fig 2(a)) (also Fig. 1);
- *Case b*: A father node and two children nodes (see Fig 2(b));
- *Case c*: A grandfather node, a father node and a child node (see Fig 2(c)).

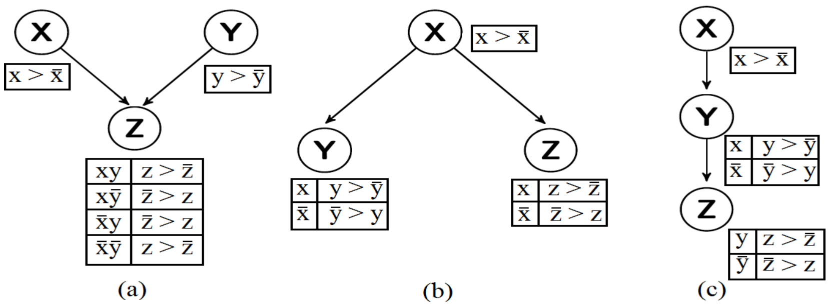

Fig. 2. Elementary cases of CP-nets

Then, any CP-net is a combination of these three elementary cases (with possibly more fathers or children). Considering these three basic structures, the following examples show in which case a particular order induced by $(\Sigma, \succ_\Sigma)$ fails to capture the ordering of interpretations induced by the CP-net.

Example 2: $V = \{X, Y, Z\}$ is the set of variables involved in the examples on Fig. 2. In these examples, preference constraints are as follows: $\phi_1 = x > \bar{x}, \phi_2 = y > \bar{y}, \phi_3 = (X \iff Y : z > \bar{z}, \neg(X \iff Y) : \bar{z} > z), \phi_4 = (x : z > \bar{z}, \bar{x} : \bar{z} > z)$, $\phi_5 = (x : y > \bar{y}, \bar{x} : \bar{y} > y)$ and $\phi_6 = (y : z > \bar{z}, \bar{y} : \bar{z} > z)$. The possibilistic logic bases obtained in the different examples in Fig 2 are:

- $\Sigma_a = \{\phi_1, \phi_2, \phi_3\}$: $\phi_1 = (x, \alpha_1), \phi_2 = (y, \alpha_2), \phi_3 = (((\neg(x \wedge y) \wedge \neg(\neg x \wedge \neg y)) \vee z) \wedge (\neg(x \wedge \neg y) \wedge \neg(\neg x \wedge y)) \vee \neg z), \alpha_3)$, and $\min(\alpha_1, \alpha_2) \succ_{\Sigma_a} \alpha_3$,
- $\Sigma_b = \{\phi_1, \phi_4, \phi_5\}$ with $\phi_4 = ((\neg x \vee z) \wedge (x \vee \neg z), \alpha_4), \phi_5 = ((\neg x \vee y) \wedge (x \vee \neg y), \alpha_5)$, and is such that $\alpha_1 \succ_{\Sigma_b} \max(\alpha_4, \alpha_5)$,
- $\Sigma_c = \{\phi_1, \phi_5, \phi_6\}$ with $\phi_6 = ((\neg y \vee z) \wedge (y \vee \neg z), \alpha_6)$ and $\alpha_1 \succ_{\Sigma_c} \alpha_5 \succ_{\Sigma_c} \alpha_6$.

Table 2. Possible alternative choices in Example 2

Ω	ϕ_1	ϕ_2	ϕ_3	ϕ_1	ϕ_4	ϕ_5	ϕ_1	ϕ_5	ϕ_6
xyz	1	1	1	1	1	1	1	1	1
$xy\bar{z}$	1	1	1-α_3	1	1-α_4	1	1	1	1-α_6
$x\bar{y}z$	1	1-α_2	1-α_3	1	1	1-α_5	1	1-α_5	1-α_6
$x\bar{y}\bar{z}$	1	1-α_2	1	1	1-α_4	1-α_5	1	1-α_5	1
$\bar{x}yz$	1-α_1	1	1-α_3	1-α_1	1-α_4	1-α_5	1-α_1	1-α_5	1
$\bar{x}y\bar{z}$	1-α_1	1	1	1-α_1	1	1-α_5	1-α_1	1-α_5	1-α_6
$\bar{x}\bar{y}z$	1-α_1	1-α_2	1	1-α_1	1-α_4	1	1-α_1	1	1-α_6
$\bar{x}\bar{y}\bar{z}$	1-α_1	1-α_2	1-α_3	1-α_1	1	1	1-α_1	1	1

