Divergence-Measure Fields on Domains with Lipschitz Boundary

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Abstract In this work we are particularly interested in analyzing some consequences of the additional assumption that the domain has a Lipschitz boundary, in the investigation of the properties of the divergence-measure fields, especially those which are vector-valued (Radon) measures whose divergence is a signed (Radon) measure.

Keywords Divergence-measure fields • Normal traces • Gauss-Green theorem • Product rule

1991 Mathematics Subject Classification Primary: 26B20,28C05, 35L65, 35B35; Secondary: 26B35, 26B12, 35L67

1 Introduction

The purpose of this paper is to establish further properties of the (extended) divergence-measure fields introduced by Chen and Frid [2–4], whose theory was further developed by Silhavý [10, 11], under the additional assumption that the underlying domain has a Lipschitz boundary. We begin by briefly reviewing the basic theory, and then we make the assumption that the domain possesses a Lipchitz deformable boundary, analyzing some consequences of this assumption. We refer to [9] for a more detailed review of the theory of the divergence-measure fields up to this date. We also refer to [6] and the papers already mentioned for a more

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G.-Q.G. Chen et al. (eds.), *Hyperbolic Conservation Laws and Related Analysis with Applications*, Springer Proceedings in Mathematics & Statistics 49,
DOI 10.1007/978-3-642-39007-4_10,

complete bibliography on the theory of divergence-measure fields, as well as many of its possible applications.

2 Divergence-Measure Fields

We begin by recalling the definition of the divergence-measure fields.

Definition 1. Let $U \subset \mathbb{R}^N$ be open. For $F \in L^p(U;\mathbb{R}^N)$, $1 \le p \le \infty$, or $F \in \mathcal{M}(U;\mathbb{R}^N)$, set

$$|\text{div}\, F|(U) := \sup\{ \int_U \nabla\varphi \cdot F \ : \ \varphi \in C_0^1(U),\ |\varphi(x)| \le 1,\ x \in U \}. \tag{1}$$

For $1 \le p \le \infty$, we say that F is an L^p-divergence-measure field over U, i.e., $F \in \mathcal{DM}^p(U)$, if $F \in L^p(U;\mathbb{R}^N)$ and

$$\|F\|_{\mathcal{DM}^p(U)} := \|F\|_{L^p(U;\mathbb{R}^N)} + |\text{div}\, F|(U) < \infty. \tag{2}$$

We say that F is an extended divergence-measure field over D, i.e., $F \in \mathcal{DM}^{ext}(U)$, if $F \in \mathcal{M}(U;\mathbb{R}^N)$ and

$$\|F\|_{\mathcal{DM}^{ext}(U)} := |F|(U) + |\text{div}\, F|(U) < \infty. \tag{3}$$

If $F \in \mathcal{DM}^*(U)$ for any open set $U \Subset \mathbb{R}^N$, then we say $F \in \mathcal{DM}^*_{loc}(\mathbb{R}^N)$.

In order to introduce notation and go directly to the heart of the matter, we recall the following product rule proved in [2], whose proof is almost entirely transposed to prove the main product rule that we will state subsequently, which is the key to establishing the Gauss-Green formula (see Theorem 3 below).

Theorem 1 (Chen and Frid [2]). *Given $F \in \mathcal{DM}^\infty(U)$ and $g \in BV(U) \cap L^\infty(U)$, then $gF \in \mathcal{DM}^\infty(U)$ and*

$$\text{div}\,(gF) = \bar{g}\text{div}\, F + \overline{F \cdot \nabla g}, \tag{4}$$

in the sense of Radon measures in U, where $\bar{g}$ (equal to g a.e.) is the limit of a mollified sequence for g through a symmetric mollifier, and $\overline{F \cdot \nabla g}$ is a Radon measure absolutely continuous with respect to $|\nabla g|$, whose absolutely continuous part with respect to the Lebesgue measure in U satisfies

$$(\overline{F \cdot \nabla g})_{ac} = F \cdot (\nabla g)_{ac}, \qquad a.e. \text{ in } U. \tag{5}$$

Moreover, $|\overline{F \cdot \nabla g}|(U) \le \|F\|_\infty |\nabla g|(U)$.

Proof. Let $g_\delta = \omega_\delta * g$, where $\omega_\delta(x) = \delta^{-N}\eta(\frac{x}{\delta})$ with a positive symmetric mollifier ω. One easily deduces that

$$\operatorname{div}(g_\delta F) = g_\delta \operatorname{div} F + F \cdot \nabla g_\delta. \tag{6}$$

Now, it is well known that g_δ converges to a Borel function $\bar{g}$, $\mathcal{H}^{N-1}$-a.e. in U (this function equals g a.e. in U).

We claim that, for a Borel set $A \subset U$, $\mathcal{H}^{N-1}(A) = 0$ implies $|\operatorname{div} F|(A) = 0$. Indeed, since $|\operatorname{div} F|$ is a Radon measure, we may assume that A is compact. Also, we may assume that $\operatorname{div} F(A) = |\operatorname{div} F|(A)$. Hence, given $\varepsilon > 0$, we may cover A with a finite number of balls $B_i = B(x_i; r_i)$, $i = 1, \ldots, J$,

$$A \subset A_\varepsilon := \cup_{i=1}^{J} B_i, \text{ such that } \sum_{i=1}^{J} r_i^{N-1} \le \varepsilon. \tag{7}$$

We may also assume that $|\operatorname{div} F|(\partial B_i) = 0$, $i = 1, \ldots, J$, since otherwise we can modify r_i slightly to satisfy this property and (7). By using an approximation of the identity sequence, we obtain a sequence $F_\delta \in C^\infty(U; \mathbb{R}^N)$ such that $F_\delta \to F$ a.e. in U, and $|\operatorname{div} F_\delta| \rightharpoonup |\operatorname{div} F|$ in $\mathcal{M}(U)$. Again, we may assume that $F_\delta \to F$ a.e. in ∂B_i, $i = 1, \ldots, J$. Now, by the usual Gauss-Green formula for smooth vector fields and domains with Lipschitz boundaries, we have

$$\int_{A_\varepsilon} \operatorname{div} F_\delta \, dx = \int_{\partial A_\varepsilon} F_\delta \cdot \nu \, d\mathcal{H}^{N-1},$$

so that, passing to the limit when $\delta \to 0$, we obtain

$$\int_{A_\varepsilon} \operatorname{div} F = \int_{\partial A_\varepsilon} F \cdot \nu \, d\mathcal{H}^{N-1} \le c\|F\|_\infty \sum_{i=1}^{J} r_i^{N-1} \le c\|F\|_\infty \varepsilon.$$

Since A is compact, $\chi_{A_\varepsilon} \to \chi_A$ everywhere in U, and by dominated convergence applied to the measure $|\operatorname{div} F|$, we get $|\operatorname{div} F|(A) = \operatorname{div} F(A) = 0$, which proves the claim.

Then, using the claim we just proved, we get

$$g_\delta \operatorname{div} F \rightharpoonup \bar{g} \operatorname{div} F, \qquad \text{in } \mathcal{M}(U),$$

as a consequence of dominated convergence applied to the measure $\operatorname{div} F$.

