Determinant versus Permanent: Salvation via Generalization?

Nicolas de Rugy-Altherre

Institut de Mathématiques de Jussieu, Université Paris Diderot, Sorbonne Paris Cité, UMR 7586 CNRS, 75205 Paris, France
nderugy@math.univ-paris-diderot.fr

Abstract. The fermionant $\mathrm{Ferm}_n^k(\bar{x}) = \sum_{\sigma \in S_n} (-k)^{c(\pi)} \prod_{i=1}^n x_{i,j}$ can be seen as a generalization of both the permanent (for $k = -1$) and the determinant (for $k = 1$). We demonstrate that it is VNP-complete for any rational $k \neq 1$. Furthermore it is $\#P$-complete for the same values of k. The immanant is also a generalization of the permanent (for a Young diagram with a single line) and of the determinant (when the Young diagram is a column). We demonstrate that the immanant of any family of Young diagrams with bounded width and at least n^ϵ boxes at the right of the first column is VNP-complete.

1 Introduction

In algebraic complexity (more specifically Valiant's model[2]) one of the main question is to know whether VP = VNP or not. Answering this is considered to be a very good step towards the resolution of $P = NP$. This question is very close to the question per vs. det, where we ask if the permanent can be computed in polynomial time in the size of the matrix, as is the determinant.

The main idea of this paper is to find a generalization of both the permanent and the determinant in order to study exactly where the difference between them lies. A generalization is here understood as a parameter, let us say t, and a function $f(t, \bar{x})$ such that for example $f(0, \bar{x}) = \det(\bar{x})$ and $f(1, \bar{x}) = \mathrm{per}(\bar{x})$. If we have a complete classification of the complexity of $f(t, \bar{x})$ for any t (with t fixed), we should be able to see where we step from VP to VNP and maybe understand a little bit more why the permanent is hard and not the determinant.

Here we study two different generalizations. First the fermionant, secondly the immanant. The fermionant was introduced by Chandrasekharan and Wiese [3] in 2011 in a context of quantum physics. It is defined with a real parameter k such that for $k = 1$ it is the determinant and for $k = -1$ it is the permanent. Mertens and Moore [7] have demonstrated its hardness for $k \geq 3$ (and with a weaker hardness for $k = 2$), in the framework of counting complexity.

Likewise, but in a different framework and with a complete different proof, we demonstrate the hardness of the fermionant seen as a polynomial for any rational $k \neq 1$ (and of course for $k \neq 0$). This give a interesting point of view on where the hardness of the permanent lies. We also get a bonus: we use a

P. Bonizzoni, V. Brattka, and B. Löwe (Eds.): CiE 2013, LNCS 7921, pp. 87–96, 2013.

technique developed by Valiant to demonstrate the hardness of the fermionant in the counting complexity framework for $k \neq 1$. We thus extend the results of Mertens and Moore [7], in particular to the case $k = 2$, which is, from what I understand, the most interesting case for physicists.

The second generalization is more classical and comes from the field of group representation. It is the immanant, introduced by Littlewood [6] in 1940. Immanants are families of polynomials indexed by Young diagrams. If the Young diagrams are a single column with n boxes, the immanant is the determinant. At the opposite end, if it is a single line of n boxes, the immanant is the permanent. The main question is: for which Young diagrams do we step from VP to VNP?

We know that if there are only a finite number of boxes on the right of the first column, the immanant is still in VP (cf [2]). On the other hand, a few hardness results have been found, fundamentally for Young diagrams in which the permanent is hidden. For example, the hook (a line of n boxes and a column of any number of boxes) and the rectangle (any number of lines each with n boxes) are hard (cf [2]), or more generally if the maximal difference between the size of two consecutive lines is as big as a power of n (cf [1]).

Here we shall demonstrate that for Young diagrams with only two columns, each with n boxes, the immanant is hard, which was an open question (cf [2] Problem 7.1). As each line of these Young diagrams has length no more than two, the permanent is not hidden in there. More generally for any family of Young diagrams with a bounded number of columns and with at least n^ϵ boxes at the right of the first column, the immanant is hard. It has been conjecture that it is still hard if we remove the bounded condition(cf [7] for example).

For a complete classification of the immanant in algebraic complexity, one "just" has to determine the complexity of the ziggurat: the Young diagrams where the first line has n boxes, the second $n-1$, the third $n-2$ etc. and the last 1 box. This immanant is most probably also hard. The complexity of the immanant with a logarithmic number of boxes to the right of the first column is also unknown.