Results are as follows:

- In the 1st case ($\mathcal{N}_a$), *symmetric Pareto* and *leximin* orders are able to capture the ordering of the CP-net exactly. Otherwise, the min order fails to distinguish between the interpretations $\{\bar{x}yz, \bar{x}y\bar{z}, \bar{x}\bar{y}z, \bar{x}\bar{y}\bar{z}\}$ and between $\{x\bar{y}\bar{z}, x\bar{y}z\}$.
- In the 2nd case ($\mathcal{N}_b$), *symmetric Pareto* order fails to capture the CP-net ordering exactly by leaving the two interpretations $\omega = x\bar{y}\bar{z}$ and $\omega' = \bar{x}\bar{y}\bar{z}$ non compared (while node X in the CP-net $\mathcal{N}_\lfloor$ ensures $x\bar{y}\bar{z} \succ_\mathcal{N} \bar{x}\bar{y}\bar{z}$). Otherwise the representation is exact. The associated vectors $\boldsymbol{\omega}(\Sigma) = (1, 1-\alpha_4, 1-\alpha_5)$ and $\boldsymbol{\omega'}(\Sigma) = (1-\alpha_1, 1, 1)$ are not comparable by *symmetric Pareto*. Indeed $\nexists\ \sigma$ s.t. $\boldsymbol{\omega}(\Sigma) \succ_{SP} \boldsymbol{\omega'}^\sigma(\Sigma)$, since $1-\alpha_1 < \min(1-\alpha_4, 1-\alpha_5)$ while $1 > \max(1-\alpha_4, 1-\alpha_5)$. Otherwise, the min order is able to compare these two interpretations $x\bar{y}\bar{z} \succ_{\min} \bar{x}\bar{y}\bar{z}$, but it fails to distinguish between the interpretations $\{\bar{x}yz, \bar{x}y\bar{z}, \bar{x}\bar{y}z, \bar{x}\bar{y}\bar{z}\}$ and between $\{x\bar{y}\bar{z}, xy\bar{z}\}$. But *leximin* is able here to capture the CP-net ordering exactly.
- In the 3rd case ($\mathcal{N}_c$), both *leximin* and min orders fail to capture the CP-net ordering: the two interpretations $\omega = x\bar{y}z$ and $\omega' = \bar{x}\bar{y}\bar{z}$ become comparable while the CP-net *cannot compare them.* Since $\boldsymbol{\omega}(\Sigma) = (1, 1-\alpha_5, 1-\alpha_6)$ and

$\boldsymbol{\omega}'(\Sigma) = (1-\alpha_1, 1, 1)$, with $\min(\boldsymbol{\omega}(\Sigma)) = 1-\alpha_5$, $\min(\boldsymbol{\omega}'(\Sigma)) = 1-\alpha_1$ and $1-\alpha_1 < 1-\alpha_5$, we have $\omega \succ_{lex} \omega'$ and $\omega \succ_{\min} \omega'$. But *symmetric Pareto* can capture the CP-net ordering exactly in this case.

To summarize, as observed in the Example, the *symmetric Pareto* order fails to compare two interpretations when the concerned variable has more than one child node as in *Case b* (Fig.2 (b)). Besides, in *Case c* (Fig.2 (c)) *leximin* and min break the incomparability of some interpretations in the CP-net.

6 Approaching CP-Net Preferences by Possibilistic Logic

As seen in Ex. 2 of Section 5, the *symmetric Pareto* relation is not fine-grained enough to capture the CP-net partial order in general, while the *lexi-min* order may make some CP-net-incomparable interpretations comparable. In this Section, we point out a class of CP-nets for which possibilistic logic with symbolic weights can capture the CP-net partial order exactly. First, we prove that any strict comparison obtained by *symmetric Pareto* is true for the CP-net order.