On the other hand, we claim that $\{\operatorname{div}(g_\delta F)\}$ is uniformly bounded in $\mathcal{M}(U)$. Indeed, this follows from

$$\begin{aligned}\langle \operatorname{div}(g_\delta F), \phi \rangle &= -\int_U g_\delta F \cdot \nabla \phi \, dx = -\int_U F \cdot \nabla(g_\delta \phi) \, dx + \int_U \phi F \cdot \nabla g_\delta \, dx \\ &\le \|g\|_\infty |\operatorname{div} F|(U) + \|F\|_\infty |\nabla g|(U),\end{aligned}$$

for all $\phi \in C_c^\infty(U)$, with $\|\phi\|_\infty = 1$.

Now, div $(g_\delta F)$ converges to div (gF), in the sense of distributions over U. Then, div $(g_\delta F) \rightharpoonup$ div (gF) in $\mathcal{M}(U)$. Hence,

$$F \cdot \nabla g_\delta \rightharpoonup \overline{F \cdot \nabla g} := \operatorname{div}(gF) - \bar{g}\operatorname{div} F.$$

Now we prove that $\overline{F \cdot \nabla g}$ is absolutely continuous w.r.t. $|\nabla g|$. Let $A \subset D$ be such that $|\nabla g|(A) = 0$. We are going to prove that $|\overline{F \cdot \nabla g}|(A) = 0$. It suffices to consider any compact set A with $|\nabla g|(A) = 0$. Given $\varepsilon > 0$, we can cover A by a finite number, J, of balls so that

$$A \subset \cup_{i=1}^J B(x_i; r_i), \quad r_i < \varepsilon; \qquad |\nabla g|\big(\cup_{i=1}^J B(x_i; r_i)\big) < \varepsilon.$$

We may assume that $|\nabla g|(\partial B(x_i; r_i)) = 0$, $i = 1, \cdots, J$. Let $\phi \in C_0(\cup_{i=1}^J B(x_i; r_i))$. Thus

$$\begin{aligned}\langle \overline{F \cdot \nabla g}, \phi \rangle &= \lim_{\delta \to 0} \int \phi(x)\, F(x) \cdot \nabla g_\delta(x)\, dx \\ &= \|\phi\|_\infty \|F\|_\infty |\nabla g|\big(\cup_{i=1}^J B(x_i; r_i)\big) \le \varepsilon \|\phi\|_\infty \|F\|_\infty,\end{aligned}$$

from the fact that $|\nabla g_\delta|(B) \to |\nabla g|(B)$, for all open sets $B \subset D$ with $|\nabla g|(\partial B) = 0$. Hence, we obtain

$$|\overline{F \cdot \nabla g}|(A) \le |\overline{F \cdot \nabla g}|\left(\cup_{i=1}^J B(x_i; r_i)\right) \le \varepsilon \|F\|_\infty.$$

The proof of (5) is a little more technical and, for that, we simply refer to [2] since it escapes our purposes here. □

We now recall a result of Silhavý in [11] that is in some sense a dual formulation for the previous result, in the sense that it compensates a relaxation on the regularity of the vector field F, which now may be just a vector measure, by imposing more regularity on the function g, which now is assumed to be in $W^{1,\infty}(U)$. As we will see, its proof follows exactly the same lines as that of Theorem 1 just recalled.

Theorem 2 (Silhavý [11]). *Given $F \in \mathcal{DM}^{ext}(U)$ and $g \in W^{1,\infty}(U)$, then $gF \in \mathcal{DM}^{ext}(U)$ and*

$$\operatorname{div}(gF) = g\operatorname{div} F + \overline{\nabla g \cdot F}, \tag{8}$$

in the sense of Radon measures in U, where $\overline{\nabla g \cdot F}$ is a Radon measure absolutely continuous with respect to $|F|$. Moreover,

(i) $|\overline{\nabla g \cdot F}|(U) \le \|\nabla g\|_\infty |F|(U)$.
(ii) *If $h \in W^{1,\infty}(U)$, $\overline{\nabla(gh) \cdot F} = h\overline{\nabla g \cdot F} + g\overline{\nabla h \cdot F} = \overline{\nabla g \cdot hF} + \overline{\nabla h \cdot gF}$.*
(iii) *If $V \subset U$ is an open set, then $(\overline{\nabla g|V \cdot F\lfloor V})_V = \overline{\nabla g \cdot F}_U \lfloor V$.*
(iv) $(\overline{\nabla g \cdot F})_{ac} = \nabla g \cdot (F)_{ac}$.

Proof. We again define g_δ as above and obtain (6). We have that g_δ converges locally uniformly to g so that the first term on the right-hand side of (6) converges to $g\,\mathrm{div}\,F$, in the sense of Radon measures. It is also easy to see that $\nabla g_\delta \cdot F$ is uniformly bounded in $\mathcal{M}(U)$. Therefore, the left-hand side of (6) is also compact in $\mathcal{M}(U)$, in the weak star topology, and since it converges to $\mathrm{div}\,(gF)$ in the sense of distributions, it follows that $\mathrm{div}\,(gF)$ is indeed a Radon measure and the whole sequence $\mathrm{div}\,(g_\delta F)$ converges to $\mathrm{div}\,(gF)$. Hence, the whole sequence $\nabla g_\delta \cdot F$ converges to the Radon measure

$$\overline{\nabla g \cdot F} := \mathrm{div}\,(gF) - g\,\mathrm{div}\ F.$$

The assertions (i)–(ii) are proved in the standard way. Assertion (iii) is called the localization property in [11]; it follows trivially from the definitions. Finally, the proof of (iv) is entirely similar to that of the analogous assertion in Theorem 1. □

We recall the Gauss-Green formula for general divergence-measure fields, first proved in [3, 4] and extended by Silhavý in [11].

Theorem 3 (Chen and Frid [3, 4], Silhavý [11]). *If $F \in \mathcal{DM}^{ext}(U)$ then there exists a linear functional $F \cdot \nu : \mathrm{Lip}(\partial U) \to \mathbb{R}$ such that*

$$F \cdot \nu(g|\partial U) = \int_U \overline{\nabla g \cdot F} + \int_U g\,\mathrm{div}\,F, \tag{9}$$

for every $g \in \mathrm{Lip}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Moreover,

$$|F \cdot \nu(h)| \le |F|_{\mathcal{DM}(U)} |h|_{Lip(\partial U)}, \tag{10}$$

for all $h \in Lip(\partial U)$, where we use the notation

$$|g|_{Lip(C)} := \sup_{x \in C} |g(x)| + Lip_C(g).$$

Proof. A major step in the proof of this result is to prove that the right-hand side of (9) depends only on the values of g restricted to ∂U, that is, that if $g \in \mathrm{Lip}(\mathbb{R}^N)$, with $g(x) = 0$, for $x \in \partial U$, then

$$\int_U \overline{\nabla g \cdot F} + \int_U g\,\mathrm{div}\,F = 0. \tag{11}$$

Clearly, we may as well assume $g(x) = 0$, for $x \in \mathbb{R}^N \setminus U$ (cf. Lemma 3.2 in [11]). We first prove (11) in the case where $\mathrm{supp}\,g$ is a compact subset of U. In this case, for $\delta > 0$ sufficiently small we have $\mathrm{supp}\,g_\delta \subset U$, where, as above, $g_\delta = g * \omega_\delta$. Then, by the definition of the divergence of the (vector-valued) distribution F, we have

$$\int_U \nabla g_\delta \cdot F + \int_U g_\delta \,\mathrm{div}\, F = 0. \tag{12}$$