2 Definitions

We work within Valiant's algebraic framework. Here is a brief introduction to this complexity theory. For a more complete overview, see [2].

An *arithmetic circuit* over $\mathbb{Q}$ is a labeled directed acyclic connected graph with vertices of indegree 0 or 2 and only one sink. The vertices with indegree 0 are called *input gates* and are labeled with variables or constants from $\mathbb{Q}$. The vertices with indegree 2 are called *computation gates* and are labeled with $\times$ or $+$. The sink of the circuit is called the *output gate*.

The polynomial computed by a gate of an arithmetic circuit is defined by induction: an input gate computes its label; a computation gate computes the product or the sum of its children's values. The polynomial computed by an arithmetic circuit is the polynomial computed by the sink of the circuit.

A *p-family* is a sequence (f_n) of polynomials such that the number of variables as well as the degree of f_n is polynomially bounded in n. The *complexity* $L(f)$ of a polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n]$ is the minimal number of computational gates of an arithmetic circuit computing f from variables $x_1, \ldots, x_n$ and constants in $\mathbb{Q}$.

Two of the main classes in this theory are: the analog of P, VP, which contains of every p-family (f_n) such that $L(f_n)$ is a function polynomially bounded in n; and the analog of NP, VNP. A p-family (f_n) is in VNP iff there exists a VP family (g_n) such that for all n,

$$f_n(x_1, \ldots, x_n) = \sum_{\bar{\epsilon} \in \{0,1\}^n} g_n(x_1, \ldots, x_n, \epsilon_1, \ldots, \epsilon_n)$$

As in most complexity theory we have a notion of reduction, the c-reduction: the oracle complexity $L^g(f)$ of a polynomial f with oracle access to g is the minimum number of computation gates and evaluations of g over previously computed values that are sufficient to compute f from the variables $x_1, \ldots x_n$ and constants from $\mathbb{Q}$. A p-family (f_n) *c-reduces* to (g_n) if there exists a polynomially bounded function p such that $L^{g_{p(n)}}(f_n)$ is a polynomially bounded function.

VNP is closed under c-reductions (See [8] for an idea of the proof). However this reduction does not distinguish lower classes. For example, 0 is VP-complete for c-reductions. In this paper we shall demonstrate hardness results, a smallest notion of reduction (as projection) is thus not needed.

The determinant is in VP. The permanent is VNP-complete for c-reductions ([2]).

3 The Fermionant

Let A be an $n \times n$ matrix. The *fermionant* of A, with parameter k is defined as

$$\mathrm{Ferm}^k A = (-1)^n \sum_{\pi \in S_n} (-k)^{c(\pi)} \prod_{i=1}^{n} A_{i,\pi(i)}$$

where S_n denotes the symmetric group of n objects and, for any permutation $\pi \in S_n$, $c(\pi)$ denotes the number of cycles of π. To study the complexity of such a function, we work within the algebraic complexity framework. The algebraic equivalent of the fermionant is the polynomial obtain where we compute the fermionant on the matrix $(x_{i,j})_{1 \le i,j \le n}$. If we write Ferm^k the p-family $(\mathrm{Ferm}_n^k)_{n \in \mathbb{N}}$, we have a complete classification of the algebraic complexity of those polynomials.

Theorem 1. *Let k be a rational.*

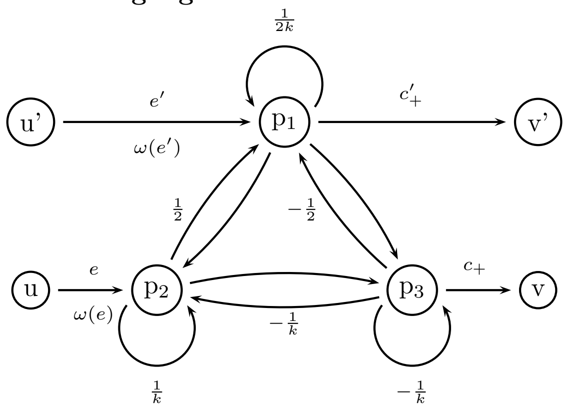
- *$Ferm^0 = 0$.*
- *$Ferm^1$ is in* VP
- *for other values of k $Ferm^k$ is* VNP*-complete for c-reductions.*