Proposition 3. *Let $\mathcal{N}$ be an acyclic CP-net and $(\Sigma, \succeq_\Sigma)$ be its associated partially ordered base. Let $\succeq_{SP}$ be the partial order associated to $(\Sigma, \succeq_\Sigma)$.*

$$\forall \omega, \omega' \in \Omega, \omega \succ_{SP} \omega' \;\Rightarrow\; \omega \succ_{\mathcal{N}} \omega'$$

Proof of Proposition 3
Suppose that $\omega \succ_{SP} \omega'$. This means that there exists a permutation σ of $\boldsymbol{\omega}'(\Sigma)$ such that when comparing the result of this permutation with $\boldsymbol{\omega}(\Sigma)$, the second vector is greater than or equal to, componentwise, the reordered one. There are two cases: either for any component, where there is no equality, the comparison between the two vectors is of the form $1 > 1-\alpha_{\sigma(i)}$, or there is at least one component where the comparison takes the form $1-\alpha_j > 1-\alpha_{\sigma(k)}$. This corresponds respectively to two different situations:

i) ω' falsifies more formulas in Σ than ω, and $\mathcal{F}_\omega \subset \mathcal{F}_{\omega'}$, where $\mathcal{F}_\omega$ (resp. $\mathcal{F}'_\omega$) denotes the set of nodes falsified by interpretation ω (resp. ω'). This corresponds to the first case above, where $\mathcal{F}_{\omega'} \setminus \mathcal{F}_\omega$ corresponds precisely to the violated formulas whose priority $\alpha_{\sigma(i)}$ is involved in the observed inequalities $1 > 1-\alpha_{\sigma(i)}$; it is known that $\mathcal{F}_\omega \subset \mathcal{F}_{\omega'}$ entails $\omega \succ_{\mathcal{N}} \omega'$.

ii) ω' falsifies at least one formula whose priority is greater than the one of another formula violated by ω, namely $1-\alpha_j > 1-\alpha_{\sigma(k)}$, equivalent to $\alpha_j < \alpha_{\sigma(k)}$. In fact, there is at least one component in $\boldsymbol{\omega}'(\Sigma)$ of the form $1-\alpha_{\sigma(r)}$ which is a minimal component among those in the two subvectors on which $\boldsymbol{\omega}(\Sigma)$ and $\boldsymbol{\omega}'(\Sigma)$ differ. It corresponds to a formula having maximal priority $(\alpha_{\sigma(r)})$ violated by ω' and not by ω. Now, the constraints $\alpha_j < \alpha_{\sigma(k)} \leq \alpha_{\sigma(r)}$ reveal that the nodes corresponding in the CP-nets to these priorities are related by a path in the CP-net linking an ancestor $X_{\sigma(r)}$ (having maximal priority) to a descendent X_j. The set of such paths can be associated with a chain of improving flips from ω' to ω, and thus $\omega \succ_{\mathcal{N}} \omega'$. □

We have noticed that there are cases where the *symmetric Pareto* order together with the possibilistic logic encoding does capture the CP-net ordering exactly. The following proposition indicates a class of CP-nets where it is indeed the case.

Proposition 4. *Let $\mathcal{N}$ be an acyclic CP-net with every node have at most one child node. Let $(\Sigma, \succeq_\Sigma)$ be its associated partially ordered base. Let $\succeq_{SP}$ be the partial order associated to $(\Sigma, \succeq_\Sigma)$. Then, $\forall \omega, \omega' \in \Omega, \omega \succ_{SP} \omega'$ iff $\omega \succ_\mathcal{N} \omega'$.*

Proof of Proposition 4

i) Suppose that $\omega \succ_\mathcal{N} \omega'$. We know that ω dominates ω' (i.e. $\omega \succ_\mathcal{N} \omega'$) if and only if there is a chain of worsening flips which consists of a change of the instantiation of one variable each time. This means that there exists a sequence $\omega_0, \cdots, \omega_k$ such that $\omega \succ \omega_0 \succ \cdots \succ \omega_k \succ \omega'$, where $\omega \succ \omega_0, \ldots, \omega_k \succ \omega'$ are ceteris paribus preferences. We have shown in Proposition 1 that such preference statements are related to the concerned variable (which corresponds here to the flip) and its children. Since we have supposed that each node has at most one child node, the associated evaluation vectors for every two interpretations in a chain of worsening flips differ on at most two components corresponding to the flipped variable and its child node. Since we give the priority to father node over the child node, the two interpretations are ordered by $\succ_{SP}$. So we have $\omega \succ_{SP} \omega_0 \succ_{SP} \cdots \succ_{SP} \omega_k \succ_{SP} \omega'$, and finally $\omega \succ_{SP} \omega'$ by transitivity.

ii) By Proposition 3, we have: if $\omega \succ_{SP} \omega'$ then $\omega \succ_\mathcal{N} \omega'$. □

We have also noticed on some examples that *leximin* order is more refined than the order induced by the considered CP-net. The following proposition establishes that any strict comparison obtained by a CP-net is also true in its possibilistic logic counterpart using *leximin* order:

Proposition 5. *Let $\mathcal{N}$ be an acyclic CP-net. Let $(\Sigma, \succeq_\Sigma)$ be its associated partially ordered base. Then: $\forall \omega, \omega' \in \Omega, \omega \succ_\mathcal{N} \omega' \Rightarrow \boldsymbol{\omega}(\Sigma) \succ_{lex} \boldsymbol{\omega'}(\Sigma)$*

Proof of Proposition 5

Since $\succ_\mathcal{N}$ is transitive, it is enough to prove that this is true for $\omega \succ_\mathcal{N} \omega'$ where there is one worsening flip which consists in a change of the instantiation of one variable, in the ceteris paribus preference style. By transitivity we get the general case where there is a chain of worsening flips since *leximin* order is also transitive. We have shown in Proposition 2 that such a ceteris paribus preference pertains to the concerned variable and its children. So for ω and ω', $\min(\{v_i \in \boldsymbol{\omega}(\Sigma)\} \cup \{v_i \in \boldsymbol{\omega'}(\Sigma)\}) \subseteq \{v_j \in (\boldsymbol{\omega'}(\Sigma)\} \setminus \{v_j \in (\boldsymbol{\omega}(\Sigma)\})$. Indeed the evaluation associated to the father node is smaller than any other evaluation associated with its children, and then the min will downrank the interpretation that violates the father node. So we have $\omega \succ_{lex} \omega'$. □

7 Related Work and Final Discussion

Possibilistic logic for preferences representation has been first advocated in [4, 5]. Its use with symbolic weights for approximating acyclic Boolean CP-nets [2] and

TCP-nets [12], has been discussed in [6, 7, 13]. Then, a representation of CP-net has been proposed using the symmetric Pareto order in [8, 9], and recalled in [10, 14] using leximin order. This representation has been presented as being faithful in the general case (without providing the proof). It turns out that the representation using the symmetric Pareto order is exact only in special cases. We have shown that it is indeed the case for the particular CP-nets where nodes have at most one child. We have also proved that in general it is a lower approximation, while the use of leximin order leads to an upper approximation.

Thus, the semantics of possibilistic logic that could lead to an exact representation of any (acyclic) CP-net in the general case is still to be found (if it exists). However, the partial ordering induced by the CP-net approach may appear somewhat questionable, as exemplified now, which in turn questions the possibility of an exact representation of the latter by means of an approach that handles preferences in a more global way.

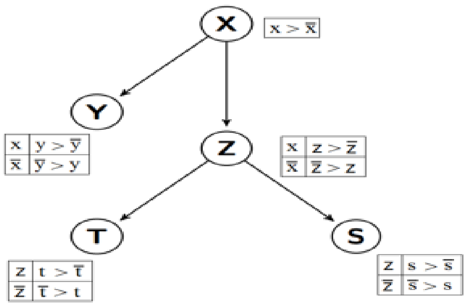

Fig. 3. CP-net related to Example 3

Example 3: Let us consider the CP-net of Fig. 3 on variables $V = \{X, Y, S, Z, T\}$. Let us consider the interpretations $\omega = xyz\bar{s}\bar{t}, \omega' = x\bar{y}\bar{z}\bar{s}\bar{t}, \omega'' = \bar{x}\bar{y}\bar{z}\bar{s}\bar{t}$ and $\omega''' = xy\bar{z}\bar{s}t$. We notice that ω violates the preferences at two grandson nodes S, T, but ω' violates the preferences at children nodes Y, Z. Moreover, ω'' violates the preference at the father node X and ω''' violates preference at a child Z and a grandson T. The CP-net order is such that $\omega \succ_{\mathcal{N}} \omega' \succ_{\mathcal{N}} \omega''$, $\omega \succ_{\mathcal{N}} \omega'''$, but it tells nothing on ω''' vs. ω'' and ω'. Thus, violating preferences at grandsons S, T (ω) is better than violating preferences at children nodes Y, Z (ω'), which is better than violating preferences at the father node X (ω''), in agreement with the CP-net implicit priorities. But it is troublesome that violating CP-net preferences at one child node Z and one grandson node T (ω''') is neither comparable with the violation of preference at the two children nodes Y, Z (ω'), let alone the father node X (ω''). This is not acknowledged by the possibilistic approach using *leximin* ordering.