Hence, taking the limit when $\delta \to 0$ in (12), using the definition of $\overline{\nabla g \cdot F}$, we obtain (11) in this case. We now consider the case where $g \in \mathrm{Lip}(\mathbb{R}^N)$ and $g(x) = 0$, for $x \in \mathbb{R}^N \setminus U$. Let $\zeta : \mathbb{R} \to \mathbb{R}$ be given by

$$\zeta(t) := \begin{cases} 0, & \text{if } t < 1/2, \\ 2(t - 1/2), & \text{if } 1/2 \le t \le 1, \\ 1, & \text{if } t > 1, \end{cases}$$

and for each $\varepsilon > 0$ let $h_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$h_\varepsilon(x) := \begin{cases} \zeta(\varepsilon^{-1}\mathrm{dist}\,(x, \partial U)), & x \in U, \\ 0, & x \in \mathbb{R}^N \setminus U. \end{cases}$$

Observe that h_ε is a Lipschitz function satisfying $h_\varepsilon(x) = 1$, if $x \in U_\varepsilon := \{x \in U : \mathrm{dist}\,(x, \partial U) \ge \varepsilon\}$. Then the function $h_\varepsilon g$ is a Lipschitz function which coincides with g on U_ε and

$$\overline{\nabla(h_\varepsilon g) \cdot F} = h_\varepsilon \overline{\nabla g \cdot F} + g \overline{\nabla h_\varepsilon \cdot F}.$$

By what we have already proved, we have

$$\int_U h_\varepsilon \overline{\nabla g \cdot F} + \int_U g \overline{\nabla h_\varepsilon \cdot F} + \int_U h_\varepsilon g \,\mathrm{div}\, F = 0. \tag{13}$$

Now, we have

$$\int_U g \overline{\nabla h_\varepsilon \cdot F} = \int_{U \setminus U_{2\varepsilon}} g \overline{\nabla h_\varepsilon \cdot F},$$

since $\nabla h_\varepsilon \equiv 0$ in U_ε. Also, $|\nabla h_\varepsilon| \le 2\varepsilon^{-1}$, and $|g(x)| \le 2\mathrm{Lip}(g)\varepsilon$, for $x \in U \setminus U_{2\varepsilon}$. Therefore,

$$\lim_{\varepsilon \to 0} \int_U g \overline{\nabla h_\varepsilon \cdot F} = \lim_{\varepsilon \to 0} \int_{U \setminus U_{2\varepsilon}} g \overline{\nabla h_\varepsilon \cdot F} = 0,$$

by dominated convergence. Hence, letting $\varepsilon \to 0$ in (13), since $h_\varepsilon \to 1$, as $\varepsilon \to 0$, everywhere in U, we finally get (11).

The assertion just proved shows that the right-hand side of (9) depends only on $g|\partial U$. Also, the inequality (10) is clear from (9), in the case where $h = H|\partial U$, where $H \in \mathrm{Lip}(\mathbb{R}^N)$, and

$$|H|_{\mathrm{Lip}(\mathbb{R}^N)} = |h|_{\mathrm{Lip}(\partial U)}.$$

Now, Kirszbraun's Theorem (see, e.g., [7, 8]) guarantees, for any $h \in \text{Lip}(\partial U)$, the existence of $H \in \text{Lip}(\mathbb{R}^N)$ such that $H|\partial U = h$ and $\text{Lip}_{\mathbb{R}^N}(H) = \text{Lip}_{\partial U}(h)$. Moreover, a trivial cut-off procedure ensures that $\|H\|_{L^\infty(\mathbb{R}^N)} = \|h\|_{L^\infty(\partial U)}$; this completes the proof. □

We now discuss a direct way of defining the normal trace functional $F \cdot \nu : \text{Lip}(\partial U) \to \mathbb{R}$. The formula was first obtained in [3, 4], under regularity restrictions on the boundary, and in [11], for general boundaries. Before stating the corresponding result, we recall the following lemma, which is a slight modification of Lemma 3.3 of [11].

Lemma 1 (Silhavý [11]). *If $F \in \mathcal{DM}^{ext}(U)$, $m \in Lip(U)$, $t \in \mathbb{R}$ and if $T \subset m^{-1}(t)$ is a compact subset of U, then the restriction $\overline{\nabla m \cdot F}\lfloor T$ of $\overline{\nabla m \cdot F}$ to T satisfies*

$$\overline{\nabla m \cdot F}\lfloor T = 0. \tag{14}$$

Proof. Clearly, we can take $t = 0$. Also, multiplying m by a suitable function in $C_0^\infty(U)$, if necessary, we can assume that m has compact support in U. Therefore, we can assume that m is a Lipshitz function vanishing on $\mathbb{R}^N \setminus W$, with $W = U \setminus T$, and, in particular, also on $\mathbb{R}^N \setminus U$. Therefore, for any $\eta \in C_0^\infty(U)$, we have

$$\int_W \overline{\nabla(\eta m) \cdot F} + \int_W \eta m \,\text{div}\, F = 0, \tag{15}$$

$$\int_U \overline{\nabla(\eta m) \cdot F} + \int_U \eta m \,\text{div}\, F = 0. \tag{16}$$

Subtracting (15) from (16), since ηm vanishes on T, we get

$$0 = \int_T \overline{\nabla(\eta m) \cdot F} = \int_T \eta \overline{\nabla m \cdot F} + \int_T m \overline{\nabla \eta \cdot F} = \int_T \eta \overline{\nabla m \cdot F},$$

and so, since η is arbitrary, we arrive at (14). □

The following result gives a simple formula to compute the normal trace of $\mathcal{DM}$-fields. This formula, displayed in (i) of the statement below, was first obtained in [3,4] under some regularity restrictions on the boundary, and later was extended to general domains in [11]. Item (ii) gives a useful necessary condition for the normal trace to be a measure over ∂U established by Silhavý [11].

Theorem 4. *Let $F \in \mathcal{DM}^{ext}(U)$ and $m : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative Lipschitz function with $\text{supp}\, m \subset \bar{U}$ which is strictly positive on U, and for each $\varepsilon > 0$ let $L_\varepsilon = \{x \in U : 0 < m(x) < \varepsilon\}$. Then:*

(i) (*cf.* [3, 4] and [11]) *If $g \in Lip(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we have*

$$F \cdot \nu(g|\partial U) = -\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{L_\varepsilon} g \, d\,(\overline{\nabla m \cdot F}). \tag{17}$$

(ii) (*cf.* [11]) *If*

$$\liminf_{\varepsilon\to 0} \varepsilon^{-1}|\overline{\nabla m\cdot F}|(L_\varepsilon) < \infty, \tag{18}$$

then $F\cdot\nu$ is a measure over ∂U.

Proof. We repeat the proof given in [11].