Similarly to the permanent we can see the fermionant as a computation on a graph G with n vertices and the edge between the vertices i and j is labeled with the variable $x_{i,j}$. A permutation $\pi \in S_n$ can be seen as a cycle cover on this graph. A *cycle cover* of G is a subset of its edges that covers all vertices of G and that form cycles. The weight of a cycle cover π is $\omega(\pi) = \prod_{e\in\pi} x_e$ and we write $c(\pi)$ its number of cycles, then

$$\text{Ferm}^k(\bar{x}) = \sum_{\pi\in CC(G)} (-k)^{c(\pi)} \prod_{e\in\pi} x_e$$

where $CC(G)$ is the set of all cycle covers of G. We shall use a so call iff-gadget, which is the labeled graph draw above. The idea of this gadget is when placed between two edges e and e' on G, any cycle cover containing exactly one of the edges e and e' will not contribute to the fermionant computed on the resulting graph.

iff-gadget for the fermionant



Lemma 1. *Let G be a graph with n vertices and $(e_1^i, e_2^i)_{1\leq i\leq l}$ be a set of pairs of edges of G such that no two edges in this set are equal. Let G' be the same graph but where we place an iff-gadget between every pair (e_1^i, e_2^i). Let π be a cycle cover of G, $\Pi(\pi)$ be the set of cycle covers of G' that match π on $E(G)$.*

– *If there is a pair (e_1^i, e_2^i) of edges such that $e_1^i \in \pi$ and $e_2^i \notin \pi$, or vice versa, then*

$$\sum_{\pi'\in\Pi(\pi)} (-k)^{c(\pi')}\omega(\pi') = 0$$

– *Else, let $d(\pi)$ be the number of pair (e_1^i, e_2^i) of edges such that $e_1^i \notin \pi$ and $e_2^i \notin \pi$. Then*

$$\sum_{\pi'\in\Pi(\pi)} (-k)^{c(\pi')}\omega(\pi') = \left(\frac{1}{2}(1-k)\right)^{d(\pi)} (-k)^{c(\pi)}\omega(\pi)$$

The proof is not given here, it is in color in the full version of the paper. Now here is the main tool of our demonstration, that allows us to interpolate the fermionant and compute the permanent.

Lemma 2. *Let G be a graph with n vertices. We make l copies of G and name them $G_1, \ldots, G_l$. Let $\tilde{F}^l$ be the disjoint union of those copies in which we label the edges of G_1 with the same weight as those of G and the edges of G_i for $i \geq 2$ with 1. If e is an edge of G, we call e_i the corresponding edge in G_i. We name F^l the graph $\tilde{F}^l$ where for any edge $e \in E(G)$ and any $1 \leq i \leq l$, we have placed an iff-gadget between e_i and e_{i+1}. Let π be a cycle cover of G and $\Pi(\pi)$ be the set of cycle covers of F^l that match π on $E(G_1)$. Then*

$$\sum_{\pi' \in \Pi(\pi)} (-k)^{c(\pi')} \omega(\pi') = \left(\frac{1}{2}(1-k)\right)^{(|E(G)|-n)(l-1)} (-k)^{l \times c(\pi)} \omega(\pi)$$

Proof. The idea is, with the help of the iff-gadget, to copy a cycle cover from G_1 to every other copies of G, without changing the weight of this cycle cover, just multiplying the number of cycles. The demonstration is by induction on l.

If $l = 2$, then we simultaneously add $|E(G)|$ iff-gadgets, but only one on each edge. By design, a cycle cover π on G_1 is repeated on G_2 (i.e., if e_1 is in π then e_2 is also in π as there is a iff-gadget between e_1 and e_2. see Lemma 1). The edges of G_2 are labeled with 1 and therefore do not contribute to the weight of the cycle cover. The number of cycles of $\pi' \in \Pi(\pi)$ is twice the number of cycles of π. There is $|E(G)|$ iff-gadgets in F^2. A cycle cover of G passes through n edges and therefore activates exactly n iff-gadgets. The other iff-gadgets are not activated and thus each contribute $\frac{1}{2}(1-k)$ to the sum.

