Some Wellposedness Results for the Ostrovsky–Hunter Equation

G.M. Coclite, L. di Ruvo, and K.H. Karlsen

Abstract The Ostrovsky-Hunter equation provides a model for small-amplitude long waves in a rotating fluid of finite depth. It is a nonlinear evolution equation. In this paper the welposedness of the Cauchy problem and of an initial boundary value problem associated to this equation is studied.

Keywords Existence • Uniqueness • Stability • Entropy solutions • Conservation laws • Ostrovsky-Hunter equation • Boundary value problems • Cauchy problems

2000 Mathematics Subject Classification 35G25, 35G15, 35L65, 35L05, 35A05

1 Introduction

The non-linear evolution equation

$$\partial_x(\partial_t u + \partial_x f(u) - \beta\partial^3_{xxx}u) = \gamma u, \tag{1}$$

with $\beta, \gamma \in \mathbb{R}$ and $f(u) = \frac{u^2}{2}$ was derived by Ostrovsky [20] to model small-amplitude long waves in a rotating fluid of finite depth. This equation generalizes the

G.M. Coclite (✉) · L. di Ruvo
Department of Mathematics, University of Bari, via E. Orabona 4, I–70125 Bari, Italy
e-mail: giuseppemaria.coclite@uniba.it; coclitegm@dm.uniba.it; diruvo@dm.uniba.it

K.H. Karlsen
Centre of Mathematics for Applications (CMA), University of Oslo, P.O. Box 1053, Blindern, N–0316 Oslo, Norway
e-mail: kennethk@math.uio.no

G.-Q.G. Chen et al. (eds.), *Hyperbolic Conservation Laws and Related Analysis with Applications*, Springer Proceedings in Mathematics & Statistics 49,
DOI 10.1007/978-3-642-39007-4_7,

Korteweg-deVries equation (corresponding to $\gamma = 0$) by an additional term induced by the Coriolis force. It is deduced by considering two asymptotic expansions of the shallow water equations, first with respect to the rotation frequency and then with respect to the amplitude of the waves [8].

Mathematical properties of the Ostrovsky equation (1) have been studied recently in great depth, including the local and global well-posedness in energy space [7, 12, 14, 25], stability of solitary waves [10, 13, 15], and convergence of solutions in the limit of the Korteweg-deVries equation [11, 15]. We shall consider the limit of the no high-frequency dispersion $\beta = 0$, therefore (1) reads

$$\partial_x(\partial_t u + \partial_x f(u)) = \gamma u. \tag{2}$$

In this form, Eq. (2) is known under various different names such as the reduced Ostrovsky equation [21, 23], the Ostrovsky-Hunter equation [3], the short-wave equation [8], and the Vakhnenko equation [18, 22].

Integrating (2) with respect to x we obtain the integro-differential formulation of (2) (see [16])

$$\partial_t u + \partial_x f(u) = \gamma \int^x u(t,y)dy,$$

which is equivalent to

$$\partial_t u + \partial_x f(u) = \gamma P, \qquad \partial_x P = u.$$

Due to the regularizing effect of the P equation we have that

$$u \in L^\infty_{loc} \Longrightarrow P \in L^\infty((0,T); W^{1,\infty}_{loc}),\ T > 0.$$

The flux f is assumed to be smooth, Lipschitz continuous, and *genuinely nonlinear*, i.e.:

$$f \in C^2(\mathbb{R}), \qquad |\{u \in \mathbb{R}; f''(u) = 0\}| = 0, \qquad f'(0) = 0, \qquad |f'(\cdot)| \leq L, \tag{3}$$

and the constant γ is assumed to be real

Since the solutions are merely locally bounded, the Lipschitz continuity of the flux f assumed in (3) guarantees the finite speed of propagation of the solutions of (2).

This paper is devoted to the wellposedness of the initial-boundary value problem (see Sect. 2) and the Cauchy problem (see Sect. 3) for (2). Our existence argument is based on a passage to the limit using a compensated compactness argument [24] in a vanishing viscosity approximation of (8):

$$\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \gamma P_\varepsilon + \varepsilon \partial^2_{xx} u_\varepsilon, \qquad \partial_x P_\varepsilon = u_\varepsilon.$$

On the other hand we use the method of [9] for the uniqueness and stability of the solutions of (2).

2 The Initial Boundary Value Problem

In this section, we augment (2) with the boundary condition

$$u(t,0) = 0, \qquad t > 0, \tag{4}$$

and the initial datum

$$u(0,x) = u_0(x), \qquad x > 0. \tag{5}$$

We assume that

$$u_0 \in L^2(0,\infty) \cap L^\infty_{loc}(0,\infty), \quad \int_0^\infty u_0(x)dx = 0. \tag{6}$$

The zero mean assumption on the initial condition is motivated by (2). Indeed, integrating both sides of (2) we have that $u(t,\cdot)$ has zero mean for every $t > 0$, therefore it is natural to assume the same on the initial condition.