(i) For each $\varepsilon > 0$ we define $m_\varepsilon(x) = \varepsilon^{-1}\min\{m(x),\varepsilon\}$. We see that m_ε is a Lipschitz function vanishing on ∂U. We have that $gm_\varepsilon \in \mathrm{Lip}(\mathbb{R}^N)$ and

$$\overline{\nabla(gm_\varepsilon)\cdot F} = m_\varepsilon\overline{\nabla g\cdot F} + g\overline{\nabla m_\varepsilon\cdot F},$$

by the properties of the pairing $\overline{\nabla g\cdot F}$. Since gm_ε vanishes on ∂U, we have

$$\int_U m_\varepsilon\, d(\overline{\nabla g\cdot F}) + \int_U g\, d(\overline{\nabla m_\varepsilon\cdot F}) + \int_U gm_\varepsilon \operatorname{div} F = 0. \tag{19}$$

Now, $m_\varepsilon(x) \to 1$ everywhere in U, so that dominated convergence implies

$$\int_U m_\varepsilon\, d(\overline{\nabla g\cdot F}) \to \int_U d(\overline{\nabla g\cdot F})$$

and $\int_U gm_\varepsilon \operatorname{div} F \to \int_U g \operatorname{div} F$. On the other hand, we have $m_\varepsilon = \varepsilon^{-1}m$ in L_ε, so $\nabla m_\varepsilon = \varepsilon^{-1}\nabla m$, a.e. in L_ε, which gives $\overline{\nabla m_\varepsilon\cdot F} = \varepsilon^{-1}\overline{\nabla m\cdot F}$, over L_ε. Moreover, since $U\setminus L_\varepsilon = m_\varepsilon^{-1}(1)$, by Lemma 1, we have $\overline{\nabla m_\varepsilon\cdot F} = 0$ on $U\setminus L_\varepsilon$. Hence, we obtain (17) from (19) when $\varepsilon\to 0$, by the definition of $F\cdot\nu(g|\partial U)$ in (9).

(ii) By (18), we have $|\overline{\nabla m\cdot F}|(L_\varepsilon) \le C\varepsilon$, for some $C > 0$ independent of ε, at least for a subsequence of $\varepsilon\to 0$, so that

$$\left|\varepsilon^{-1}\int_{L_\varepsilon} g\, d(\overline{\nabla m\cdot F})\right| \le C\|g\|_{L^\infty(\mathbb{R}^N)},$$

for each $g\in \mathrm{Lip}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Therefore, given $g\in\mathrm{Lip}(\partial U)$, we may extend g to a Lipschitz function on $\mathbb{R}^N$ so that $\|g|\partial U\|_{L^\infty(\partial U)} = \|g\|_{L^\infty(\mathbb{R}^N)}$, and so, by (17), we deduce that $|F\cdot\nu(g)| \le C\|g\|_{L^\infty(\partial U)}$, which implies, by the Riesz representation theorem, that $F\cdot\nu$ is a measure on ∂U, as asserted. □

Remark 1. A typical example of m in the statement of Theorem 4 is provided by $m(x) = \operatorname{dist}(x,\partial U)$, for $x\in U$, and $m(x) = 0$, for $x\in\mathbb{R}^N\setminus U$.

Remark 2. The following interesting example from [11] shows cases where $F\cdot\nu$ is a measure over ∂U and cases where $F\cdot\nu$ fails to be a measure. Namely, for $1\le\alpha<3$, let $F:\mathbb{R}^2\setminus\{0\}\to\mathbb{R}^2$ be defined by

$$F(x) = \frac{1}{|x|^\alpha}(x_2, -x_1),$$

and let $U = \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, |x| < 1,\ x_2 < 0\}$. Clearly, $\operatorname{div} F = 0$, in $\mathbb{R}^2 \setminus \{0\}$, and we easily verify that

$$F \in \mathcal{DM}^p(U; \mathbb{R}^2) \quad \text{with } 1 \le p < 2/(\alpha - 1), \text{ for } 1 < \alpha < 3, \text{ and } p = \infty, \text{ for } \alpha = 1.$$

Now, if $g \in \operatorname{Lip}(\partial U)$ and $\operatorname{supp} g \subset \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 = 0,\ |x_1| < 1\}$, we may use (18) with m satisfying $m(x) = -x_2$, for $-\varepsilon_0 < x_2 \le 0$, and $|x_1| < 1 - \varepsilon_0$, with $\varepsilon_0 > 0$ small enough so that $g(x) = 0$, if $|x_1| \ge 1 - \varepsilon_0$. Also, we may consider an extension of g to $\mathbb{R}^2$ such that $g(x_1, x_2) = g(x_1, 0)$, for $|x_2| < \varepsilon_0$. Applying (18) with m and the extension of g so defined, we get

$$F \cdot \nu(g) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{-1}^1 g(t, s) \frac{t}{(t^2 + s^2)^{\frac{\alpha}{2}}}\, dt\, ds,$$

which gives

$$F \cdot \nu(g) = \int_{-1}^1 g(t, 0) \operatorname{sgn}(t) |t|^{1-\alpha}\, dt, \qquad \text{for } 1 \le \alpha < 2,$$

and

$$F \cdot \nu(g) = \lim_{\varepsilon \to 0} \int_{|t| > \varepsilon} g(t, 0) \operatorname{sgn}(t) |t|^{1-\alpha}\, dt, \qquad \text{for } 2 \le \alpha < 3.$$

This shows that, for $1 \le \alpha < 2$, $F \cdot \nu$ is a measure, while, for $2 \le \alpha < 3$, $F \cdot \nu$ is not a measure on ∂U.

Remark 3. For $\varepsilon_0 > 0$ sufficiently small and $0 < s < \varepsilon_0$, we may consider the open set $U_s := \{x \in U \,:\, m(x) > s\}$, for m as in Theorem 4. By Theorem 4, for the normal trace $F \cdot \nu | \partial U_s$, we have the following formula similar to (17),

$$F \cdot \nu(g | \partial U_s) = -\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{\{s < m(x) < s + \varepsilon\}} g\, d\, (\overline{\nabla m \cdot F}), \tag{20}$$

and, again, we have that the condition

$$\liminf_{\varepsilon \to 0} \varepsilon^{-1} |\overline{\nabla m \cdot F}|(\{s < m(x) < s + \varepsilon\}) < \infty, \tag{21}$$

implies that $F \cdot \nu | \partial U_s$ is a measure on ∂U_s. If we consider the monotone function $V(s) = |\overline{\nabla m \cdot F}|(\{0 < m(x) < s\})$, for $s \in (0, \varepsilon_0)$, we see that the left-hand side of (21) is the right-derivative of V at s, except possibly for a countable subset of $(0, \varepsilon_0)$, and we know that it exists for a.e. $s \in (0, \varepsilon_0)$. Therefore, $F \cdot \nu | \partial U_s$ is a measure for a.e. $s \in (0, \varepsilon_0)$, and in this sense we may assert that "for almost all boundaries" ∂U the normal trace $F \cdot \nu|\partial U$ is a measure.

3 Domains with a Lipschitz Deformable Boundary

We now enter into the main subject of the present paper, beginning with the definition of a deformable Lipschitz boundary.

Definition 2. Let $\Omega \subset \mathbb{R}^N$ be an open set. We say that $\partial\Omega$ is a *deformable Lipschitz boundary* if the following hold:

(i) For each $x \in \partial\Omega$, there exist an $r > 0$ and a Lipschitz mapping $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, upon rotating and relabeling the coordinate axis if necessary,

$$\Omega \cap Q(x,r) = \{\, y \in \mathbb{R}^N \ : \ \gamma(y_1, \cdots, y_{N-1}) < y_N \,\} \cap Q(x,r),$$

where $Q(x,r) = \{\, y \in \mathbb{R}^N \ : \ |y_i - x_i| \le r,\ i = 1, \cdots, N \,\}$. We denote by $\tilde{\gamma}$ the map $\tilde{y} \mapsto (\tilde{y}, \gamma(\tilde{y}))$, $\tilde{y} = (y_1, \cdots, y_{N-1})$.