Suppose the lemma true for $l-1$ copies. Let F^{l-1} be the disjoint union of $l-1$ copies of G with iff-gadgets. We add a new copy G_l of G linked to F^{l-1} with iff-gadgets to obtain F^l. Let π be a cycle cover of G, $\Pi^l(\pi)$ the set of every cycle covers of F^l that match π on $E(G_1)$ and $\Pi^{l-1}(\pi)$ the same but on F^{l-1}. By induction,

$$\sum_{\pi' \in \Pi^{l-1}(\pi)} (-k)^{c(\pi')} \omega(\pi') = \left(\frac{1}{2}(1-k)\right)^{(|E(g)|-n)(l-2)} (-k)^{(l-1) \times c(\pi)} \omega(\pi)$$

Let $\hat{F}^l$ be the disjoint union of F^{l-1} and G_l. To obtain F^l from this graph, one has just to add a iff-gadget between every edge e_{l-1} and e_l. We can apply then Lemma 1 to this graph. If π'' is a cycle cover of $\hat{F}^l$ that match π on G_1, let $\Lambda(\pi'')$ be the set of cycle covers of F^l that match π'' on $E(\hat{F}^l)$. Then, if we call $d(\pi'')$ the number of pairs (e_{l-1}, e_l) that are not in π'',

$$\sum_{\lambda \in \Lambda(\pi'')} (-k)^{c(\lambda)} \omega(\lambda) = \left(\frac{1}{2}(1-k)\right)^{d(\pi'')} (-k)^{c(\pi'')} \omega(\pi'')$$

Let us study a little bit more π''. It is a cycle cover of two disjoint graphs, F^{l-1} and G_l. Therefore it is composed of two sub cycle covers: σ' a cycle cover of F^{l-1} which by induction is in a $\Pi^{l-1}(\pi)$ and a cycle cover λ of G_l. However, as every edge of G_l is linked with an iff-gadget to its image in G_{l-1} in F^l, the cycle cover π'' will contribute to the last sum if and only if it contain both e_{l-1} and e_l, or neither e_{l-1} and e_l. Thus, λ must be the copy of π in G_l, which we write λ_π and $c(\pi'') = c(\sigma) + c(\lambda_\pi) = c(\sigma) + c(\pi)$.

There are n edges in the last image G_l that are passed through by π''. Therefore, there are $(|E(G)| - n)$ iff-gadgets that are not activated by π'' (i.e., $d(\pi'') = |E(G)| - n$). Thus,

$$\begin{aligned}
\sum_{\pi' \in \Pi(\pi)} (-k)^{c(\pi')}\omega(\pi') &= \sum_{\pi'' \in \Pi^l(\pi)} \sum_{\lambda \in \Lambda(\pi'')} (-k)^{c(\lambda)}\omega(\lambda) \\
&= \sum_{\pi'' \in \Pi^l(\pi)} \left(\frac{1}{2}(1-k)\right)^{|E(G)|-n} (-k)^{c(\pi'')}\omega(\pi'') \\
&= \left(\frac{1}{2}(1-k)\right)^{|E(G)|-n} (-k)^{c(\lambda_\pi)} \sum_{\sigma \in \Pi^{l-1}(\pi)} (-k)^{c(\sigma)}\omega(\sigma) \\
&= \left(\frac{1}{2}(1-k)\right)^{(|E(G)-n)\times(l-1)} (-k)^{l\times c(\pi)}\omega(\pi)
\end{aligned}$$

Where $\Pi(\pi)$ is the set of cycle covers of F^l that match π on $E(G)$; $\Pi^l(\pi)$ the set of cycle covers of $\tilde{F}^l$ that match π on $E(G)$ and for $\pi'' \in \Pi^l(\pi)$, $\Lambda(\pi'')$ the set of cycle covers that match π'' on $E(\tilde{F}^l)$. We have $\Pi(\pi) = \bigcup_{\pi'' \in \Pi^l(\pi)} \Lambda(\pi'')$ which completes our demonstration.

□

Proof of Theorem 1. The first case is trivial. For the second, it is a well known result, as $\mathrm{Ferm}_n^1(\bar{x}) = \det_n(\bar{x})$. Now, let k be a rational different than 0 and 1.