Integrating (2) on $(0,x)$ we obtain the integro-differential formulation of the initial-boundary value problem (2), (4), (5) (see [16])

$$\begin{cases} \partial_t u + \partial_x f(u) = \gamma \int_0^x u(t,y)dy, & t > 0,\ x > 0, \\ u(t,0) = 0, & t > 0, \\ u(0,x) = u_0(x), & x > 0. \end{cases} \tag{7}$$

This is equivalent to

$$\begin{cases} \partial_t u + \partial_x f(u) = \gamma P, & t > 0,\ x > 0, \\ \partial_x P = u, & t > 0,\ x > 0, \\ u(t,0) = P(t,0) = 0, & t > 0, \\ u(0,x) = u_0(x), & x > 0. \end{cases} \tag{8}$$

Due to the regularizing effect of the P equation in (8) we have that

$$u \in L^\infty_{loc}((0,\infty)^2) \Longrightarrow P \in L^\infty_{loc}((0,\infty); W^{1,\infty}_{loc}(0,\infty)). \tag{9}$$

Therefore, if a map $u \in L^\infty_{loc}((0,\infty)^2)$ satisfies, for every convex map $\eta \in C^2(\mathbb{R})$,

$$\partial_t \eta(u) + \partial_x q(u) - \gamma \eta'(u) P \leq 0, \qquad q(u) = \int^u f'(\xi)\eta'(\xi)\, d\xi, \tag{10}$$

in the sense of distributions, then [5, Theorem 1.1] provides the existence of a strong trace u_0^τ on the boundary $x = 0$.

Definition 1. We say that $u \in L^\infty_{loc}((0,\infty)^2)$ is an entropy solution of the initial-boundary value problem (2), (4), and (5) if:

(i) u is a distributional solution of (7) or equivalently of (8);
(ii) for every convex function $\eta \in C^2(\mathbb{R})$ the entropy inequality (10) holds in the sense of distributions in $(0,\infty) \times (0,\infty)$;
(iii) for every convex function $\eta \in C^2(\mathbb{R})$ with corresponding q defined by $q' = f'\eta'$, the boundary entropy condition

$$q(u_0^\tau(t)) - q(0) - \eta'(0)\frac{(u_0^\tau(t))^2}{2} \leq 0 \tag{11}$$

holds for a.e. $t \in (0,\infty)$, where $u_0^\tau(t)$ is the trace of u on the boundary $x = 0$.

We observe that the previous definition is equivalent to the following inequality (see [2]):

$$\begin{aligned}\int_0^\infty \int_0^\infty (|u-c|\partial_t\phi + \operatorname{sign}(u-c)\,(f(u)-f(c))\partial_x\phi) dt\, dx \\ + \gamma \int_0^\infty \int_0^\infty \operatorname{sign}(u-c)\, P\, dt\, dx \\ - \int_0^\infty \operatorname{sign}(c)\,(f(u_0^\tau(t)) - f(c)) dt \\ + \int_0^\infty |u_0(x) - c|\phi(0,x) dx \geq 0,\end{aligned}$$

for every non-negative $\phi \in C^\infty(\mathbb{R}^2)$ with compact support, and for every $c \in \mathbb{R}$.

The main result of this section is the following theorem.

Theorem 1. *Assume (3), (5), and (6). The initial-boundary value problem (2), (4), and (5) possesses a unique entropy solution u in the sense of Definition 1. Moreover, if u and v are two entropy solutions (2), (4), (5) in the sense of Definition 1 the following inequality holds*

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(0,R)} \leq e^{Ct}\, \|u(0,\cdot) - v(0,\cdot)\|_{L^1(0,R+Lt)}, \tag{12}$$

for almost every $t > 0$, $R, T > 0$, and a suitable constant $C > 0$.

Our existence argument is based on a passage to the limit in a vanishing viscosity approximation of (8). Fix a small number $\varepsilon > 0$, and let $u_\varepsilon = u_\varepsilon(t, x)$ be the unique classical solution of the following mixed problem

$$\begin{cases} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \gamma P_\varepsilon + \varepsilon \partial_{xx}^2 u_\varepsilon, & t > 0,\ x > 0, \\ \partial_x P_\varepsilon = u_\varepsilon, & t > 0,\ x > 0, \\ u_\varepsilon(t, 0) = P_\varepsilon(t, 0) = 0, & t > 0, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x > 0, \end{cases} \tag{13}$$

where $u_{\varepsilon,0}$ is a $C^\infty(0, \infty)$ approximation of u_0 such that

$$\|u_{\varepsilon,0}\|_{L^2(0,\infty)} \le \|u_0\|_{L^2(0,\infty)}, \quad \int_0^\infty u_{\varepsilon,0}(x)dx = 0. \tag{14}$$

Clearly, (13) is equivalent to the integro-differential problem

$$\begin{cases} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \gamma \int_0^x u_\varepsilon(t, y)dy + \varepsilon \partial_{xx}^2 u_\varepsilon, & t > 0,\ x > 0, \\ u_\varepsilon(t, 0) = 0, & t > 0, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x > 0. \end{cases} \tag{15}$$

The existence of such solutions can be obtained by fixing a small number $\delta > 0$ and considering the further approximation of (13) (see [4])

$$\begin{cases} \partial_t u_{\varepsilon,\delta} + \partial_x f(u_{\varepsilon,\delta}) = \gamma P_{\varepsilon,\delta} + \varepsilon \partial_{xx}^2 u_{\varepsilon,\delta}, & t > 0,\ x > 0, \\ -\delta \partial_{xx}^2 P_{\varepsilon,\delta} + \partial_x P_{\varepsilon,\delta} = u_{\varepsilon,\delta}, & t > 0,\ x > 0, \\ u_{\varepsilon,\delta}(t, 0) = P_{\varepsilon,\delta}(t, 0) = \partial_x P_{\varepsilon,\delta}(t, 0) = 0, & t > 0, \\ u_{\varepsilon,\delta}(0, x) = u_{\varepsilon,0}(x), & x > 0, \end{cases}$$

and then sending $\delta \to 0$.