(ii) There exists a map $\Psi : \partial\Omega \times [0,1] \to \bar{\Omega}$ such that Ψ is a bi-Lipschitz homeomorphism over its image and $\Psi(x,0) = x$, for all $x \in \partial\Omega$. For $s \in [0,1]$, we denote by Ψ_s the mapping from $\partial\Omega$ to $\bar{\Omega}$ given by $\Psi_s(x) = \Psi(x,s)$, and set $\partial\Omega_s := \Psi_s(\partial\Omega)$.

We say that the Lipschitz deformation $\Psi : \partial\Omega \times [0,1] \to \bar{\Omega}$ is *regular*, and that Ω has a *regular Lipschitz deformable boundary*, if, besides (i) and (ii), we have

(iii) $J[\nabla\Psi_s \circ \tilde{\gamma}] \rightharpoonup J[\nabla\tilde{\gamma}]$, as $s \to 0$, in the weak star topology of $L^\infty(B)$ for any bounded open set $B \subset \mathbb{R}^{N-1}$ such that $\tilde{\gamma}(B) \subset \partial\Omega$, with $\tilde{\gamma}$ as in (i), where $J[\nabla g]$ denotes the Jacobian of ∇g (see, e.g., [7]).

Remark 4. In [2] the additional condition (iii) for defining a regular Lipschitz deformation was stated in a slightly stronger way, asking that $\nabla\Psi_s \circ \tilde{\gamma} \to \nabla\tilde{\gamma}$, as $s \to 0$, in $L^1(B)$. Nevertheless, the weak convergence of the Jacobian is already enough to guarantee the validity of the formula

$$F \cdot \nu\big|_{\partial\Omega} = \text{ess.}\lim (F \cdot \nu_s) \circ \Psi_s, \quad \text{in the weak star topology of } L^\infty(\partial\Omega, \mathcal{H}^{N-1}), \tag{22}$$

which holds for $\mathcal{DM}^\infty$-fields, as established in [2].

Actually, condition (iii) is equivalent to $J[d\Psi_s] := \det(d\Psi_s^* \, d\Psi_s)^{1/2} \rightharpoonup 1$ in the weak star topology of $L^\infty(\partial\Omega)$, where, for each $\omega \in \partial\Omega$, $d\Psi_s(\omega) : T_\omega(\partial\Omega) \to \mathbb{R}^N$ is the differential mapping of Ψ_s at $\omega \in \partial\Omega$ and $d\Psi_s^*(\omega) : \mathbb{R}^N \to T_\omega(\partial\Omega)$ denotes the adjoint mapping. This follows from the Cauchy-Binet formula for the Jacobian (see, e.g., [7]).

We start our discussion by introducing the *level set function* $h : \mathbb{R}^N \to \mathbb{R}$, defined by

$$h(x) = \begin{cases} 0, & \text{for } x \in \mathbb{R}^N \setminus \bar{\Omega}, \\ s, & \text{for } x \in \partial\Omega_s, \\ 1, & \text{for } x \in \Omega \setminus \Psi(\partial\Omega \times [0,1]). \end{cases}$$

By formula (17) we have

$$F \cdot \nu(g|\partial U) = -\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{L_\varepsilon} g\, d\,(\overline{\nabla h \cdot F}), \tag{23}$$

for any $F \in \mathcal{DM}^{ext}(\Omega)$, and any $g \in \mathrm{Lip}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, with $L_\varepsilon = \{x \in \Omega : 0 < h(x) < \varepsilon\}$.

Remark 5. The following standard example of a domain with a regular deformable Lipschitz boundary shows that, for the sake of studying local properties of the normal trace operator, any domain with a Lipschitz boundary may be viewed as a domain with a regular deformable Lipschitz boundary. So, let

$$U := \{\, x \in \mathbb{R}^N \,:\, \gamma(x_1, \cdots, x_{N-1}) < x_N \,\}, \tag{24}$$

where $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ is a Lipschitz function. U is then an unbounded open set, ∂U is the graph of γ, $\partial U = \{(\tilde{x}, x_N) \in \mathbb{R}^N : \tilde{x} \in \mathbb{R}^{N-1},\ x_N = \gamma(\tilde{x})\}$, and it is very easy to define a regular Lipschitz deformation for ∂U by simply setting $\Psi((\tilde{x}, \gamma(\tilde{x})), s) = (\tilde{x}, \gamma(\tilde{x}) + s\delta)$, $\tilde{x} \in \mathbb{R}^{N-1}$, $s \in [0,1]$, where $\delta > 0$ is arbitrary. It turns out that, by property (i) in Definition 2, for test functions g, as in (9), with support contained in a sufficiently small neighborhood, say, a neighborhood like those appearing in Definition 2(i), the normal trace operator given by (9) may be defined using (23) where h is the level set function associated to this trivial standard deformation. More specifically, in this case, the level set function is simply defined by

$$h(x) = \begin{cases} 0, & \text{if } x_N < \gamma(\tilde{x}), \\ s, & \text{if } x_N = \gamma(\tilde{x}) + s\delta, \text{ for } s \in [0,1], \\ 1, & \text{if } x_N \geq \gamma(\tilde{x}) + \delta. \end{cases}$$

Therefore, considered as distributions in $\mathbb{R}^N$ with support contained in $\partial\Omega$, the normal trace operators associated to $\mathcal{DM}^{ext}$-fields can always be split in a countable sum of distributions, whose supports possess the finite intersection property, each of which may be defined like the normal trace operator for a standard domain as just described. Indeed, it suffices to employ a partition of unity subordinate to a suitable covering of $\partial\Omega$.

The above remark is important in connection, for instance, with the theory of hyperbolic systems of conservation laws (see, e.g., [6]). Namely, if $\mathbb{R}^N$ is the space-time space $\mathbb{R}^{n+1}$, so $N = n + 1$, with points denoted (x, t), suppose $F(x, t) = (\eta(u(x, t)), q(u(x, t)))$ where $\eta : \mathbb{R}^m \to \mathbb{R}$ and $q : \mathbb{R}^m \to \mathbb{R}^n$ form an entropy-entropy flux pair for a hyperbolic system of conservation laws, and $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}^m$ is a weak entropy solution for this system, and let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. It is an important question to determine whether there is a measurable function $u_\tau : \partial\Omega \to \mathbb{R}^m$ such that the normal trace operator may be represented by $(\eta(u_\tau(\omega)), q(u_\tau(\omega))) \cdot \nu(\omega)$, where $\nu(\omega)$ is the outer unit normal vector at $\omega \in \partial\Omega$. Through the splitting of the normal trace operator mentioned in the above remark, this question, for a general domain with Lipschitz boundary, may be reduced to the corresponding one for a hyper-graph domain as U in (24).

For simplicity, in what follows, we will always assume that Ω is a bounded open set with a regular deformable Lipschitz boundary. We emphasize that, for the purpose of getting local information about the normal trace operator, as has been already mentioned, this assumption does not represent any additional restriction beyond that of possessing a Lipschitz boundary. The fact that Ω is bounded allows us to restrict our discussion to just two cases, namely, that for fields in $\mathcal{DM}^1(\Omega)$ and that for fields in $\mathcal{DM}^{ext}(\Omega)$, since the boundedess of Ω implies $\mathcal{DM}^p(\Omega) \subset \mathcal{DM}^1(\Omega)$, for all $1 < p \le \infty$. Let us then focus our attention in these two cases.