Let us write $(P_l G)$ the graph obtained in the previous lemma, when we duplicate l times G and add iff-gadgets to repeat every cycle cover l times. We have seen that

$$\mathrm{Ferm}_{ln}^k(P_l G)(\bar{x}) = \sum_{\pi \in \mathrm{CC}(G)} (-k)^{l\times c(\pi)} \prod_{e \in \pi} \omega(e) \left(\frac{1}{2}(1-k)\right)^{(l-1)\times(|E(G)|-n)}$$

Let us write $c_m = \sum_{\pi \in \mathrm{CC}(G) | c(\pi) = m} \prod_{e\in\pi} \omega(e)$, $\alpha = \left(\frac{1}{2}(1-k)\right)^{|E(G)|-n}$, $f_l = \mathrm{Ferm}_{ln}^k(P_l(G))$ and $\omega_l = (-k)^l$, then

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha^n \end{pmatrix} \begin{pmatrix} \omega_1 & \omega_1^2 & \dots & \omega_1^n \\ \omega_2 & \omega_2^2 & \dots & \omega_2^n \\ \vdots & & \ddots & \vdots \\ \omega_n & \omega_n^2 & \dots & \omega_n^n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

This system of equation is a Vandermonde system and therefore is invertible (if $k \neq 1$ and $k \neq -1$, because in these cases, some ω_i are equal and the matrix is not invertible): there exists some rationals $\omega^*_{l,m}$ such that for any m, $c_m = \sum_{l=1}^{n} \omega^*_{l,m} f_l(\bar{x})$.

Therefore, for any m, we have a c-reduction from c_m to the fermionant, $(c_m) \leq_c (\mathrm{Ferm}^k)$. But, $c_1 := \sum_{\pi \in S_n | c(\pi)=1} \prod_{i+1}^{n} x_{i,\pi(i)} = \mathrm{Ham}_n(\bar{x})$, where Ham_n is the Hamiltonian, which is known to be VNP-complete ([2], Corollary 3.19). □

The fermionant can be expressed as a linear combination of polynomial size of the Hamiltonian. From that we have concluded that the fermionant is VNP-complete. However, the Hamiltonian is also $\#P$-complete, when considered as a counting problem. This gives us a Turing reduction from the Hamiltonian to the fermionant and thus it is also $\#P$-complete, but only when computed on rational matrix; the Turing reductions requires rationals ($\frac{1}{2}$, $-\frac{1}{k}$, etc). We can adapt the proof of Valiant for the $\#P$-completeness of the permanent to replace those rationals by some gadgets only using 0 and 1. And thus we have the following non trivial corollary. The proof is in the full version of the paper.

Corollary 1. *For every $k \neq 1$ and $k \neq 0$, Ferm^k is $\#P$-complete for matrices over* $\{0, 1\}$.

4 Immanant with Constant Length

Immanants are defined with characters of representations of S_n. Such characters can be indexed by Young diagrams of n boxes (i.e., collections of boxes arranged in left-adjusted rows with a decreasing row length). As all the work of representation theory has already be done (Lemma 3), I will not define more those characters. We shall only work on Young diagrams.

The *immanant* associated with a Young diagram Y (and its associate character χ_Y) is

$$\mathrm{im}_\chi(\bar{x}) = \sum_{\pi \in S_n} \chi_Y(\pi) \prod_{i=1}^{n} x_{i,\pi(i)}$$

For example, if the Young diagram is a single row of n boxes, then for any $\sigma \in S_n$, $\chi_Y(\sigma) = 1$ and thus $\mathrm{im}_Y = \mathrm{per}$. At the opposite end, if Y is a single column with n boxes, $\chi_Y(\sigma) = sg(\sigma)$ and $\mathrm{im}_Y = \det$. For more details (and for a nice demonstration of the Murnaghan-Nakayama rule, one of the main parts of our demonstration), see [5].

A classical theorem states that the irreducible characters of the symmetric group form a basis for the class functions on S_n. Class functions are real functions defined on S_n and stabled under conjugation (i.e., $\forall \pi, \sigma \in S_n, f(\pi\sigma\pi^{-1}) = f(\sigma)$). The function $\pi \longmapsto (-k)^{c(\pi)}$ is such a class function and thus is a linear combination of characters. Mertens and Moore [7] have computed those characters, and applied to the immanant we get:

Lemma 3. *For any integers k and n, if we write Λ_k^n for the set of every Young diagram with n boxes and at most k columns, then there exists some constants d_Y^k such that for any matrix A:*

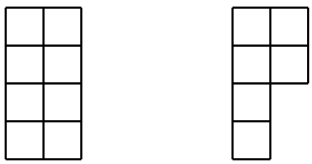
$$Ferm_n^k(A) = \sum_{Y \in \Lambda_k^n} d_Y^k \, im_Y(A)$$

Intuitively this suggests that the family of every immanants of bounded width is VNP-complete. In algebraic complexity this is not that interesting, as this family is very large. But if we prove that with a certain family of immanant we can compute every immanants of width less than a certain k, then this family will be VNP-complete. It is exactly what we are going to do for the demonstration of the following proposition.