Let us prove some a priori estimates on u_ε.

Lemma 1. *The following statements are equivalent*

$$\int_0^\infty u_\varepsilon(t, x)dx = 0, \quad t \ge 0, \tag{16}$$

$$\frac{d}{dt}\int_0^\infty u_\varepsilon^2 dx + 2\varepsilon \int_0^\infty (\partial_x u_\varepsilon)^2 dx = 0, \quad t > 0. \tag{17}$$

Proof. Let $t > 0$. We begin by proving that (16) implies (17). Multiplying (15) by u_ε and integrating over $(0, \infty)$ gives

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty u_\varepsilon^2 dx = \int_0^\infty u_\varepsilon \partial_t u_\varepsilon dx$$
$$= \varepsilon \int_0^\infty u_\varepsilon \partial_{xx}^2 u_\varepsilon dx - \int_0^\infty u_\varepsilon f'(u_\varepsilon)\partial_x u_\varepsilon dx + \gamma \int_0^\infty u_\varepsilon \Big(\int_0^x u_\varepsilon dy\Big) dx$$
$$= -\varepsilon \int_0^\infty (\partial_x u_\varepsilon)^2 dx + \gamma \int_0^\infty u_\varepsilon \Big(\int_0^x u_\varepsilon dy\Big) dx.$$

For (13),

$$\int_0^\infty u_\varepsilon \Big(\int_0^x u_\varepsilon dy\Big) dx = \int_0^\infty P_\varepsilon(t,x)\partial_x P_\varepsilon(t,x) dx = \frac{1}{2} P_\varepsilon^2(t,\infty).$$

Then,

$$\frac{d}{dt}\int_0^\infty u_\varepsilon^2 dx + 2\varepsilon \int_0^\infty (\partial_x u_\varepsilon)^2 dx = \gamma P_\varepsilon^2(t,\infty). \tag{18}$$

Thanks to (16),

$$\lim_{x\to\infty} P_\varepsilon^2(t,x) = \Big(\int_0^\infty u_\varepsilon(t,x) dx\Big)^2 = 0. \tag{19}$$

Now (18) and (19) give (17).

Let us show that (17) implies (16). We assume by contraddiction that (16) does not hold, namely:

$$\int_0^\infty u_\varepsilon(t,x) dx \neq 0.$$

For (13),

$$P_\varepsilon^2(t,\infty) = \Big(\int_0^\infty u_\varepsilon(t,x) dx\Big)^2 \neq 0.$$

Therefore, (18) gives

$$\frac{d}{dt}\int_0^\infty u_\varepsilon^2 dx + 2\varepsilon \int_0^\infty (\partial_x u_\varepsilon)^2 dx \neq 0,$$

which contradicts (17). □

Lemma 2. *For each $t \geq 0$, (16) holds true. In particular, we have that*

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(0,\infty)} + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(0,\infty)} ds \leq \|u_0\|^2_{L^2(0,\infty)}. \tag{20}$$

Proof. We begin by observing that $u_\varepsilon(t,0)=0$ implies $\partial_t u_\varepsilon(t,0)=0$. Thus, thanks to (3),

$$\varepsilon\partial_{xx}^2 u_\varepsilon(t,0)=\partial_t u_\varepsilon(t,0)+f'(u_\varepsilon(t,0))\partial_x u_\varepsilon(t,0)-\gamma\int_0^0 u_\varepsilon(t,x)dx=0. \quad (21)$$

Differentiating (15) with respect to x, we have

$$\partial_x(\partial_t u_\varepsilon+\partial_x f(u_\varepsilon)-\varepsilon\partial_{xx}^2 u_\varepsilon)=\gamma u_\varepsilon.$$

For (21) and the smoothness of u_ε, an integration over $(0,\infty)$ gives (16). Lemma 1 says that (17) also holds true. Therefore, integrating (17) on $(0,t)$, for (14), we have (20). □

Lemma 3. *We have that*

$$\{u_\varepsilon\}_{\varepsilon>0} \quad \textit{is bounded in} \quad L^\infty_{loc}((0,\infty)^2). \quad (22)$$

Consequently,

$$\{P_\varepsilon\}_{\varepsilon>0} \quad \textit{is bounded in} \quad L^\infty_{loc}((0,\infty)^2). \quad (23)$$

Proof. Thanks to (15), (20), and the Hölder inequality,

$$\begin{aligned}\partial_t u_\varepsilon+\partial_x f(u_\varepsilon)-\varepsilon\partial_{xx}^2 u_\varepsilon&=\gamma\int_0^x u_\varepsilon(t,y)dy\le\gamma\left|\int_0^x u_\varepsilon(t,y)dy\right|\\&\le\gamma\int_0^x|u_\varepsilon(t,y)|dy\le\gamma\sqrt{x}\,\|u_\varepsilon(t,\cdot)\|_{L^2(0,\infty)}\\&\le\gamma\sqrt{x}\,\|u_0\|_{L^2(0,\infty)}\,.\end{aligned}$$