Theorem 5. *Let Ω be a bounded open set with a deformable Lipschitz boundary and $F \in \mathcal{DM}^1(\Omega)$. Let $\Psi : \partial\Omega \times [0, 1] \to \bar{\Omega}$ be a Lipschitz deformation of $\partial\Omega$. Then, for almost all $s \in [0, 1]$, and all $\phi \in C_0^\infty(\mathbb{R}^N)$,*

$$\int_{\Omega_s} \phi \, div \, F = \int_{\partial\Omega_s} \phi(\omega) F(\omega) \cdot \nu_s(\omega) \, d\mathcal{H}^{N-1}(\omega) - \int_{\Omega_s} F(x) \cdot \nabla\phi(x) \, dx, \tag{25}$$

where ν_s is the unit outward normal field defined $\mathcal{H}^{N-1}$-almost everywhere in $\partial\Omega_s$, and Ω_s is the open subset of Ω bounded by $\partial\Omega_s$.

Proof. For $\phi \in C_0^\infty(\mathbb{R}^N)$, let

$$\zeta_\phi(s) = \int_{\partial\Omega_s} \phi(\omega) F(\omega) \cdot \nu_s(\omega) \, d\mathcal{H}^{N-1}(\omega), \quad s \in [0, 1],$$

where ν_s is as in the statement. Let $s_0 \in [0, 1]$ be a Lebesgue point for ζ_ϕ, for ϕ in a countable dense set in $C_0^\infty(\mathbb{R}^N)$. For $\delta > 0$ sufficiently small, let $g_\delta : \mathbb{R} \to \mathbb{R}$ be defined as

$$g_\delta(s) = \begin{cases} 0, & s < s_0 - \delta, \\ \frac{s - s_0 + \delta}{2\delta}, & s_0 - \delta \le s \le s_0 + \delta, \\ 1, & s > s_0 + \delta. \end{cases}$$

Set $\psi_\delta = g_\delta \circ h \, \phi$, where h is the level set function associated to the Lipschitz deformation Ψ. By the Gauss-Green formula, we have

$$
\begin{aligned}
0 &= \int_{\Omega} F \cdot \nabla \psi_\delta \, dx + \int_{\Omega} \psi_\delta \operatorname{div} F \\
&= \int_{\Omega} \phi g_\delta'(h(x)) F \cdot \nabla h \, dx + \int_{\Omega} g_\delta(h(x)) F \cdot \nabla \phi \, dx + \int_{\Omega} \psi_\delta \operatorname{div} F,
\end{aligned}
$$

which gives, by the coarea formula,

$$
0 = -\frac{1}{2\delta} \int_{s_0-\delta}^{s_0+\delta} \int_{\partial\Omega_s} \phi F \cdot \nu_s \, d\mathcal{H}^{N-1}(\omega) \, ds + \int_{\Omega} g_\delta(h(x)) F \cdot \nabla \phi \, dx + \int_{\Omega} \psi_\delta \operatorname{div} F.
$$

Letting $\delta \to 0$, we obtain (25) for $s = s_0$, where s_0 is an arbitrary Lebesgue point of ζ_ϕ, for ϕ in a countable dense subset of $C_0^\infty(\mathbb{R}^N)$, and, so, (25) holds for almost all $s \in [0, 1]$ as was to be proved. □

From Theorem 5 and the Gauss-Green formula (9), when $F \in \mathcal{DM}^1(\Omega)$, it follows that, for any $g \in \mathrm{Lip}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we have the following formula for the normal trace functional $F \cdot \nu : \mathrm{Lip}(\partial\Omega) \to \mathbb{R}$,

$$
\langle F \cdot \nu, g|\partial\Omega \rangle = \operatorname{ess.}\lim_{s \to 0} \int_{\partial\Omega_s} g F(\omega) \cdot \nu(\omega) \, d\mathcal{H}^{N-1}(\omega), \tag{26}
$$

where the limit on the right-hand side exists by applying dominated convergence to the other two terms in (25). Therefore, for any $\phi \in \mathrm{Lip}(\partial\Omega)$, we have

$$
\langle F \cdot \nu, \phi \rangle = \operatorname{ess.}\lim_{s \to 0} \int_{\partial\Omega_s} \phi \circ \Psi_s^{-1}(\omega) F(\omega) \cdot \nu_s(\omega) \, d\mathcal{H}^{N-1}(\omega),
$$

or, by using the area formula,

$$
\begin{aligned}
\langle F \cdot \nu, \phi \rangle &= \operatorname{ess.}\lim_{s \to 0} \int_{\partial\Omega} \phi(\omega) F \circ \Psi_s(\omega) \cdot \nu_s(\Psi_s(\omega)) J[\Psi_s] \, d\mathcal{H}^{N-1}(\omega) \\
&= \operatorname{ess.}\lim_{s \to 0} \int_{\partial\Omega} \phi(\omega) F \circ \Psi_s(\omega) \cdot \nu(\Psi_s(\omega)) \, d\mathcal{H}^{N-1}(\omega),
\end{aligned}
$$

where we have used the fact that Ψ is a regular Lipschitz deformation. Therefore we have proved the following formula for the normal trace for a $\mathcal{DM}^1$-field.

Theorem 6. *Let $F \in \mathcal{DM}^1(\Omega)$, where Ω is a bounded open set with a Lipschitz boundary admitting a regular deformation $\Psi : \partial\Omega \times [0, 1] \to \bar{\Omega}$. Denoting by $F \cdot \nu|_{\partial\Omega}$ the continuous linear functional Lip$(\partial\Omega) \to \mathbb{R}$ given by the normal trace of F at $\partial\Omega$, we have the formula*

$$
F \cdot \nu|_{\partial\Omega} = \operatorname{ess.}\lim_{s \to 0} F \circ \Psi_s(\cdot) \cdot \nu_s(\Psi_s(\cdot)), \tag{27}
$$

with equality in the sense of $(Lip(\partial\Omega))^*$, *where on the right-hand side the functionals are given by ordinary functions in* $L^1(\partial\Omega)$.

We now turn to the case where $F \in \mathcal{DM}^{ext}(\Omega)$. Let us again consider the level set function h associated to the regular Lipschitz deformation $\Psi : \partial\Omega \times [0,1] \to \bar{\Omega}$. Let us consider the measure μ over $\Psi(\partial\Omega \times [0,1])$ given by

$$\mu := |\overline{\nabla h \cdot F}| \lfloor \Psi(\partial\Omega \times [0,1]).$$

We consider the pull back of μ by Ψ, $\Psi^\sharp\mu$, which is the measure on $\partial\Omega \times [0,1]$ defined by

$$\langle \Psi^\sharp\mu, \varphi\rangle = \langle \mu, \varphi \circ \Psi^{-1}\rangle, \qquad \forall\varphi \in C(\partial\Omega \times [0,1]).$$