Proposition 1. *Let $[n, n]$ be the square Young diagram with two columns, each with n rows. Then $(im_{[n,n]})_{n \in \ltimes}$ is* VNP*-complete for c-reductions.*

Young diagrams

$[4, 4]$ $[4, 2]$



Proof. More generally, let $[l_1, l_2]$ be the two columns Young diagram with l_1 boxes in the first column and l_2 in the second. More specifically, the Young diagrams of width a most 2 and of n boxes are $([l, n-l])_{l \in [n/2, n]}$. Each of them can be obtained from the square diagram $[l, l]$ by removing a skew hook of size $\delta = (l - (n - l)) = 2l - n$. A skew hook in a Young diagram is a connected collection of boxes in the border of the diagram such that if you remove this hook it is still a Young diagram (i.e., the row sizes are still decreasing). Furthermore, if you remove a skew hook of size δ to $[l, l]$, you can obtain only $[l, n - l]$ and $[l - 1, n - l + 1]$. The Murnaghan-Nakayama rule (c.f. [2] chap. 7.2 for more details) states that:

$$\mathrm{im}_{[l,l]}(\bar{x}, \boldsymbol{l}) = (-1)^{l-1} \mathrm{im}_{[l,n-l]}(\bar{x}) + (-1)^l \mathrm{im}_{[l-1,n-l+1]}(\bar{x})$$

Where $\boldsymbol{l}$ is an encoding of a cycle of length l. We know that, from Lemma 3:

$$\mathrm{Ferm}_n^2(\bar{x}) = \sum_{Y \in \Lambda_2^n} d_Y^2 \mathrm{im}_Y(\bar{x}) = \sum_{l=n/2}^{n} d_{[l,n-l]}^2 \mathrm{im}_{[l,n-l]}(\bar{x})$$

From those two facts, we can compute the fermionant from the square immanant. We just have to take new constants: let $\alpha_{[n-1,1]} = d_{[n-1,1]}(-1)^n$ and for any $2 \leq l \leq \frac{n}{2}$, $\alpha_{[n-l,l]} = (-1)^l (d_{[l,n-1]} - \alpha_{[l+1,n-l-1]}(-a)^{l+1})$. For simplicity, we write $\alpha_{\boldsymbol{l}} = \alpha_{[l,n-l]}$. If n is even

$$\sum_{l=n/2}^{n-1} \alpha_{\mathbf{l}} \mathrm{im}_{[l,l]}(\bar{x}, \boldsymbol{2l-n})$$
$$= \sum_{l=n/2}^{n-1} \alpha_{\mathbf{l}} \left((-1)^{l-1} \mathrm{im}_{[l,n-l]}(\bar{x}) + (-1)^{l} \mathrm{im}_{[l-1,n-l+1]}(\bar{x})\right)$$
$$= d_{[n-1,1]}\mathrm{im}_{[n-1,1]}(\bar{x}) + \sum_{l=n/2}^{n-2} d_{\mathbf{l}}\mathrm{im}_{[l,n-l]}(\bar{x})$$
$$- \sum_{l=n/2}^{n-2} \alpha_{\mathbf{l+1}}(-1)^{l+1}\mathrm{im}_{[l,n-l]}(\bar{x}) + \sum_{l=n/2}^{n-1} \alpha_{\mathbf{l}}(-1)^{l}\mathrm{im}_{[l-1,n-l+1]}(\bar{x})$$
$$= \sum_{l=n/2}^{n-1} d_{\mathbf{l}}\mathrm{im}_{[l,n-l]}(\bar{x}) - \sum_{l=n/2+1}^{n-1} \alpha_{\mathbf{l}}(-1)^{l}\mathrm{im}_{[l-1,n-l+1]}(\bar{x})$$
$$+ \sum_{l=n/2}^{n-1} \alpha_{\mathbf{l}}(-1)^{l}\mathrm{im}_{[l-1,n-l+1]}(\bar{x})$$
$$= \sum_{l=n/2}^{n-1} d_{\mathbf{l}}\mathrm{im}_{[l,n-l]}(\bar{x}) + \alpha_{\mathbf{n/2+1}}(-1)^{n/2}\mathrm{im}_{[n/2,n/2]}(\bar{x})$$