Let v, w, v_ε, and w_ε be the solutions of the following equations:

$$\begin{cases}\partial_t v+\partial_x f(v)=\gamma\,\|u_0\|_{L^2(0,\infty)}\sqrt{x}, & t>0, x>0,\\ v(t,0)=0, & t>0,\\ v(0,x)=u_0(x), & x>0,\end{cases}$$

$$\begin{cases}\partial_t w+\partial_x f(w)=-\gamma\,\|u_0\|_{L^2(0,\infty)}\sqrt{x}, & t>0, x>0,\\ w(t,0)=0, & t>0,\\ w(0,x)=u_0(x), & x>0,\end{cases}$$

$$\begin{cases} \partial_t v_\varepsilon + \partial_x f(v_\varepsilon) = \gamma \left\| u_0 \right\|_{L^2(0,\infty)} \sqrt{x} + \varepsilon \partial^2_{xx} v_\varepsilon, & t > 0, x > 0, \\ v_\varepsilon(t,0) = 0, & t > 0, \\ v_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x > 0, \end{cases}$$

$$\begin{cases} \partial_t w_\varepsilon + \partial_x f(w_\varepsilon) = -\gamma \left\| u_0 \right\|_{L^2(0,\infty)} \sqrt{x} + \varepsilon \partial^2_{xx} w_\varepsilon, & t > 0, x > 0, \\ w_\varepsilon(t,0) = 0, & t > 0, \\ w_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x > 0, \end{cases}$$

respectively. Then u_ε, v_ε, and w_ε are respectively a solution, a supersolution, and a subsolution of the parabolic problem

$$\begin{cases} \partial_t q + \partial_x f(q) = \gamma \int_0^x u_\varepsilon(t,y)dy + \varepsilon \partial^2_{xx} q, & t > 0,\ x > 0, \\ q(t,0) = 0, & t > 0, \\ q(0,x) = u_{\varepsilon,0}(x), & x > 0. \end{cases}$$

Thus, see [6, Chap. 2, Theorem 9],

$$w_\varepsilon \le u_\varepsilon \le v_\varepsilon.$$

Moreover, $\{w_\varepsilon\}_{\varepsilon>0}$ and $\{v_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded in $L^\infty_{loc}((0,\infty)^2)$ and converge to w and v respectively, see [1, 17]. Therefore the two functions

$$W = \inf_{\varepsilon>0} w_\varepsilon, \qquad V = \sup_{\varepsilon>0} v_\varepsilon$$

belong to $L^\infty_{loc}((0,\infty)^2)$ and satisfy

$$W \le w_\varepsilon \le u_\varepsilon \le v_\varepsilon \le V. \tag{24}$$

This gives (22). Since

$$|P_\varepsilon(t,x)| = \left| \int_0^x u_\varepsilon(t,y)dy \right| \le \int_0^x |u_\varepsilon(t,y)|dy,$$

(23) follows from (22). □

Let us continue by proving the existence of a distributional solution to (2), (4), and (5) satisfying (10).

Lemma 4. *There exists a function* $u \in L^\infty_{loc}((0,\infty)^2)$ *that is a distributional solution of* (8) *and satisfies* (10) *for every convex entropy* $\eta \in C^2(\mathbb{R})$.

We construct a solution by passing to the limit in a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of viscosity approximations (13). We use the compensated compactness method [24].

Lemma 5. *There exist a subsequence* $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ *of* $\{u_\varepsilon\}_{\varepsilon>0}$ *and a limit function* $u \in L^\infty_{loc}((0,\infty)^2)$ *such that*

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0,\infty)^2),\ 1 \le p < \infty. \tag{25}$$

Moreover, we have

$$P_{\varepsilon_k} \to P \text{ a.e. and in } L^p_{loc}(0,\infty; W^{1,p}_{loc}(0,\infty)),\ 1 \le p < \infty, \tag{26}$$

where

$$P(t,x) = \int_0^x u(t,y)dy, \qquad t \ge 0, \quad x \ge 0.$$

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}$ be any convex C^2 entropy function, and $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q' = f'\eta'$. By multiplying the first equation in (13) by $\eta'(u_\varepsilon)$ and using the chain rule, we get

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \underbrace{\varepsilon\partial^2_{xx}\eta(u_\varepsilon)}_{=:\mathcal{L}_{1,\varepsilon}} \underbrace{-\varepsilon\eta''(u_\varepsilon)\,(\partial_x u_\varepsilon)^2}_{=:\mathcal{L}_{2,\varepsilon}} \underbrace{+\gamma\eta'(u_\varepsilon)P_\varepsilon}_{=:\mathcal{L}_{3,\varepsilon}},$$

where $\mathcal{L}_{1,\varepsilon}$, $\mathcal{L}_{2,\varepsilon}$, $\mathcal{L}_{3,\varepsilon}$ are distributions.