We may apply the disintegration process to $\Psi^\sharp\mu$ (see, e.g., Theorem 2.28, p. 57 in [1]) to write $\Psi^\sharp\mu = \sigma \otimes \tilde{\mu}_s$, for the Radon measure σ on $[0,1]$ given by the projection of $\Psi^\sharp\mu$ onto $[0,1]$, and so $\sigma(E) = \Psi^\sharp\mu(\partial\Omega \times E)$ for any Borel set $E \subset [0,1]$, and Radon measures $\tilde{\mu}_s$ such that $s \mapsto \tilde{\mu}_s$ is σ-measurable, $\tilde{\mu}_s(\partial\Omega) = 1$, σ-a.e. in $[0,1]$, so that we have

$$\int_{\partial\Omega\times[0,1]} \varphi(\omega, s)\, d\Psi^\sharp\mu = \int_{[0,1]} \left(\int_{\partial\Omega} \varphi(\omega, s)\, d\tilde{\mu}_s(\omega)\right) d\sigma(s),\ \forall\varphi \in C(\partial\Omega\times[0,1]). \tag{28}$$

Therefore, by pushing forward the equation $\Psi^\sharp\mu = \sigma \otimes \tilde{\mu}_s$ by Ψ, we obtain

$$\mu = \sigma \otimes \mu_s, \quad \mu_s := (\Psi_s)_\sharp\tilde{\mu}_s,$$

where, for any $\zeta \in C(\partial\Omega_s)$,

$$\langle (\Psi_s)_\sharp\tilde{\mu}_s, \zeta\rangle = \langle \tilde{\mu}_s, \zeta \circ \Psi_s\rangle.$$

In particular, for any $\phi \in C_0^\infty(\mathbb{R}^N)$, with $\operatorname{supp}\phi \cap \Omega \subset \Psi(\partial\Omega \times [0,1])$, we have

$$\int_\Omega \phi(x)\, d(\overline{\nabla h \cdot F}) = \int_{[0,1]} \left(\int_{\partial\Omega_s} \phi(x)\theta(x)\, d\mu_s\right) d\sigma(s), \tag{29}$$

where θ is the μ-measurable function, with $|\theta| = 1$, μ-a.e., such that

$$\theta\mu = \overline{\nabla h \cdot F} \lfloor \Psi(\partial\Omega \times [0,1]).$$

Now, we have the decomposition $\sigma = H(s)\, ds + \sigma_{\text{sing}}$, for some non-negative $H \in L^1([0,1])$, and $\sigma_{\text{sing}} = \sigma\lfloor\mathcal{N}$, for some Borel set $\mathcal{N} \subset [0,1]$ of one-dimensional Lebesgue measure zero, by the Lebesgue decomposition theorem (see, e.g., [7], p. 42). We then define

$$(\overline{\nabla h \cdot F})_s := \theta\, H(s)\, \mu_s. \tag{30}$$

We have the following analogue of Theorem 5 when $F \in \mathcal{DM}^{ext}(\Omega)$.

Theorem 7. *Let Ω be a bounded open set with a deformable Lipschitz boundary and $F \in \mathcal{DM}^{ext}(\Omega)$. Let $\Psi : \partial\Omega \times [0,1] \to \bar{\Omega}$ be a Lipschitz deformation of $\partial\Omega$. Then, for almost all $s \in [0,1]$, and all $\phi \in C_0^\infty(\mathbb{R}^N)$,*

$$\int_{\Omega_s} \phi\, div\, F = \int_{\partial\Omega_s} \phi(\omega)\, d(\overline{\nabla h \cdot F})_s - \int_{\Omega_s} \nabla\phi(x) \cdot F. \tag{31}$$

Proof. The proof is nearly identical to that of Theorem 5, the only difference being that now we must choose $s_0 \in [0,1]\backslash\mathcal{N}$, with $\mathcal{N}$ as above, such that s_0 is a Lebesgue point of

$$\zeta_\phi(s) := \int_{\partial\Omega_s} \phi\, d(\overline{\nabla h \cdot F})_s$$

for ϕ in a countable dense set in $C_0^\infty(\mathbb{R}^N)$, and we take g_δ only for small $\delta > 0$ such that $|(\overline{\nabla h \cdot F})|(\partial\Omega_{s_0 \pm \delta}) = 0$. □

Similarly to what was done for $\mathcal{DM}^1$-fields, from Theorem 7 we get the following result.

Theorem 8. *Let $F \in \mathcal{DM}^{ext}(\Omega)$, where Ω is a bounded open set with a Lipschitz boundary admitting a regular deformation $\Psi : \partial\Omega \times [0,1] \to \bar{\Omega}$. Denoting by $F \cdot \nu|_{\partial\Omega}$ the continuous linear functional $Lip(\partial\Omega) \to \mathbb{R}$ given by the normal trace of F at $\partial\Omega$, we have the formula*

$$F \cdot \nu|_{\partial\Omega} = \text{ess.}\lim_{s\to 0} \Psi_s^\sharp\, d(\overline{\nabla h \cdot F})_s, \tag{32}$$

with equality in the sense of $(Lip(\partial\Omega))^*$, where on the right-hand side the functionals are given by the pull back by Ψ_s of the measures $d(\overline{\nabla h \cdot F})_s$, resulting from the disintegration of $d(\overline{\nabla h \cdot F})$.

4 Application to Time-Regularity of Entropy Solutions to Hyperbolic Conservation Laws

Let $n, d \in \mathbb{N}$, $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times (0,\infty)$, and $\mathcal{U} \subset \mathbb{R}^n$ be an open and convex set. We consider the N-dimensional system of n conservation laws

$$\partial_t U + \sum_{\alpha=1}^{d} \partial_\alpha F^\alpha(U) = 0, \qquad \text{in } \mathbb{R}_+^{N+1}, \tag{33}$$

with $U(x,t) \in \mathcal{U}$ and $F^\alpha : \mathcal{U} \to \mathbb{R}^n$, where ∂_α denotes the partial derivative with respect to x_α.

Together with (33), we consider the initial data

$$U(x,0) = U_0(x). \tag{34}$$

The following result provides time-regularity information about entropy solutions of the problem (33) and (34). It extends a result established in [6] (Theorem 4.5.1), which follows from the theory for L^∞ divergence-measure fields.

Theorem 9. *Let $U_0 \in L^1_{loc}(\mathbb{R}^d)$, and let $U \in L^1_{loc}(\mathbb{R}^d \times [0,\infty))$ be a weak solution of (33) and (34), in the sense that, for any $\phi \in C_c^\infty(\mathbb{R}^{d+1})$, we have*

$$\int_{\mathbb{R}^{d+1}_+} U(x,t)\partial_t\phi + \sum_{\alpha=1}^d F^\alpha(U)\partial_\alpha\phi \, dx\,dt + \int_{\mathbb{R}^d} U_0(x)\phi(x,0)\,dx = 0. \tag{35}$$

Let $\eta : \mathcal{U} \to [0,\infty)$ be a strictly convex function, with $\eta(U) \geq c_1|U| + c_2$, for some $c_1 > 0$, $c_2 \in \mathbb{R}$, such that $\eta(U(x,t)) \in L^p(K \cap \mathbb{R}^{d+1}_+)$, for any compact set $K \subset \mathbb{R}^{d+1}$, for some $p > 1$. Suppose that there exists a vector measure $Q \in \mathcal{M}(K \cap \mathbb{R}^{d+1}_+; \mathbb{R}^d)$, for any compact set $K \subset \mathbb{R}^{d+1}$, such that $\eta(U(x,t))$ satisfies

$$\partial_t \eta(U) + div_x Q \leq 0, \qquad in\ \mathbb{R}^{d+1}_+, \tag{36}$$

in the sense of distributions, where $\mathcal{M}(\Omega;\mathbb{R}^d)$ denotes the $\mathbb{R}^d$-valued Radon measures with finite total variation on Ω. Then,

$$U \in C((0,\infty)\setminus S; L^1_{loc}(\mathbb{R}^d)), \tag{37}$$

for some at most countable set $S \subset (0,\infty)$. Moreover, if we have, for all nonnegative $\psi \in C_c^\infty(\mathbb{R}^{d+1})$,

$$\int_{\mathbb{R}^{d+1}_+} \{\eta(U(x,t))\,\partial_t\psi\,dx\,dt + \nabla_x\psi \cdot dQ\} + \int_{\mathbb{R}^d} \eta(U_0(x))\psi(x,0)\,dx \geq 0, \tag{38}$$

then the above strong continuity holds on the right for $t = 0$.