8 Concluding Remarks

The interest for preference representation of the possibilistic logic framework relies first on the logical nature of the representation and constitutes an alternative to the introduction of a preference relation inside the representation language, as in, e.g., [15]. Moreover, the possibilistic representation is expressive (see [10] for an introductory survey), and can capture partial orders thanks to the use of symbolic weights, without being obliged to impose greater priority weights to any preference (as it is the case for father node preferences in CP nets). Still much remains to be done. First, the question of an exact representation of any CP-net remains open. Moreover, an attempt has been made recently [10] for representing more general CP-theories [16] in the possibilistic logic approach (by introducing further inequalities between symbolic weights in order to take into account the CP-theory idea that some preferences hold irrespective of the values of some variables), where the leximin order seems to provide an upper approximation. This remains to be confirmed and developed further. Comparing CP-nets with Bayesian possibilistic nets would be also of interest.

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References

1. Domshlak, C., Hüllermeier, E., Kaci, S., Prade, H.: Preferences in AI: An overview. Artif. Intell. 175, 1037–1052 (2011)
2. Boutilier, C., Brafman, R.I., Domshlak, C., Hoos, H., Poole, D.: CP-nets: A tool for representing and reasoning with conditional ceteris paribus preference statements. J. Artificial Intelligence Research (JAIR) 21, 135–191 (2004)
3. Dubois, D., Prade, H.: Possibilistic logic: a retrospective and prospective view. Fuzzy Sets and Systems 144, 3–23 (2004)
4. Benferhat, S., Dubois, D., Prade, H.: Towards a possibilistic logic handling of preferences. Applied Intelligence 14, 303–317 (2001)
5. Benferhat, S., Dubois, D., Kaci, S., Prade, H.: Possibilistic logic representation of preferences: relating prioritized goals and satisfaction levels expressions. In: Proc. 15th. Europ. Conf. on Artificial Intelligence (ECAI 2002), Lyon, July 21-26, pp. 685–689. IOS Press (2002)
6. Dubois, D., Kaci, S., Prade, H.: CP-nets and possibilistic logic: Two approaches to preference modeling. Steps towards a comparison. In: Proc. of IJCAI 2005 Workshop on Advances in Preference Handling, Edinburg, July 31-August 1 (2005)
7. Dubois, D., Kaci, S., Prade, H.: Approximation of conditional preferences networks "CP-nets" in possibilistic logic. In: Proc. Inter. Conf. on Fuzzy Systems (FUZZ-IEEE), Vancouver, July 16-21, pp. 2337–2342 (2006)
8. Kaci, S., Prade, H.: Mastering the processing of preferences by using symbolic priorities. In: Proc. 18th Europ. Conf. on Artif. Intel. (ECAI 2008), pp. 376–380 (2008)
9. Kaci, S.: Working With Preferences: Less Is More. Springer (2012)
10. Dubois, D., Prade, H., Touazi, F.: Handling partially ordered preferences in possibilistic logic - A survey discussion. In: Proc. ECAI 2012 Workshop on Weighted Logics for AI, pp. 91–98 (2012)
11. Benferhat, S., Prade, H.: Encoding formulas with partially constrained weights in a possibilistic-like many-sorted propositional logic. In: Kaelbling, L.P., Saffiotti, A. (eds.) Proc. 19th Int. Joint Conf. on Artif. Intellig. (IJCAI 2005), Edinburgh, July 30-August 5, pp. 1281–1286 (2005)

12. Wilson, N.: Extending CP-nets with stronger conditional preference statements. In: Proc. 19th National Conference on Artificial Intelligence (AAAI 2004), pp. 735–741 (2004)
13. Kaci, S., Prade, H.: Relaxing ceteris paribus preferences with partially ordered priorities. In: Mellouli, K. (ed.) ECSQARU 2007. LNCS (LNAI), vol. 4724, pp. 660–671. Springer, Heidelberg (2007)
14. HadjAli, A., Kaci, S., Prade, H.: Database preference queries - A possibilistic logic approach with symbolic priorities. Ann. Math. Artif. Intell. 63, 357–383 (2011)
15. Bienvenu, M., Lang, J., Wilson, N.: From preference logics to preference languages, and back. In: Lin, F.Z., Sattler, U., Truszczynski, M. (eds.) Proc. 12th Int. Conf. on Principles of Knowled. Represent. and Reas. (KR 2010), Toronto, pp. 414–424. AAAI (2010)
16. Wilson, N.: Computational techniques for a simple theory of conditional preferences. Artif. Intell. 175, 1053–1091 (2011)