Thanks to Lemma 2

$$\mathcal{L}_{1,\varepsilon} \to 0 \text{ in } H^{-1}_{loc}((0,\infty)^2),$$

$$\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1_{loc}((0,\infty)^2).$$

We prove that

$$\{\mathcal{L}_{3,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1_{loc}((0,\infty)^2).$$

Let K be a compact subset of $(0,\infty)^2$. For Lemma 3,

$$\begin{aligned}\left\|\gamma\eta'(u_\varepsilon)P_\varepsilon\right\|_{L^1(K)} &= \gamma \iint_K |\eta'(u_\varepsilon)||P_\varepsilon| dt\,dx \\ &\le \gamma \left\|\eta'(u_\varepsilon)\right\|_{L^\infty(K)} \|P_\varepsilon\|_{L^\infty(K)} |K|.\end{aligned}$$

Therefore, Murat's lemma [19] implies that

$$\{\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon)\}_{\varepsilon>0} \text{ lies in a compact subset of } H^{-1}_{\mathrm{loc}}((0,\infty)^2). \tag{27}$$

The L^∞_{loc} bound stated in Lemma 3, (27), and Tartar's compensated compactness method [24] give the existence of a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and a limit function $u \in L^\infty_{loc}((0,\infty)^2)$ such that (25) holds.

Finally, (26) follows from (25), the Hölder inequality, and the identities

$$P_{\varepsilon_k}(t,x) = \int_0^x u_{\varepsilon_k}(t,y)dy, \qquad \partial_x P_{\varepsilon_k} = u_{\varepsilon_k}.$$

Moreover, [5, Theorem 1.1] tells us that the limit u admits a strong boundary trace u_0^τ at $(0,\infty) \times \{x = 0\}$. Since, arguing as in [5, Sect. 3.1] (indeed our solution is obtained as the vanishing viscosity limit of (8)), [5, Lemma 3.2] and the boundedness of the source term P (cf. (9)) imply (11). □

We are now ready for the proof of Theorem 1.

Proof (Proof of Theorem 1). Lemma (5) gives the existence of an entropy solution $u(t,x)$ of (7), or equivalently (8).

Let us show that $u(t,x)$ is unique, and that (12) holds true. Since our solutions is only locally bounded we use the doubling of variables method and get local estimates based on the finite speed of propagation of the waves generated by (2). Let u, v be two entropy solutions of (7), or equivalently of (8), and $0 < t < T$. By arguing as in [2,9], using the fact that the two solutions satisfy the same boundary conditions, we can prove that

$$\partial_t(|u-v|) + \partial_x((f(u) - f(v))\mathrm{sign}\,(u-v)) - \gamma \mathrm{sign}\,(u-v)\,(P_u - P_v) \leq 0$$

holds in the sense of distributions in $(0,\infty) \times (0,\infty)$, and

$$\begin{aligned}\|u(t,\cdot) - v(t,\cdot)\|_{I(t)} &\leq \|u_0 - v_0\|_{I(0)} \\ &\quad + \gamma \int_0^t \int_{I(s)} \mathrm{sign}\,(u-v)\,(P_u - P_v) ds\, dx,\end{aligned} \qquad 0 < t < T, \tag{28}$$

where

$$P_u(t,x) = \int_0^x u(t,y)dy, \quad P_v = \int_0^x v(t,y)dy, \quad I(s) = (0, R + L(t-s)),$$

and L is the Lipschitz constant of the flux f.

Since

$$\begin{aligned}\gamma \int_0^t \int_{I(s)} \mathrm{sign}\,(u-v)\,(P_u - P_v) ds\, dx &\leq \gamma \int_0^t \int_{I(s)} |P_u - P_v| ds\, dx \\ &\leq \gamma \int_0^t \int_{I(s)} \left(\left| \int_0^x |u-v| dy \right|\right) ds\, dx\end{aligned}$$

$$\leq \gamma \int_0^t \int_{I(s)} \left(\left| \int_{I(s)} |u - v| dy \right| \right) ds\, dx$$
$$= \gamma \int_0^t |I(s)| \, \|u(s, \cdot) - v(s, \cdot)\|_{L^1(I(s))} \, ds, \tag{29}$$

and

$$|I(s)| = R + L(t - s) \leq R + Lt \leq R + LT, \tag{30}$$

we can consider the following continuous function:

$$G(t) = \|u(t, \cdot) - v(t, \cdot)\|_{L^1(I(t))}, \quad t \geq 0. \tag{31}$$

Using this notation, it follows from (28)–(30) that

$$G(t) \leq G(0) + C \int_0^t G(s) ds,$$

where $C = \gamma(R + LT)$. Gronwall's inequality and (31) give

$$\|u(t, \cdot) - v(t, \cdot)\|_{L^1(0,R)} \leq e^{Ct} \|u_0 - v_0\|_{L^1(0,R+Lt)},$$

that is (12). □

3 The Cauchy Problem

Let us consider now the Cauchy problem associated to (2). Since the arguments are similar to those of the previous section we simply sketch them, highlighting only the differences between the two problems.

In this section we augment (2) with the initial datum

$$u(0, x) = u_0(x), \qquad x \in \mathbb{R}. \tag{32}$$

We assume that

$$u_0 \in L^2(\mathbb{R}) \cap L^\infty_{loc}(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x) dx = 0. \tag{33}$$

Indeed, integrating both sides of (2) we have that $u(t, \cdot)$ has zero mean for every $t > 0$, therefore it is natural to assume the same on the initial condition. We rewrite the Cauchy problem (2), (32) in the following way

$$\begin{cases} \partial_t u + \partial_x f(u) = \gamma \int_0^x u(t,y)dy, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{34}$$

or equivalently

$$\begin{cases} \partial_t u + \partial_x f(u) = \gamma P, & t > 0,\ x \in \mathbb{R}, \\ \partial_x P = u, & t > 0,\ x \in \mathbb{R}, \\ P(t,0) = 0, & t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{35}$$