Proof. The result follows by applying Theorem 8 to the domains $\Omega[t_0+] := \{(x,t) : t > t_0\}$, $t_0 > 0$, with regular Lipschitz deformation $\Omega[t_0+]_s = \Omega[(t_0+s)+]$, $s \in [0,1]$, and $\Omega[t_0-] := \{(x,t) : -\infty < t < t_0\}$, $t_0 > 0$, with regular Lipschitz deformation $\Omega[t_0-]_s = \Omega[(t_0-s)-]$, $s \in [0,1]$. Here, for simplicity, we may view $U(x,t)$ as extended to $t<0$ as 0, as well as $\eta(U(x,t))$, as $\eta(0)$, and Q as the null measure, for $t<0$. We then obtain that, for a.e. $t_0 > 0$,

$$\begin{cases} \int_{\mathbb{R}^d} U(x,t_0)\,\phi(x)\,dx = \text{ess.}\lim_{t\to t_0}\int_{\mathbb{R}^d} U(x,t)\phi(x)\,dx, \\ \int_{\mathbb{R}^d} \eta(U(x,t_0))\phi(x)\,dx = \text{ess.}\lim_{t\to t_0}\int_{\mathbb{R}^d}\eta(U(x,t))\phi(x)\,dx, \end{cases} \tag{39}$$

for all $\phi \in C_0^\infty(\mathbb{R}^d)$. Now, using (37), for almost all $0 < \delta < s < t < T$, and $R > 0$, there exists an $A(R, \delta, T) > 0$ such that

$$\int_{|x|<R} \eta(U(x,t))\,dx \le \int_{|x|<R} \eta(U(x,s))\,dx + A(R,\delta,T). \tag{40}$$

This gives that, for any $\delta > 0$, $\int_{|x|<R} \eta(U(x,t))\,dx$ is uniformly bounded for $\delta < t < T$, for almost all $R > 0$. Using the assumptions on η, we conclude that $\int_{|x|<R} |U(x,t)|\,dx$ is also uniformly bounded for $t > \delta$, for almost all $R > 0$. Hence, we may take $\phi \in L^{p'}(\mathbb{R}^d)$, with compact support, in (41), with $p' = p/(p-1)$. Now, since η is strictly convex, we conclude the proof in a standard way. □

As an example, in [5], Chen and Perepelitsa prove the convergence of the solutions $(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)$ to the Cauchy problem for the Navier-Stokes equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + \kappa\rho^\gamma)_x = \varepsilon u_{xx}, \end{cases} \tag{41}$$

with initial data

$$\rho(x,0) = \rho_0(x), \qquad u(x,0) = u_0(x), \tag{42}$$

where $\gamma > 1$ and $\kappa > 0$, by a scaling defined by $\kappa = (\gamma-1)^2/4\gamma$. Using energy estimates and compensated compactness with Young measures with unbounded support, they prove the convergence in $L^1(K \cap \mathbb{R}^2_+)$ of $(\rho^\varepsilon, m^\varepsilon)$, with $m^\varepsilon = \rho^\varepsilon u^\varepsilon$, to some $(\rho(x,t), m(x,t)) \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$, and also the convergence in $L^1(K \cap \mathbb{R}^2_+)$ of $\eta^*(\rho^\varepsilon(x,t), m^\varepsilon(x,t))$ to $\eta^*(\rho(x,t), m(x,t))$, where

$$\eta^*(\rho,m) = \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho), \qquad e(\rho) = \frac{\kappa}{\gamma-1}\rho^{\gamma-1},$$

for any compact K. Nevertheless, passing to the limit in the inequality

$$\partial_t \eta^*(\rho^\varepsilon, m^\varepsilon) + \partial_x q^*(\rho^\varepsilon, m^\varepsilon) \le \frac{\varepsilon}{2}\partial_x^2 (u^\varepsilon)^2, \tag{43}$$

which holds for each $\varepsilon > 0$, with

$$q^*(\rho,m) = \frac{1}{2}\frac{m^3}{\rho^2} + me(\rho) + \rho m e'(\rho),$$

we obtain an inequality of the form

$$\partial_t \eta^*(\rho(x,t), m(x,t)) + \partial_x Q(x,t) \le 0, \qquad \text{in } \mathbb{R}^2_+, \tag{44}$$

in the sense of distributions, for some $Q \in \mathcal{M}(K \cap \mathbb{R}^2_+)$, for any compact $K \subset \mathbb{R}^2$. Because of the presence of the term $\rho^\varepsilon(u^\varepsilon)^3$ in $q^*(\rho^\varepsilon, m^\varepsilon)$, whose estimates obtained in [5] only guarantee the uniform boundedness in $L^1(K \cap \mathbb{R}^2_+)$, for any compact $K \subset \mathbb{R}^2$, we can only deduce that

$$\langle q^*(\rho^\varepsilon, m^\varepsilon), \partial_x \psi\rangle \to \langle Q, \partial_x \psi\rangle,$$

for any $\psi \in C_c^\infty(\mathbb{R}^2_+)$, for some (signed) Radon measure Q in $\mathbb{R}^2_+$, with finite total variation in $K \cap \mathbb{R}^2_+$, for any compact $K \subset \mathbb{R}^2$.

Actually, from the results in [5] we obtain

$$\int_{\mathbb{R}^2_+} \{\psi_t \eta^*(\rho, m)\, dx\, dt + \psi_x\, dQ\} + \int_{\mathbb{R}} \psi(x,0)\eta^*(\rho_0(x), m_0(x))\, dx \ge 0, \quad (45)$$

for all nonnegative $\psi \in C_c^\infty(\mathbb{R}^2)$, under suitable conditions on the initial data.

We can then apply Theorem 9 to conclude that the weak solution of the compressible isentropic Euler equations, $(\rho(x,t), m(x,t))$, obtained in [5] as the limit of the vanishing viscosity solutions of the corresponding Navier-Stokes equations, satisfies

$$(\rho, m) \in C((0,\infty) \setminus S; L^1_{\mathrm{loc}}(\mathbb{R})),$$

for some at most countable subset $S \subset (0,\infty)$, and, moreover, $(\rho(\cdot,t), m(\cdot,t)) \to (\rho_0(\cdot), m(\cdot))$, as $t \downarrow 0$, in $L^1_{\mathrm{loc}}(\mathbb{R})$.

Acknowledgements The author gratefully acknowledges the support from CNPq, through grant proc. 303950/2009-9, and FAPERJ, through grant E-26/103.019/2011.

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