Furthermore, $\mathrm{im}_{[n,0]}(\bar{x}) = \det_n(\bar{x})$ and then can be computed with only a polynomial number of arithmetic operations. Thus,

$$\sum_{l=n/2}^{n-1} \alpha_{\mathbf{l}}\mathrm{im}_{[l,l]}(\bar{x}, \boldsymbol{2l-n}) + \det_n(\bar{x}) + (-1)^{\frac{n}{2}}\alpha_{\frac{n}{2}+1}\mathrm{im}_{[\frac{n}{2},\frac{n}{2}]}(\bar{x})$$
$$= \sum_{l=n/2}^{n} d_{[l,n-l]}\mathrm{im}_{[l,n-l]} = \mathrm{Ferm}_n^2(\bar{x})$$

We obtain an arithmetic circuit of polynomial size that compute Ferm_n^2 with $n/2$ oracles that can compute $\mathrm{im}_{[l,l]}$ for $l \in [n/2, n]$. To obtain a c-reduction from the fermionant to the immant, we just have to notice that $\mathrm{im}_{[l,l]} \leq_p \mathrm{im}_{[l',l']}$ as soon as $l' \geq l$. Indeed, we just have to erase the first $l' - l$-th rows, which can be done by Corollary 3.2 of [1].

The demonstration for n odd works the same, the border cases must just be studied a little bit more closer. □

We can generalize this result to almost every family of bounded width. The proof is similar and is in the annex.

Theorem 2. *Let* (Y_n) *be a family of Young diagrams of length bounded by* $k \geq 2$ *such that* $|Y_n| = \Omega(n)$. *Then*

– *if the number of boxes in the right of the first column if bounded by a constant c, then* (im_{Y_n}) *is in* VP.
– *otherwise, if there is an* $\epsilon > 0$ *and at least* n^ϵ *boxes at the right of the first column,* (im_{Y_n}) *is* VNP*-complete for c-reductions.*

5 Conclusion and Perspectives

The generalization via the fermionant tell us that the determinant is really special: the coefficients 1 and −1 allows us, in a simplify way, to cancel some monomials and not to have to compute everything. The k in the fermionant, even thinly different than 1, separates these monomials and prevents the cancelations.

As for the immanant, the interpretation of the result is harder. Especially as our theorem does not completely classify immanants of constant width, what about the immanant of $[n, \log n]$? Bürgisser's algorithm gives a subexponentiel upper bound, but does not put it in VP. Howerver, under the extended Valiant hypothesis (end of chapter 2 in [2]), it can not be VNP-complete. Is it a good candidate to be neither VP nor VNP-complete? Or even VP-complete? Or is it as hard as the determinant? This is unknown.

Other generalizations also can be imagined. For example generating functions of a graph property are polynomials that generalize the permanent and some of them can be computed as fast as the determinant. This framework allows us to use our knowledge on graph theory to understand where we step from VP to VNP. There is no classification of these generation functions, but some results have been found [2,4].

I thank to both of my doctoral advisors, A. Durand and G. Malod as well to M. Casula and E. Boulat who try to explain to me the physic behind the fermionant.

References

1. Brylinski, J.-L., Brylinski, R.: Complexity and Completeness of Immanants. CoRR, cs.CC/0301024 (2003)
2. Bürgisser, P.: Completeness and Reduction in Algebraic Complexity Theory. Springer (2000)
3. Chandrasekharan, S., Wiese, U.-J.: Partition Functions of Strongly Correlated Electron Systems as "Fermionants". ArXiv e-prints (August 2011)
4. de Rugy-Altherre, N.: A dichotomy theorem for homomorphism polynomials, pp. 308–322 (2012)
5. Greene, C.: A rational-function identity related to the Murnaghan-Nakayama formula for the characters of Sn. J. Algebraic Comb. 1(3), 235–255 (1992)
6. Littlewood, D.E.: The theory of group characters and matrix representations of groups. The Clarendon Press (1940)
7. Mertens, S., Moore, C.: The complexity of the fermionant, and immanants of constant width. ArXiv e-prints (October 2011)
8. Poizat, B.: À la recherche de la définition de la complexité d'espace pour le calcul des polynômes à la manière de valiant. Journal of Symbolic Logic 73 (2008)