Due to the regularizing effect of the P equation in (35) we have that

$$u \in L^\infty_{loc}((0,\infty)\times\mathbb{R}) \Longrightarrow P \in L^\infty_{loc}((0,\infty); W^{1,\infty}_{loc}(\mathbb{R})).$$

Definition 2. We say that $u \in L^\infty_{loc}((0,\infty)\times\mathbb{R})$ is an entropy solution of the initial value problem (2), and (32) if:

(i) u is a distributional solution of (34) or equivalently of (35);
(ii) For every convex function $\eta \in C^2(\mathbb{R})$ the entropy inequality

$$\partial_t \eta(u) + \partial_x q(u) - \gamma\eta'(u)P \le 0, \qquad q(u) = \int^u f'(\xi)\eta'(\xi)\, d\xi, \tag{36}$$

holds in the sense of distributions in $(0,\infty)\times\mathbb{R}$.

The main result of this section is the following theorem.

Theorem 2. *Assume* (32) *and* (33)*. The initial value problem* (2) *and* (32) *possesses a unique entropy solution u in the sense of Definition* 2*. Moreover, if u and v are two entropy solutions* (2) *and* (32)*, in the sense of Definition* 2 *the following inequality holds*

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(-R,R)} \le e^{Ct}\, \|u(0,\cdot) - v(0,\cdot)\|_{L^1(-R-Lt,R+Lt)}\,, \tag{37}$$

for almost every $t > 0$*,* $R, T > 0$*, and a suitable constant* $C > 0$*.*

Our existence argument is based on a passage to the limit in a vanishing viscosity approximation of (35).

Fix a small number $\varepsilon > 0$, and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following mixed problem

$$\begin{cases} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \gamma P_\varepsilon + \varepsilon\partial^2_{xx} u_\varepsilon, & t > 0,\ x \in \mathbb{R}, \\ \partial_x P_\varepsilon = u_\varepsilon, & t > 0,\ x \in \mathbb{R}, \\ P_\varepsilon(t,0) = 0, & t > 0, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{38}$$

where $u_{\varepsilon,0}$ is a $C^\infty(\mathbb{R})$ approximation of u_0 such that

$$\|u_{\varepsilon,0}\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \quad \int_{\mathbb{R}} u_{\varepsilon,0}(x)dx = 0. \tag{39}$$

Clearly, (38) is equivalent to the integro-differential problem

$$\begin{cases} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \gamma \int_0^x u_\varepsilon(t,y)dy + \varepsilon\partial_{xx}^2 u_\varepsilon, & t > 0,\ x \in \mathbb{R}, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}. \end{cases} \tag{40}$$

The existence of such solutions can be obtained by fixing a small number $\delta > 0$ and considering the further approximation of (38) (see [4])

$$\begin{cases} \partial_t u_{\varepsilon,\delta} + \partial_x f(u_{\varepsilon,\delta}) = \gamma P_{\varepsilon,\delta} + \varepsilon\partial_{xx}^2 u_{\varepsilon,\delta}, & t > 0,\ x \in \mathbb{R}, \\ -\delta\partial_{xx}^2 P_{\varepsilon,\delta} + \partial_x P_{\varepsilon,\delta} = u_{\varepsilon,\delta}, & t > 0,\ x \in \mathbb{R}, \\ P_{\varepsilon,\delta}(t,0) = 0, & t > 0, \\ u_{\varepsilon,\delta}(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases}$$

and then sending $\delta \to 0$.

Let us prove some a priori estimates on u_ε. Arguing as in Lemma 1 we have the following.

Lemma 6. *Let us suppose that*

$$P_\varepsilon(t,-\infty) = 0, \quad t \geq 0, \quad (or \quad P_\varepsilon(t,\infty) = 0), \tag{41}$$

where $P_\varepsilon(t,x)$ is defined in (38). Then the following statements are equivalent

$$\int_{\mathbb{R}} u_\varepsilon(t,x)dx = 0, \quad t \geq 0, \tag{42}$$

$$\frac{d}{dt}\int_{\mathbb{R}} u_\varepsilon^2 dx + 2\varepsilon\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 dx = 0, \quad t > 0. \tag{43}$$

Lemma 7. *For each $t \geq 0$, (42) holds true, and*

$$P_\varepsilon(t,\infty) = P_\varepsilon(t,-\infty) = 0. \tag{44}$$

In particular, we have that

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \leq \|u_0\|^2_{L^2(\mathbb{R})}. \tag{45}$$

Proof. Differentiating (40) with respect to x, we have

$$\partial_x(\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) - \varepsilon\partial_{xx}^2 u_\varepsilon) = u_\varepsilon.$$

Since u_ε is a smooth solution of (40), an integration over $\mathbb{R}$ gives (42).
Again for the regularity of u_ε, from (38), we get

$$\lim_{x\to-\infty}(\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) - \varepsilon\partial_{xx}^2 u_\varepsilon) = \gamma\int_0^{-\infty} u_\varepsilon(t,x)dx = \gamma P_\varepsilon(t,-\infty) = 0,$$

$$\lim_{x\to\infty}(\partial_t s u_\varepsilon + \partial_x f(u_\varepsilon) - \varepsilon\partial_{xx}^2 u_\varepsilon) = \gamma\int_0^{\infty} u_\varepsilon(t,x)dx = \gamma P_\varepsilon(t,\infty) = 0,$$

that is (44).

Lemma 6 says that (43) also holds true. Therefore, integrating (43) on $(0,t)$, for (39), we have (45). □

Arguing as in Lemma 3 we obtain the following lemma:

Lemma 8. *We have that*

$$\{u_\varepsilon\}_{\varepsilon>0} \quad \textit{is bounded in} \quad L^\infty_{loc}((0,\infty)\times\mathbb{R}). \tag{46}$$

Consequently,

$$\{P_\varepsilon\}_{\varepsilon>0} \quad \textit{is bounded in} \quad L^\infty_{loc}((0,\infty)\times\mathbb{R}). \tag{47}$$

Let us continue by proving the existence of a distributional solution to (2) and (5) satisfying (36).

Lemma 9. *There exists a function $u \in L^\infty_{loc}((0,\infty)\times\mathbb{R})$ that is a distributional solution of (35) and satisfies (36) for every convex entropy $\eta \in C^2(\mathbb{R})$.*

We construct a solution by passing to the limit in a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of viscosity approximations (38). We use the compensated compactness method [24].

Lemma 10. *There exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and a limit function $u \in L^\infty_{loc}((0,\infty)\times\mathbb{R})$ such that*

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty. \tag{48}$$

Moreover, we have

$$P_{\varepsilon_k} \to P \text{ a.e. and in } L^p_{loc}((0,\infty); W^{1,p}_{loc}(\mathbb{R})),\ 1 \le p < \infty, \tag{49}$$

where

$$P(t,x) = \int_0^x u(t,y)dy, \qquad t \ge 0, \quad x \in \mathbb{R}.$$

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}$ be any convex C^2 entropy function, and $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q' = f'\eta'$. By multiplying the first equation in (38) by $\eta'(u_\varepsilon)$ and using the chain rule, we get

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \underbrace{\varepsilon \partial_{xx}^2 \eta(u_\varepsilon)}_{=:\mathcal{L}_{1,\varepsilon}} \underbrace{-\varepsilon \eta''(u_\varepsilon)\,(\partial_x u_\varepsilon)^2}_{=:\mathcal{L}_{2,\varepsilon}} \underbrace{+\gamma \eta'(u_\varepsilon) P_\varepsilon}_{=:\mathcal{L}_{3,\varepsilon}},$$

where $\mathcal{L}_{1,\varepsilon}$, $\mathcal{L}_{2,\varepsilon}$, $\mathcal{L}_{3,\varepsilon}$ are distributions.

Arguing as in Lemma 5, we have that

$$\mathcal{L}_{1,\varepsilon} \to 0 \text{ in } H_{loc}^{-1}((0,\infty)\times\mathbb{R}),$$
$$\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon>0} \text{ and } \{\mathcal{L}_{3,\varepsilon}\}_{\varepsilon>0} \text{ are uniformly bounded in } L_{loc}^{1}((0,\infty)\times\mathbb{R}).$$

Therefore, Murat's lemma [19] implies that

$$\{\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon)\}_{\varepsilon>0} \text{ lies in a compact subset of } H_{\mathrm{loc}}^{-1}((0,\infty)\times\mathbb{R}). \tag{50}$$

The L_{loc}^∞ bound stated in Lemma 8, (50), and Tartar's compensated compactness method [24] imply the existence of a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and a limit function $u \in L_{loc}^\infty((0,\infty)\times\mathbb{R})$ such that (48) holds.

Finally, (49) follows from (48), the Hölder inequality, and the identities

$$P_{\varepsilon_k}(t,x) = \int_0^x u_{\varepsilon_k}(t,y)dy, \qquad \partial_x P_{\varepsilon_k} = u_{\varepsilon_k}. \qquad \square$$

We are now ready for the proof of Theorem 2.

Proof (Proof of Theorem 2). Lemma (10) gives the existence of an entropy solution u of (7), or equivalently (35).

Let us show that u is unique, and that (37) holds true. Let u, v be two entropy solutions of (7) or equivalently of (35) and $0 < t < T$. Arguing as in [9] we can prove that

$$\begin{aligned}\|u(t,\cdot) - v(t,\cdot)\|_{I(t)} &\le \|u_0 - v_0\|_{I(0)} \\ &\quad + \gamma \int_0^t \int_{I(s)} \mathrm{sign}\,(u-v)\,(P_u - P_v) ds\, dx\end{aligned} \qquad 0 < t < T, \tag{51}$$

where

$$P_u(t,x) = \int_0^x u(t,y)dy, \quad P_v = \int_0^x v(t,y)dy, \quad I(s) = (-R - L(t-s), R + L(t-s)),$$

and L is the Lipschitz constant of the flux f.

Since

$$\begin{aligned}\gamma \int_0^t \int_{I(s)} \operatorname{sign}(u-v)\,(P_u - P_v) ds\,dx &\le \gamma \int_0^t \int_{I(s)} |P_u - P_v| ds\,dx \\ &\le \gamma \int_0^t \int_{I(s)} \left(\left| \int_0^x |u-v| dy \right|\right) ds\,dx \\ &\le \gamma \int_0^t \int_{I(s)} \left(\left| \int_{I(s)} |u-v| dy \right|\right) ds\,dx \\ &= \gamma \int_0^t |I(s)|\, \|u(s,\cdot) - v(s,\cdot)\|_{L^1(I(s))}\, ds,\end{aligned} \tag{52}$$

and

$$|I(s)| = 2R + 2L(t-s) \le 2R + 2Lt \le 2R + 2LT, \tag{53}$$

we can consider the following continuous function:

$$G(t) = \|u(t,\cdot) - v(t,\cdot)\|_{L^1(I(t))}, \quad t \ge 0. \tag{54}$$

It follows from (51) to (53) that

$$G(t) \le G(0) + C \int_0^t G(s) ds,$$

where $C = \gamma(2R + 2LT)$.

Gronwall's inequality and (54) give

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(-R,R)} \le e^{Ct} \|u_0 - v_0\|_{L^1(-R-Lt,R+Lt)},$$

that is (37). □

References

1. D. Amadori, L. Gosse, G. Guerra, Godunov-type approximation for a general resonant balance law with large data. J. Differ. Equ. **198**, 233–274 (2004)
2. C. Bardos, A.Y. le Roux, J.C. Nédélec, First order quasilinear equations with boundary conditions. Commun. Partial Differ. Equ. 4 **9**, 1017–1034 (1979)
3. J. Boyd, Ostrovsky and Hunters generic wave equation for weakly dispersive waves: matched asymptotic and pseudospectral study of the paraboloidal travelling waves (corner and near-corner waves). Eur. J. Appl. Math. **16**(1), 65–81 (2005)

4. G.M. Coclite, H. Holden, K.H. Karlsen, Wellposedness for a parabolic-elliptic system. Discret. Contin. Dyn. Syst. **13**(3), 659–682 (2005)
5. G.M. Coclite, K.H. Karlsen, Y.-S. Kwon, Initial-boundary value problems for conservation laws with source terms and the Degasperis-Procesi equation. J. Funct. Anal. **257**(12), 3823–3857 (2009)
6. A. Friedman, *Partial Differential Equations of Parabolic Type* (Dover Books on Mathematics, New York, 2008)
7. G. Gui, Y. Liu, On the Cauchy problem for the Ostrovsky equation with positive dispersion. Commun. Partial Differ. Equ. **32**(10–12), 1895–1916 (2007)
8. J. Hunter, Numerical solutions of some nonlinear fispersive wave equations. (Computational solution of nonlinear systems of equations (Fort Collins, CO, 1988)). Lect. Appl. Math. (American Mathematical Society, Providence) **26**, 301–316 (1990)
9. S.N. Kružkov, First order quasilinear equations with several independent variables. Mat. Sb. (N.S.), **81**(123), 228–255 (1970)
10. S. Levandosky, Y. Liu, Stability of solitary waves of a generalized Ostrovsky equation. SIAM J. Math. Anal. **38**(3), 985–1011 (2006)
11. S. Levandosky, Y. Liu, Stability and weak rotation limit of solitary waves of the Ostrovsky equation. Discret. Contin. Dyn. Syst. B **7**(7), 793–806 (2007)
12. F. Linares, A. Milanes, Local and global well-posedness for the Ostrovsky equation. J. Differ. Equ. **222**(2), 325–340 (2006)
13. Y. Liu, On the stability of solitary waves for the Ostrovsky equation. Quart. Appl. Math. **65**(3), 571–589 (2007)
14. Y. Liu, V. Varlamov, Cauchy problem for the Ostrovsky equation. Discret. Contin. Dyn. Syst. **10**(3), 731–753 (2004)
15. Y. Liu, V. Varlamov, Stability of solitary waves and weak rotation limit for the Ostrovsky equation. J. Differ. Equ. **203**(1), 159–183 (2004)
16. Y. Liu, D. Pelinovsky, A. Sakovich, Wave breaking in the Ostrovsky–Hunter equation. SIAM J. Math. Anal. **42**(5), 1967–1985 (2010)
17. J. Málek, J. Nevcas, M. Rokyta, M. Rocircuvzivcka, *Weak and Measure-Valued Solutions to Evolutionary PDEs*. Applied Mathematics and Mathematical Computation, vol. 13 (Chapman-Hall, London, 1996)
18. A.J. Morrison, E.J. Parkes, V.O. Vakhnenko, The N loop soliton solutions of the Vakhnenko equation. Nonlinearity **12**(5), 1427–1437 (1999)
19. F. Murat, L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$. J. Math. Pures Appl. (9), **60**(3), 309–322 (1981)
20. L.A. Ostrovsky, Nonlinear internal waves in a rotating ocean. Okeanologia **18**, 181–191 (1978)
21. E.J. Parkes, Explicit solutions of the reduced Ostrovsky equation. Chaos Solitons and Fractals **31**(3), 602–610 (2007)
22. E.J. Parkes, V.O. Vakhnenko, The calculation of multi-soliton solutions of the Vakhnenko equation by the inverse scattering method. Chaos, Solitons and Fractals **13**(9), 1819–1826 (2002)
23. Y.A. Stepanyants, On stationary solutions of the reduced Ostrovsky equation: periodic waves, compactons and compound solitons. Chaos, Solitons and Fractals **28**(1), 193–204 (2006)
24. L. Tartar, Compensated compactness and applications to partial differential equations. In: *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium*, vol. IV (Pitman, Boston, 1979), pp. 136–212
25. K. Tsugawa, Well-posedness and weak rotation limit for the Ostrovsky equation. J. Differ. Equ. **247**(12), 3163–3180 (2009)